

# Intro to misere game theory

Rebecca Milley

Grenfell Campus, Memorial University of Newfoundland, Canada

August 17, 2020

# Combinatorial games

# Combinatorial games

- Normal play: First player unable to move loses.

# Combinatorial games

- Normal play: First player unable to move loses.
  - Non-trivial equality, inequality, addition, negation.

# Combinatorial games

- Normal play: First player unable to move loses.
  - Non-trivial equality, inequality, addition, negation.
  - Positions form partially ordered abelian group.

# Combinatorial games

- Normal play: First player unable to move loses.
  - Non-trivial equality, inequality, addition, negation.
  - Positions form partially ordered abelian group.
  - Game reductions (give unique canonical form):

# Combinatorial games

- Normal play: First player unable to move loses.
  - Non-trivial equality, inequality, addition, negation.
  - Positions form partially ordered abelian group.
  - Game reductions (give unique canonical form):
    - **Domination:** If Left has two options and one is “always better” ( $\geq$ ) than the other, then remove the *dominated option*.

# Combinatorial games

- Normal play: First player unable to move loses.
  - Non-trivial equality, inequality, addition, negation.
  - Positions form partially ordered abelian group.
  - Game reductions (give unique canonical form):
    - **Domination:** If Left has two options and one is “always better” ( $\geq$ ) than the other, then remove the *dominated option*.
    - **Reversibility:** If Left can “predict” that Right will respond to the Left option  $A$  by moving to  $B$ , then replace the *reversible option*  $A$  with the left moves from  $B$ .

# Combinatorial games

- Normal play: First player unable to move loses.
  - Non-trivial equality, inequality, addition, negation.
  - Positions form partially ordered abelian group.
  - Game reductions (give unique canonical form):
    - **Domination:** If Left has two options and one is “always better” ( $\geq$ ) than the other, then remove the *dominated option*.
    - **Reversibility:** If Left can “predict” that Right will respond to the Left option  $A$  by moving to  $B$ , then replace the *reversible option*  $A$  with the left moves from  $B$ .
- Misere play: First player unable to move wins.

# Combinatorial games

- Normal play: First player unable to move loses.
  - Non-trivial equality, inequality, addition, negation.
  - Positions form partially ordered abelian group.
  - Game reductions (give unique canonical form):
    - **Domination:** If Left has two options and one is “always better” ( $\geq$ ) than the other, then remove the *dominated option*.
    - **Reversibility:** If Left can “predict” that Right will respond to the Left option  $A$  by moving to  $B$ , then replace the *reversible option*  $A$  with the left moves from  $B$ .
- Misere play: First player unable to move wins.
  - Everything is awful!

# General misere play

## General misere play

- Not merely the “opposite” of normal play.

## General misere play

- Not merely the “opposite” of normal play.
  - No relationship between misere, normal outcome (Mesdal & Ottaway 2007).

# General misere play

- Not merely the “opposite” of normal play.
  - No relationship between misere, normal outcome (Mesdal & Ottaway 2007).

Normal → Misère ↓	P	L	R	N
P				
L				
R				
N				

## General misere play

- Not merely the “opposite” of normal play.
  - No relationship between misere, normal outcome (Mesdal & Ottaway 2007).
- Addition is less intuitive.

# General misere play

- Not merely the “opposite” of normal play.
  - No relationship between misere, normal outcome (Mesdal & Ottaway 2007).
- Addition is less intuitive.

Normal outcomes:

$+$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{N}$	$\mathcal{P}$
$\mathcal{L}$	$\mathcal{L}$	$?$	$\mathcal{LUN}$	$\mathcal{L}$
$\mathcal{R}$	$?$	$\mathcal{R}$	$\mathcal{RUN}$	$\mathcal{R}$
$\mathcal{N}$	$\mathcal{LUN}$	$\mathcal{RUN}$	$?$	$\mathcal{N}$
$\mathcal{P}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{N}$	$\mathcal{P}$

# General misere play

- Not merely the “opposite” of normal play.
  - No relationship between misere, normal outcome (Mesdal & Ottaway 2007).
- Addition is less intuitive.

Misere outcomes:

$+$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{N}$	$\mathcal{P}$
$\mathcal{L}$	?	?	?	?
$\mathcal{R}$	?	?	?	?
$\mathcal{N}$	?	?	?	?
$\mathcal{P}$	?	?	?	?

# General misere play

- Zero is trivial.

# General misere play

- Zero is trivial.
  - Only game equal to zero is  $\{\cdot|\cdot\}$ .

# General misere play

- Zero is trivial.
  - Only game equal to zero is  $\{\cdot|\cdot\}$ .
  - $G - G \neq 0$  for all  $G \neq 0$ ; i.e., no inverses.

# General misere play

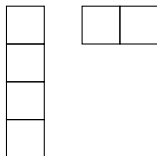
- Zero is trivial.
  - Only game equal to zero is  $\{\cdot|\cdot\}$ .
  - $G - G \neq 0$  for all  $G \neq 0$ ; i.e., no inverses.
- Equality, inequality is rare and difficult to prove.

# General misere play

- Zero is trivial.
  - Only game equal to zero is  $\{\cdot|\cdot\}$ .
  - $G - G \neq 0$  for all  $G \neq 0$ ; i.e., no inverses.
- Equality, inequality is rare and difficult to prove.
  - Can't use " $G = H \Leftrightarrow G - H \in \mathcal{P}$ " as normal play.

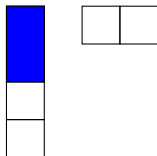
# General misere play

- Zero is trivial.
  - Only game equal to zero is  $\{\cdot|\cdot\}$ .
  - $G - G \neq 0$  for all  $G \neq 0$ ; i.e., no inverses.
- Equality, inequality is rare and difficult to prove.
  - Can't use " $G = H \Leftrightarrow G - H \in \mathcal{P}$ " as normal play.
  - Less simplification (domination, reversibility).



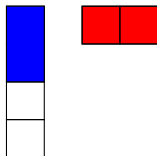
# General misere play

- Zero is trivial.
  - Only game equal to zero is  $\{\cdot|\cdot\}$ .
  - $G - G \neq 0$  for all  $G \neq 0$ ; i.e., no inverses.
- Equality, inequality is rare and difficult to prove.
  - Can't use " $G = H \Leftrightarrow G - H \in \mathcal{P}$ " as normal play.
  - Less simplification (domination, reversibility).



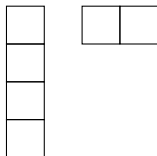
# General misere play

- Zero is trivial.
  - Only game equal to zero is  $\{\cdot|\cdot\}$ .
  - $G - G \neq 0$  for all  $G \neq 0$ ; i.e., no inverses.
- Equality, inequality is rare and difficult to prove.
  - Can't use " $G = H \Leftrightarrow G - H \in \mathcal{P}$ " as normal play.
  - Less simplification (domination, reversibility).



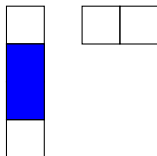
# General misere play

- Zero is trivial.
  - Only game equal to zero is  $\{\cdot|\cdot\}$ .
  - $G - G \neq 0$  for all  $G \neq 0$ ; i.e., no inverses.
- Equality, inequality is rare and difficult to prove.
  - Can't use " $G = H \Leftrightarrow G - H \in \mathcal{P}$ " as normal play.
  - Less simplification (domination, reversibility).



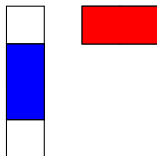
# General misere play

- Zero is trivial.
  - Only game equal to zero is  $\{\cdot|\cdot\}$ .
  - $G - G \neq 0$  for all  $G \neq 0$ ; i.e., no inverses.
- Equality, inequality is rare and difficult to prove.
  - Can't use " $G = H \Leftrightarrow G - H \in \mathcal{P}$ " as normal play.
  - Less simplification (domination, reversibility).



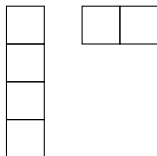
# General misere play

- Zero is trivial.
  - Only game equal to zero is  $\{\cdot|\cdot\}$ .
  - $G - G \neq 0$  for all  $G \neq 0$ ; i.e., no inverses.
- Equality, inequality is rare and difficult to prove.
  - Can't use " $G = H \Leftrightarrow G - H \in \mathcal{P}$ " as normal play.
  - Less simplification (domination, reversibility).



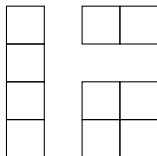
# General misere play

- Zero is trivial.
  - Only game equal to zero is  $\{\cdot|\cdot\}$ .
  - $G - G \neq 0$  for all  $G \neq 0$ ; i.e., no inverses.
- Equality, inequality is rare and difficult to prove.
  - Can't use " $G = H \Leftrightarrow G - H \in \mathcal{P}$ " as normal play.
  - Less simplification (domination, reversibility).



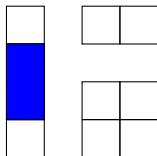
# General misere play

- Zero is trivial.
  - Only game equal to zero is  $\{\cdot|\cdot\}$ .
  - $G - G \neq 0$  for all  $G \neq 0$ ; i.e., no inverses.
- Equality, inequality is rare and difficult to prove.
  - Can't use " $G = H \Leftrightarrow G - H \in \mathcal{P}$ " as normal play.
  - Less simplification (domination, reversibility).



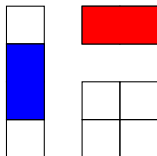
# General misere play

- Zero is trivial.
  - Only game equal to zero is  $\{\cdot|\cdot\}$ .
  - $G - G \neq 0$  for all  $G \neq 0$ ; i.e., no inverses.
- Equality, inequality is rare and difficult to prove.
  - Can't use " $G = H \Leftrightarrow G - H \in \mathcal{P}$ " as normal play.
  - Less simplification (domination, reversibility).



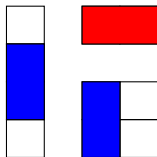
# General misere play

- Zero is trivial.
  - Only game equal to zero is  $\{\cdot|\cdot\}$ .
  - $G - G \neq 0$  for all  $G \neq 0$ ; i.e., no inverses.
- Equality, inequality is rare and difficult to prove.
  - Can't use " $G = H \Leftrightarrow G - H \in \mathcal{P}$ " as normal play.
  - Less simplification (domination, reversibility).



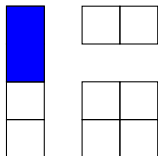
# General misere play

- Zero is trivial.
  - Only game equal to zero is  $\{\cdot|\cdot\}$ .
  - $G - G \neq 0$  for all  $G \neq 0$ ; i.e., no inverses.
- Equality, inequality is rare and difficult to prove.
  - Can't use " $G = H \Leftrightarrow G - H \in \mathcal{P}$ " as normal play.
  - Less simplification (domination, reversibility).



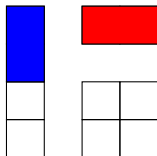
# General misere play

- Zero is trivial.
  - Only game equal to zero is  $\{\cdot|\cdot\}$ .
  - $G - G \neq 0$  for all  $G \neq 0$ ; i.e., no inverses.
- Equality, inequality is rare and difficult to prove.
  - Can't use " $G = H \Leftrightarrow G - H \in \mathcal{P}$ " as normal play.
  - Less simplification (domination, reversibility).



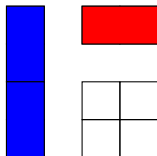
# General misere play

- Zero is trivial.
  - Only game equal to zero is  $\{\cdot|\cdot\}$ .
  - $G - G \neq 0$  for all  $G \neq 0$ ; i.e., no inverses.
- Equality, inequality is rare and difficult to prove.
  - Can't use " $G = H \Leftrightarrow G - H \in \mathcal{P}$ " as normal play.
  - Less simplification (domination, reversibility).



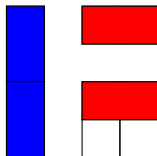
# General misere play

- Zero is trivial.
  - Only game equal to zero is  $\{\cdot|\cdot\}$ .
  - $G - G \neq 0$  for all  $G \neq 0$ ; i.e., no inverses.
- Equality, inequality is rare and difficult to prove.
  - Can't use " $G = H \Leftrightarrow G - H \in \mathcal{P}$ " as normal play.
  - Less simplification (domination, reversibility).



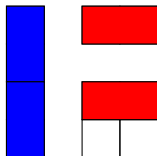
# General misere play

- Zero is trivial.
  - Only game equal to zero is  $\{\cdot|\cdot\}$ .
  - $G - G \neq 0$  for all  $G \neq 0$ ; i.e., no inverses.
- Equality, inequality is rare and difficult to prove.
  - Can't use " $G = H \Leftrightarrow G - H \in \mathcal{P}$ " as normal play.
  - Less simplification (domination, reversibility).



# General misere play

- Zero is trivial.
  - Only game equal to zero is  $\{ \cdot | \cdot \}$ .
  - $G - G \neq 0$  for all  $G \neq 0$ ; i.e., no inverses.
- Equality, inequality is rare and difficult to prove.
  - Can't use " $G = H \Leftrightarrow G - H \in \mathcal{P}$ " as normal play.
  - Less simplification (domination, reversibility).  
e.g., 0 and  $1 = \{ 0 | \cdot \}$  are incomparable.



# Restricted misere theory (Plambeck, Seigel)

## Restricted misere theory (Plambeck, Seigel)

- (Recall) **Equality:**  $G = H$  means  $o(G + X) = o(H + X)$  for all games  $X$ .

## Restricted misere theory (Plambeck, Seigel)

- (Recall) **Equality:**  $G = H$  means  $o(G + X) = o(H + X)$  for all games  $X$ .
- Define a **universe** as a set of games closed under addition, negation, and options.

## Restricted misere theory (Plambeck, Seigel)

- (Recall) **Equality:**  $G = H$  means  $o(G + X) = o(H + X)$  for all games  $X$ .
- Define a **universe** as a set of games closed under addition, negation, and options. Let  $\mathcal{U}$  be a universe.

## Restricted misere theory (Plambeck, Seigel)

- (Recall) **Equality:**  $G = H$  means  $o(G + X) = o(H + X)$  for all games  $X$ .
- Define a **universe** as a set of games closed under addition, negation, and options. Let  $\mathcal{U}$  be a universe.
  - **Equivalence mod  $\mathcal{U}$ :**  $G \equiv_{\mathcal{U}} H$  means

## Restricted misere theory (Plambeck, Seigel)

- (Recall) **Equality:**  $G = H$  means  $o(G + X) = o(H + X)$  for all games  $X$ .
- Define a **universe** as a set of games closed under addition, negation, and options. Let  $\mathcal{U}$  be a universe.
  - **Equivalence mod  $\mathcal{U}$ :**  $G \equiv_{\mathcal{U}} H$  means  $o(G + X) = o(H + X)$  for all  $X \in \mathcal{U}$ .

## Restricted misere theory (Plambeck, Seigel)

- (Recall) **Equality:**  $G = H$  means  $o(G + X) = o(H + X)$  for all games  $X$ .
- Define a **universe** as a set of games closed under addition, negation, and options. Let  $\mathcal{U}$  be a universe.
  - **Equivalence mod  $\mathcal{U}$ :**  $G \equiv_{\mathcal{U}} H$  means  $o(G + X) = o(H + X)$  for all  $X \in \mathcal{U}$ .
  - **Inequality modulo  $\mathcal{U}$ :**  $G \geq_{\mathcal{U}} H$  means  $o(G + X) \geq o(H + X)$  for all  $X \in \mathcal{U}$ .

# Restricted misere theory (Plambeck, Seigel)

- (Recall) **Equality:**  $G = H$  means  $o(G + X) = o(H + X)$  for all games  $X$ .
- Define a **universe** as a set of games closed under addition, negation, and options. Let  $\mathcal{U}$  be a universe.
  - **Equivalence mod  $\mathcal{U}$ :**  $G \equiv_{\mathcal{U}} H$  means  $o(G + X) = o(H + X)$  for all  $X \in \mathcal{U}$ .
  - **Inequality modulo  $\mathcal{U}$ :**  $G \geq_{\mathcal{U}} H$  means  $o(G + X) \geq o(H + X)$  for all  $X \in \mathcal{U}$ .  
ie, Left wins  $G + X$  if she wins  $H + X$ ,  $\forall X \in \mathcal{U}$ .

## Restricted misere theory (Plambeck, Seigel)

- (Recall) **Equality:**  $G = H$  means  $o(G + X) = o(H + X)$  for all games  $X$ .
- Define a **universe** as a set of games closed under addition, negation, and options. Let  $\mathcal{U}$  be a universe.
  - **Equivalence mod  $\mathcal{U}$ :**  $G \equiv_{\mathcal{U}} H$  means  $o(G + X) = o(H + X)$  for all  $X \in \mathcal{U}$ .
  - **Inequality modulo  $\mathcal{U}$ :**  $G \geq_{\mathcal{U}} H$  means  $o(G + X) \geq o(H + X)$  for all  $X \in \mathcal{U}$ .  
ie, Left wins  $G + X$  if she wins  $H + X$ ,  $\forall X \in \mathcal{U}$ .
- These definitions are natural and useful.

# Restricted misere theory (Plambeck, Seigel)

- (Recall) **Equality**:  $G = H$  means  $o(G + X) = o(H + X)$  for all games  $X$ .
- Define a **universe** as a set of games closed under addition, negation, and options. Let  $\mathcal{U}$  be a universe.
  - **Equivalence mod  $\mathcal{U}$** :  $G \equiv_{\mathcal{U}} H$  means  $o(G + X) = o(H + X)$  for all  $X \in \mathcal{U}$ .
  - **Inequality modulo  $\mathcal{U}$** :  $G \geq_{\mathcal{U}} H$  means  $o(G + X) \geq o(H + X)$  for all  $X \in \mathcal{U}$ .  
ie, Left wins  $G + X$  if she wins  $H + X$ ,  $\forall X \in \mathcal{U}$ .
- These definitions are natural and useful.
  - If we are analyzing some rule set (e.g., Domineering), all of the positions we compare are in the universe of that rule set.

# Restricted misere theory (Plambeck, Seigel)

- (Recall) **Equality:**  $G = H$  means  $o(G + X) = o(H + X)$  for all games  $X$ .
- Define a **universe** as a set of games closed under addition, negation, and options. Let  $\mathcal{U}$  be a universe.
  - **Equivalence mod  $\mathcal{U}$ :**  $G \equiv_{\mathcal{U}} H$  means  $o(G + X) = o(H + X)$  for all  $X \in \mathcal{U}$ .
  - **Inequality modulo  $\mathcal{U}$ :**  $G \geq_{\mathcal{U}} H$  means  $o(G + X) \geq o(H + X)$  for all  $X \in \mathcal{U}$ .  
ie, Left wins  $G + X$  if she wins  $H + X$ ,  $\forall X \in \mathcal{U}$ .
- These definitions are natural and useful.
  - If we are analyzing some rule set (e.g., Domineering), all of the positions we compare are in the universe of that rule set.
  - A position may have algebraic structure *modulo*  $\mathcal{U}$  that it doesn't have in general (e.g., invertibility, simplification).

# Some definitions and nice universes.

## Some definitions and nice universes.

- **Dicot**: at every point, either both players can move or neither can.  
Let  $\mathcal{D}$  be the universe of dicot games.

## Some definitions and nice universes.

- **Dicot:** at every point, either both players can move or neither can.  
Let  $\mathcal{D}$  be the universe of dicot games.
- **Left end:** Left has no move now.

## Some definitions and nice universes.

- **Dicot:** at every point, either both players can move or neither can.  
Let  $\mathcal{D}$  be the universe of dicot games.
- **Left end:** Left has no move now.  
**Dead left end:** Left has no move now and no move in any follower.

## Some definitions and nice universes.

- **Dicot:** at every point, either both players can move or neither can. Let  $\mathcal{D}$  be the universe of dicot games.
  - **Left end:** Left has no move now.  
**Dead left end:** Left has no move now and no move in any follower.  
**Dead-ending game:** All the end followers are dead ends. Let  $\mathcal{E}$  be the universe of dead-ending games.
-

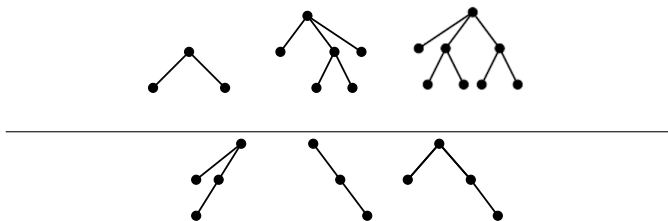
## Some definitions and nice universes.

- **Dicot:** at every point, either both players can move or neither can. Let  $\mathcal{D}$  be the universe of dicot games.
- **Left end:** Left has no move now.
- **Dead left end:** Left has no move now and no move in any follower.
- **Dead-ending game:** All the end followers are dead ends. Let  $\mathcal{E}$  be the universe of dead-ending games.



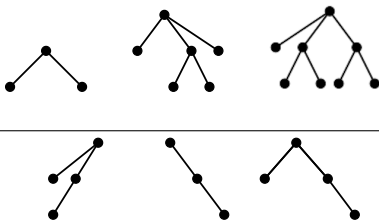
## Some definitions and nice universes.

- **Dicot:** at every point, either both players can move or neither can. Let  $\mathcal{D}$  be the universe of dicot games.
- **Left end:** Left has no move now.
- **Dead left end:** Left has no move now and no move in any follower.
- **Dead-ending game:** All the end followers are dead ends. Let  $\mathcal{E}$  be the universe of dead-ending games.



## Some definitions and nice universes.

- **Dicot:** at every point, either both players can move or neither can. Let  $\mathcal{D}$  be the universe of dicot games.
- **Left end:** Left has no move now.
- **Dead left end:** Left has no move now and no move in any follower.
- **Dead-ending game:** All the end followers are dead ends. Let  $\mathcal{E}$  be the universe of dead-ending games.



$\hookrightarrow$  Domineering, Hackenbush, NoGo, Snort  $\subset \mathcal{E}, \mathcal{D} \subset \mathcal{E}$ .

# Invertibility

Write  $\overline{G}$  instead of  $-G$ , call it the **conjugate**.

# Invertibility

Write  $\overline{G}$  instead of  $-G$ , call it the **conjugate**.

## Theorem (Milley 2013)

*If Left wins first on  $G + \overline{G} + Y$  for any left end  $Y \in \mathcal{U}$ , and if all followers of  $G$  are invertible modulo  $\mathcal{U}$ , then  $G + \overline{G} \equiv_{\mathcal{U}} 0$ .*

# Invertibility

Write  $\overline{G}$  instead of  $-G$ , call it the **conjugate**.

## Theorem (Milley 2013)

*If Left wins first on  $G + \overline{G} + Y$  for any left end  $Y \in \mathcal{U}$ , and if all followers of  $G$  are invertible modulo  $\mathcal{U}$ , then  $G + \overline{G} \equiv_{\mathcal{U}} 0$ .*

## Theorem (Milley and Renault, 2013)

*If  $G$  is an end then  $G + \overline{G} \equiv_{\mathcal{E}} 0$*

# Invertibility

Write  $\overline{G}$  instead of  $-G$ , call it the **conjugate**.

## Theorem (Milley 2013)

*If Left wins first on  $G + \overline{G} + Y$  for any left end  $Y \in \mathcal{U}$ , and if all followers of  $G$  are invertible modulo  $\mathcal{U}$ , then  $G + \overline{G} \equiv_{\mathcal{U}} 0$ .*

## Theorem (Milley and Renault, 2013)

*If  $G$  is an end then  $G + \overline{G} \equiv_{\mathcal{E}} 0$*

$\hookrightarrow$  e.g., all  $1 \times n$  Domineering boards are invertible modulo dead-ending games (and therefore modulo domineering games).

# Invertibility

Write  $\overline{G}$  instead of  $-G$ , call it the **conjugate**.


## Theorem (Milley 2013)

*If Left wins first on  $G + \overline{G} + Y$  for any left end  $Y \in \mathcal{U}$ , and if all followers of  $G$  are invertible modulo  $\mathcal{U}$ , then  $G + \overline{G} \equiv_{\mathcal{U}} 0$ .*

## Theorem (Milley and Renault, 2013)

*If  $G$  is an end then  $G + \overline{G} \equiv_{\mathcal{E}} 0$*

$\hookrightarrow$  e.g., all  $1 \times n$  Domineering boards are invertible modulo dead-ending games (and therefore modulo domineering games).

$\hookrightarrow$  also,   $\equiv_{\mathcal{E}} 0$

# Absolute Game Theory

# Absolute Game Theory

- New results for restricted misere play appear in a submitted paper by Larsson, Milley, Nowakowski, Renault, Santos.

# Absolute Game Theory

- New results for restricted misere play appear in a submitted paper by Larsson, Milley, Nowakowski, Renault, Santos.
- The paper applies results from *absolute* CGT (Larsson, Nowakowski, Santos) to misere play, specifically to the dicotic and dead-ending universes.

# Absolute Game Theory

- New results for restricted misere play appear in a submitted paper by Larsson, Milley, Nowakowski, Renault, Santos.
- The paper applies results from *absolute* CGT (Larsson, Nowakowski, Santos) to misere play, specifically to the dicotic and dead-ending universes.
- The main results are (1) a comparison test for these universes, and (2) reversibility reductions leading to (3) unique canonical forms.

# Strong outcome

## Strong outcome

- The **strong outcome** of  $G$ , denoted  $\hat{o}(G)$ , is the outcome of  $G$  if the second player is allowed to 'pass' an unlimited number of times.

## Strong outcome

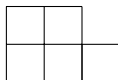
- The **strong outcome** of  $G$ , denoted  $\hat{o}(G)$ , is the outcome of  $G$  if the second player is allowed to 'pass' an unlimited number of times.
  - $\hat{o}_L(G)$  is the outcome of  $G$  if Left plays first and Right can pass.

## Strong outcome

- The **strong outcome** of  $G$ , denoted  $\hat{o}(G)$ , is the outcome of  $G$  if the second player is allowed to 'pass' an unlimited number of times.
  - $\hat{o}_L(G)$  is the outcome of  $G$  if Left plays first and Right can pass.
  - $\hat{o}_R(G)$  is the outcome of  $G$  if Right plays first and Left can pass.

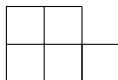
## Strong outcome

- The **strong outcome** of  $G$ , denoted  $\hat{o}(G)$ , is the outcome of  $G$  if the second player is allowed to 'pass' an unlimited number of times.
  - $\hat{o}_L(G)$  is the outcome of  $G$  if Left plays first and Right can pass.
  - $\hat{o}_R(G)$  is the outcome of  $G$  if Right plays first and Left can pass.
- Example 1:



## Strong outcome

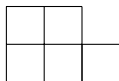
- The **strong outcome** of  $G$ , denoted  $\hat{o}(G)$ , is the outcome of  $G$  if the second player is allowed to 'pass' an unlimited number of times.
  - $\hat{o}_L(G)$  is the outcome of  $G$  if Left plays first and Right can pass.
  - $\hat{o}_R(G)$  is the outcome of  $G$  if Right plays first and Left can pass.
- Example 1:



Under misere play, this game has outcome  $\mathcal{N}$  and strong outcome  $\mathcal{P}$ .

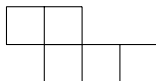
## Strong outcome

- The **strong outcome** of  $G$ , denoted  $\hat{o}(G)$ , is the outcome of  $G$  if the second player is allowed to 'pass' an unlimited number of times.
  - $\hat{o}_L(G)$  is the outcome of  $G$  if Left plays first and Right can pass.
  - $\hat{o}_R(G)$  is the outcome of  $G$  if Right plays first and Left can pass.
- Example 1:



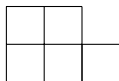
Under misere play, this game has outcome  $\mathcal{N}$  and strong outcome  $\mathcal{P}$ .

- Example 2:



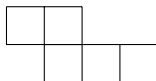
## Strong outcome

- The **strong outcome** of  $G$ , denoted  $\hat{o}(G)$ , is the outcome of  $G$  if the second player is allowed to 'pass' an unlimited number of times.
  - $\hat{o}_L(G)$  is the outcome of  $G$  if Left plays first and Right can pass.
  - $\hat{o}_R(G)$  is the outcome of  $G$  if Right plays first and Left can pass.
- Example 1:



Under misere play, this game has outcome  $\mathcal{N}$  and strong outcome  $\mathcal{P}$ .

- Example 2:



Under misere play, this game has outcome  $\mathcal{N}$  and strong outcome  $\mathcal{L}$ .

## Comparison test

Recall: In general, to show  $G \succeq_{\mathcal{U}} H$ , we must show

$$o(G + X) \geq o(H + X) \quad \forall X \in \mathcal{U}.$$

## Comparison test

Recall: In general, to show  $G \geq_{\mathcal{U}} H$ , we must show

$$o(G + X) \geq o(H + X) \quad \forall X \in \mathcal{U}.$$

### Theorem ( Larsson,M,N,R,S)

Let  $G, H \in \mathcal{E}$ . Then  $G \geq_{\mathcal{E}} H$  if and only if

- (i)  $\hat{o}(G) \geq \hat{o}(H)$ ; and
- (ii) for all left options  $H^L$ , there exists EITHER a  $G^L$  with  $G^L \geq_{\mathcal{E}} H^L$  OR an  $H^{LR}$  with  $G \geq_{\mathcal{E}} H^{LR}$ ; and
- (iii) for all right options  $G^R$ , there exists EITHER an  $H^R$  with  $G^R \geq_{\mathcal{E}} H^R$  OR a  $G^{RL}$  with  $G^{RL} \geq_{\mathcal{E}} H$ .

## Comparison test

Recall: In general, to show  $G \geq_{\mathcal{U}} H$ , we must show

$$o(G + X) \geq o(H + X) \quad \forall X \in \mathcal{U}.$$

### Theorem ( Larsson,M,N,R,S)

Let  $G, H \in \mathcal{E}$ . Then  $G \geq_{\mathcal{E}} H$  if and only if

- (i)  $\hat{o}(G) \geq \hat{o}(H)$ ; and
- (ii) for all left options  $H^L$ , there exists EITHER a  $G^L$  with  $G^L \geq_{\mathcal{E}} H^L$  OR an  $H^{LR}$  with  $G \geq_{\mathcal{E}} H^{LR}$ ; and
- (iii) for all right options  $G^R$ , there exists EITHER an  $H^R$  with  $G^R \geq_{\mathcal{E}} H^R$  OR a  $G^{RL}$  with  $G^{RL} \geq_{\mathcal{E}} H$ .

To prove 'if': need to show that Left winning  $H + X \Rightarrow$  Left wins  $G + X$ .

## Comparison test

Recall: In general, to show  $G \geq_{\mathcal{U}} H$ , we must show

$$o(G + X) \geq o(H + X) \quad \forall X \in \mathcal{U}.$$

### Theorem ( Larsson,M,N,R,S)

Let  $G, H \in \mathcal{E}$ . Then  $G \geq_{\mathcal{E}} H$  if and only if

- (i)  $\hat{o}(G) \geq \hat{o}(H)$ ; and
- (ii) for all left options  $H^L$ , there exists EITHER a  $G^L$  with  $G^L \geq_{\mathcal{E}} H^L$  OR an  $H^{LR}$  with  $G \geq_{\mathcal{E}} H^{LR}$ ; and
- (iii) for all right options  $G^R$ , there exists EITHER an  $H^R$  with  $G^R \geq_{\mathcal{E}} H^R$  OR a  $G^{RL}$  with  $G^{RL} \geq_{\mathcal{E}} H$ .

To prove 'if': need to show that Left winning  $H + X \Rightarrow$  Left wins  $G + X$ .

Base case:  $X$  is a left end.

Then the outcome of Left playing first on  $G + X$  is  $\hat{o}_L(G)$ .

## Comparisons in misere domineering

$$G \geq_{\varepsilon} H \Leftrightarrow \begin{array}{l} \text{(i) } \hat{o}(G) \geq \hat{o}(H) \\ \text{(ii) } \forall H^L, \exists G^L \text{ with } G^L \geq_{\varepsilon} H^L \text{ OR } H^{LR} \text{ with } G \geq_{\varepsilon} H^{LR} \\ \text{(iii) } \forall G^R, \exists H^R \text{ with } G^R \geq_{\varepsilon} H^R \text{ OR } G^{RL} \text{ with } G^{RL} \geq_{\varepsilon} H. \end{array}$$

# Comparisons in misere domineering

- $G \geq_{\varepsilon} H \Leftrightarrow$  (i)  $\hat{o}(G) \geq \hat{o}(H)$   
 (ii)  $\forall H^L, \exists G^L$  with  $G^L \geq_{\varepsilon} H^L$  OR  $H^{LR}$  with  $G \geq_{\varepsilon} H^{LR}$   
 (iii)  $\forall G^R, \exists H^R$  with  $G^R \geq_{\varepsilon} H^R$  OR  $G^{RL}$  with  $G^{RL} \geq_{\varepsilon} H$ .

$$0 > \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}$$

$$0 > \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}$$

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} > \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}$$

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} > \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} > \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline \square \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} > \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}$$

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} > \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}$$

# Domination & reversibility

# Domination & reversibility

- Domination works as normal in restricted misere play.

# Domination & reversibility

- Domination works as normal in restricted misere play.
- Reversibility does not!

# Domination & reversibility

- Domination works as normal in restricted misere play.
- Reversibility does not!
  - The problem is when an otherwise reversible option would reverse through an end...

# Domination & reversibility

- Domination works as normal in restricted misere play.
- Reversibility does not!
  - The problem is when an otherwise reversible option would reverse through an end...
  - Other (“open”) reversibility works as normal in any misere universe (LMNRS).

# Domination & reversibility

- Domination works as normal in restricted misere play.
- Reversibility does not!
  - The problem is when an otherwise reversible option would reverse through an end...
  - Other (“open”) reversibility works as normal in any misere universe (LMNRS).
  - “End” reversibility has been solved for dicotic (DRSS 2015) and dead-ending (LMNRS) universes.

# Domination & reversibility

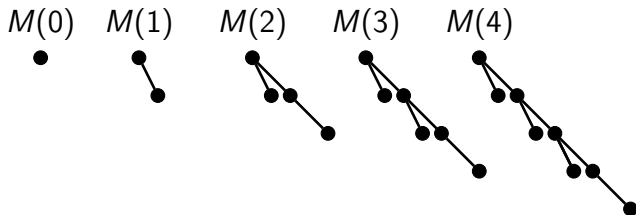
- Domination works as normal in restricted misere play.
- Reversibility does not!
  - The problem is when an otherwise reversible option would reverse through an end...
  - Other (“open”) reversibility works as normal in any misere universe (LMNRS).
  - “End” reversibility has been solved for dicotic (DRSS 2015) and dead-ending (LMNRS) universes.

Before we get to end-reversible reductions, we need **perfect murder** games:  $M(n) = \{\cdot \mid 0, M(n-1)\}$ .

# Domination & reversibility

- Domination works as normal in restricted misere play.
- Reversibility does not!
  - The problem is when an otherwise reversible option would reverse through an end...
  - Other (“open”) reversibility works as normal in any misere universe (LMNRS).
  - “End” reversibility has been solved for dicotic (DRSS 2015) and dead-ending (LMNRS) universes.

Before we get to end-reversible reductions, we need **perfect murder** games:  $M(n) = \{\cdot \mid 0, M(n-1)\}$ .



# End-reversible reductions

# End-reversible reductions

In any universe:

# End-reversible reductions

In any universe:

1. Remove dominated options.

# End-reversible reductions

In any universe:

1. Remove dominated options.
2. Reverse open-reversible options.

# End-reversible reductions

In any universe:

1. Remove dominated options.
2. Reverse open-reversible options.

In  $\mathcal{D}$ :

# End-reversible reductions

In any universe:

1. Remove dominated options.
2. Reverse open-reversible options.

In  $\mathcal{D}$ :

3. Replace end-reversible options by  $*$ .

# End-reversible reductions

In any universe:

1. Remove dominated options.
2. Reverse open-reversible options.

In  $\mathcal{D}$ :

3. Replace end-reversible options by  $*$ .
4. Replace  $\{* \mid *\}$  by 0.

In  $\mathcal{E}$ :

# End-reversible reductions

In any universe:

1. Remove dominated options.
2. Reverse open-reversible options.

In  $\mathcal{D}$ :

3. Replace end-reversible options by  $*$ .
4. Replace  $\{* \mid *\}$  by 0.

In  $\mathcal{E}$ :

3. Remove non-fundamental end-reversible options, including removal of lone options as long as the result is in  $\mathcal{E}$ .

# End-reversible reductions

In any universe:

1. Remove dominated options.
2. Reverse open-reversible options.

In  $\mathcal{D}$ :

3. Replace end-reversible options by  $*$ .
4. Replace  $\{* \mid *\}$  by 0.

In  $\mathcal{E}$ :

3. Remove non-fundamental end-reversible options, including removal of lone options as long as the result is in  $\mathcal{E}$ .
4. Simultaneously remove lone left and lone right end-reversible options; i.e., replace  $\{A \mid C\}$  with 0 if  $A$  and  $C$  are end-reversible.

# End-reversible reductions

In any universe:

1. Remove dominated options.
2. Reverse open-reversible options.

In  $\mathcal{D}$ :

3. Replace end-reversible options by  $*$ .
4. Replace  $\{* \mid *\}$  by 0.

In  $\mathcal{E}$ :

3. Remove non-fundamental end-reversible options, including removal of lone options as long as the result is in  $\mathcal{E}$ .
4. Simultaneously remove lone left and lone right end-reversible options; i.e., replace  $\{A \mid C\}$  with 0 if  $A$  and  $C$  are end-reversible.
5. Replace other end-reversible options by  $\{\cdot \mid M(n)\}$  for left options or  $\{-M(n) \mid \cdot\}$  for right.

# Unique canonical form

## Unique canonical form

- If  $G, H \in \mathcal{D}$  are reduced and  $G \equiv_{\mathcal{D}} H$  then  $G$  and  $H$  are identical. (Dorbec et al.)

## Unique canonical form

- If  $G, H \in \mathcal{D}$  are reduced and  $G \equiv_{\mathcal{D}} H$  then  $G$  and  $H$  are identical. (Dorbec et al.)
- If  $G, H \in \mathcal{E}$  are reduced and  $G \equiv_{\mathcal{E}} H$  then  $G$  and  $H$  are identical. (Larsson et al.)

# The conjugate property

# The conjugate property

- Recall, in general, we have  $G + \overline{G} \neq 0$ .

# The conjugate property

- Recall, in general, we have  $G + \overline{G} \neq 0$ .
- For some  $\mathcal{U}$ , we might have  $G + \overline{G} \equiv_{\mathcal{U}} 0$ .

## The conjugate property

- Recall, in general, we have  $G + \overline{G} \neq 0$ .
- For some  $\mathcal{U}$ , we might have  $G + \overline{G} \equiv_{\mathcal{U}} 0$ .
- We could maybe even have  $G + H \equiv_{\mathcal{U}} 0$ , with  $H \not\equiv_{\mathcal{U}} \overline{G}$ .

## The conjugate property

- Recall, in general, we have  $G + \overline{G} \neq 0$ .
- For some  $\mathcal{U}$ , we might have  $G + \overline{G} \equiv_{\mathcal{U}} 0$ .
- We could maybe even have  $G + H \equiv_{\mathcal{U}} 0$ , with  $H \not\equiv_{\mathcal{U}} \overline{G}$ . !?

# The conjugate property

- Recall, in general, we have  $G + \overline{G} \neq 0$ .
- For some  $\mathcal{U}$ , we might have  $G + \overline{G} \equiv_{\mathcal{U}} 0$ .
- We could maybe even have  $G + H \equiv_{\mathcal{U}} 0$ , with  $H \not\equiv_{\mathcal{U}} \overline{G}$ . !?  
↔ Impartial example in Plambeck 2008.

# The conjugate property

- Recall, in general, we have  $G + \overline{G} \neq 0$ .
- For some  $\mathcal{U}$ , we might have  $G + \overline{G} \equiv_{\mathcal{U}} 0$ .
- We could maybe even have  $G + H \equiv_{\mathcal{U}} 0$ , with  $H \not\equiv_{\mathcal{U}} \overline{G}$ . !?
  - ↔ Impartial example in Plambeck 2008.
  - ↔ Not-really-an-example in Milley 2015.

# The conjugate property

- Recall, in general, we have  $G + \overline{G} \neq 0$ .
- For some  $\mathcal{U}$ , we might have  $G + \overline{G} \equiv_{\mathcal{U}} 0$ .
- We could maybe even have  $G + H \equiv_{\mathcal{U}} 0$ , with  $H \not\equiv_{\mathcal{U}} \overline{G}$ . !?  
     $\hookrightarrow$  Impartial example in Plambeck 2008.  
     $\hookrightarrow$  Not-really-an-example in Milley 2015.
- We say  $\mathcal{U}$  has the **conjugate property** if this cannot happen.

# The conjugate property

- Recall, in general, we have  $G + \overline{G} \neq 0$ .
- For some  $\mathcal{U}$ , we might have  $G + \overline{G} \equiv_{\mathcal{U}} 0$ .
- We could maybe even have  $G + H \equiv_{\mathcal{U}} 0$ , with  $H \not\equiv_{\mathcal{U}} \overline{G}$ . !?  
     $\hookrightarrow$  Impartial example in Plambeck 2008.  
     $\hookrightarrow$  Not-really-an-example in Milley 2015.
- We say  $\mathcal{U}$  has the **conjugate property** if this cannot happen.
- Larsson et al: dicotic and dead-ending universes have the conjugate property:

# The conjugate property

- Recall, in general, we have  $G + \overline{G} \neq 0$ .
- For some  $\mathcal{U}$ , we might have  $G + \overline{G} \equiv_{\mathcal{U}} 0$ .
- We could maybe even have  $G + H \equiv_{\mathcal{U}} 0$ , with  $H \not\equiv_{\mathcal{U}} \overline{G}$ . !?  
     $\hookrightarrow$  Impartial example in Plambeck 2008.  
     $\hookrightarrow$  Not-really-an-example in Milley 2015.
- We say  $\mathcal{U}$  has the **conjugate property** if this cannot happen.
- Larsson et al: dicotic and dead-ending universes have the conjugate property:  
    i.e., if  $G$  has an additive inverse, then it is  $\overline{G}$ .

# Ongoing work

## Ongoing work

- Apply results for dead-ending universe to specific rule sets within  $\mathcal{E}$ , such as domineering, to analyze these games under misère play.

## Ongoing work

- Apply results for dead-ending universe to specific rule sets within  $\mathcal{E}$ , such as domineering, to analyze these games under misère play.
- Extend algebraic results to other universes besides  $\mathcal{D}$  and  $\mathcal{E}$ .

## Ongoing work

- Apply results for dead-ending universe to specific rule sets within  $\mathcal{E}$ , such as domineering, to analyze these games under misère play.
- Extend algebraic results to other universes besides  $\mathcal{D}$  and  $\mathcal{E}$ .
  - Comparison tests.

## Ongoing work

- Apply results for dead-ending universe to specific rule sets within  $\mathcal{E}$ , such as domineering, to analyze these games under misère play.
- Extend algebraic results to other universes besides  $\mathcal{D}$  and  $\mathcal{E}$ .
  - Comparison tests.
  - Reversibility through ends.

## Ongoing work

- Apply results for dead-ending universe to specific rule sets within  $\mathcal{E}$ , such as domineering, to analyze these games under misère play.
- Extend algebraic results to other universes besides  $\mathcal{D}$  and  $\mathcal{E}$ .
  - Comparison tests.
  - Reversibility through ends.
  - Unique canonical forms.

## Ongoing work

- Apply results for dead-ending universe to specific rule sets within  $\mathcal{E}$ , such as domineering, to analyze these games under misère play.
- Extend algebraic results to other universes besides  $\mathcal{D}$  and  $\mathcal{E}$ .
  - Comparison tests.
  - Reversibility through ends.
  - Unique canonical forms.
  - Conjugate property?

# References

- Allen2009 M.R. Allen, An investigation of partizan misère games, PhD thesis, Dalhousie, 2009.
- DorbecRSS P. Dorbec, G. Renault, A.N. Siegel, E. Sopena, Dicots, and a taxonomic ranking for misère games, *Journal of Combinatorial Theory, Series A* **130** (2015), 42–63.
- Ettinger J.M. Ettinger, Topics in Combinatorial Games, PhD Thesis, University of Wisconsin–Madison, 1996.
- LarssonNS2016A U. Larsson, R.J. Nowakowski, C.P. Santos, Absolute combinatorial game theory, (2016).
- LarssonNPS U. Larsson, R.J. Nowakowski, J.P. Neto, C.P. Santos, Guaranteed scoring games, *Electronic Journal of Combinatorics* **23(2)** (2016), #P3.27.
- MckayMN N.A. McKay, R. Milley, R.J. Nowakowski, Misère-play hackenbush sprigs, *International Journal of Game Theory* **45** (2015), 731–742.
- MesdaO2007 G.A. Mesdal, P. Ottaway, Simplification of partizan games in misère play, *Integers* **7** (2007).
- Milley2015 R. Milley, Partizan kayles and misere invertibility, *Integers* **15** (2015), #G3.
- MilleyRenault R. Milley, G. Renault, Dead ends in misère play: the misère monoid of canonical numbers, *Discrete Mathematics* **313** (2013), 2223–2231.
- MilleyRenault2017 R. Milley, G. Renault, Restricted developments in partizan mis'ere game theory, in *Games of No Chance 5*, MSRI Book Series, Cambridge University Press, Massachusetts, 2017.
- Ottaw2009 P. Ottaway, Combinatorial games with restricted options under normal and misère play, PhD thesis, Dalhousie University, 2009.
- Plamb2005 T.E. Plambeck, Taming the wild in impartial combinatorial games, *Integers* **5** (2005), #G5.
- Plamb2008 T.E. Plambeck, Advances in losing, in M.H. Albert and R.J. Nowakowski (Eds.) *Games of No Chance 3*, MSRI Book Series **56**, Cambridge University Press, Massachusetts, 2008.
- PlambS2008 T.E. Plambeck, A.N. Siegel, Misère quotients for impartial games, *Journal of Combinatorial Theory, Series A* **115** (2008), 593–622.
- Siege A.N. Siegel, Misère canonical forms of partizan games, in R.J. Nowakowski (Ed.) *Games of No Chance 4*, MSRI Book Series **63**, Cambridge University Press, Massachusetts, 2015.
- Si A.N. Siegel, *Combinatorial Game Theory*, American Math. Soc. (2013).