Loopy Games

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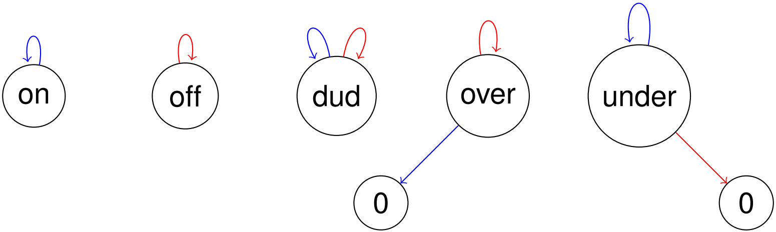
Outline

Definition of a Loopy Game

A loopy game is a graph where

- Each vertex represents a position;
- There is a specified starting vertex;
- Moves are indicated by directed edges labeled Left, Right



Assumptions for this Tutorial

- *finite* loopy games (finitely many vertices)
- *normal* play convention (last move wins)
- Any infinite play is a draw

References

[WW]

Berlekamp, Conway, and Guy. *Winning Ways for your Mathematical Plays*, 2nd ed., Vol 2, Chapter 11 "Games Infinite and Indefinite".

[CGT]

Siegel, Aaron. *Combinatorial Game Theory*, Chapter VI "Loopy Games".

References Errata

[WW]

- Bottom of p. 353 (boxed claim) only applies to stoppers.
- p. 361–363, all games should be assumed to be stoppers.
- Extras, p. 369–370, contrary to the claim, the dominated options cannot be omitted (cf. [CGT] Exercise VI.4.8).

[CGT]

- Exercise VI.1.4(c) is wrong (part (d) is suspect as well). OK if G is a stopper.
- In sections VI.2 and VI.5, all games should be assumed to be stoppers.

Comparison of Approaches

The definitions in these slides are equivalent to those in [WW], but incorporate ideas from [CGT].

$G \geq_{WW} H \Rightarrow G \geq_{CGT} H$ (and similarly for other operators). I do not know if the converse holds. To establish equivalence, we would need to prove the converses of both parts of [CGT] Theorem VI.4.4.

Let G be a loopy game.

Play

A **play**, or **run**, of G is any sequence of moves starting at G. It may be finite or infinite, and is **not necessarily alternating**.

Enders and stoppers

Let G be a loopy game.

Ender

G is an **ender** if there is no infinite play starting at any vertex in G.

Finite enders are the traditional finite loopfree games.

Stopper

G is a **stopper** if there is no infinite *alternating* play starting at any vertex in G.

Sums, Negatives, Zero

Let G and H be loopy games.

$$G + H = \{G^L + H, G + H^L | G^R + H, G + H^R\}.$$

Choose one of the components G or H, then make a move in that component.

$$-G = \{-G^R | -G^L\}.$$

The two sides are reversed.

$$0 = \{|\}.$$

Warning: $G - G$ is not necessarily equal to 0 (equality to be defined later).

Survival

Survival

A player **survives** a game if they win or draw.

Comparison of on, off, and dud

Recall that

$$\text{on} = \{\text{on}|\}$$

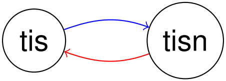
$$\text{off} = \{|\text{off}\}$$

$$\text{dud} = \{\text{dud}|\text{dud}\}$$

We have, for any loopy game G:

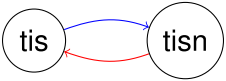
- $\text{off} \leq G \leq \text{on}$.
- $\text{on} + \text{off} = \text{dud}$.
- $\text{dud} + G = \text{dud}$.

Sidling



Define $\mathrm{tis} = \{\mathrm{tisn}|\}$, $\mathrm{tisn} = \{|\mathrm{tis}\}$.

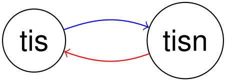
Sidling



Define $\text{tis} = \{\text{tisn}|\}$, $\text{tisn} = \{|\text{tis}\}$.

- $\text{tis} \leq \text{on}$, $\text{tisn} \leq \text{on}$.

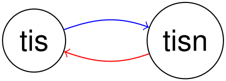
Sidling



Define $\text{tis} = \{\text{tisn}|\}$, $\text{tisn} = \{|\text{tis}\}$.

- $\text{tis} \leq \text{on}$, $\text{tisn} \leq \text{on}$.
- $\text{tis} \leq \{\text{on}|\} = \text{on}$, $\text{tisn} \leq \{|\text{on}\} = 0$.

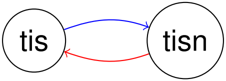
Sidling



Define $\text{tis} = \{\text{tisn}|\}$, $\text{tisn} = \{|\text{tis}\}$.

- $\text{tis} \leq \text{on}$, $\text{tisn} \leq \text{on}$.
- $\text{tis} \leq \{\text{on}|\} = \text{on}$, $\text{tisn} \leq \{|\text{on}\} = 0$.
- $\text{tis} \leq \{0|\} = 1$, $\text{tisn} \leq \{|\text{on}\} = 0$.

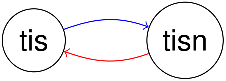
Sidling



Define $\mathrm{tis} = \{\mathrm{tisn}|\}$, $\mathrm{tisn} = \{|\mathrm{tis}\}$.

- $\mathrm{tis} \le \mathrm{on}$, $\mathrm{tisn} \le \mathrm{on}$.
- $\mathrm{tis} \le \{\mathrm{on}|\} = \mathrm{on}$, $\mathrm{tisn} \le \{|\mathrm{on}\} = 0$.
- $\mathrm{tis} \le \{0|\} = 1$, $\mathrm{tisn} \le \{|\mathrm{on}\} = 0$.
- $\mathrm{tis} \le \{0|\} = 1$, $\mathrm{tisn} \le \{|1\} = 0$.

Sidling



Define $\text{tis} = \{\text{tisn}|\}$, $\text{tisn} = \{|\text{tis}\}$.

- $\text{tis} \leq \text{on}$, $\text{tisn} \leq \text{on}$.
- $\text{tis} \leq \{\text{on}|\} = \text{on}$, $\text{tisn} \leq \{|\text{on}\} = 0$.
- $\text{tis} \leq \{0|\} = 1$, $\text{tisn} \leq \{|\text{on}\} = 0$.
- $\text{tis} \leq \{0|\} = 1$, $\text{tisn} \leq \{|1\} = 0$.

Conclusion:

$$\text{tis} \leq 1, \text{tisn} \leq 0.$$

Similarly:

$$\text{tis} \geq 0, \text{tisn} \geq -1.$$

Sidling gives stopper-sides

Bounds for tis and tisn

$$1 \geq \text{tis} \geq 0$$

$$0 \geq \text{tisn} \geq -1$$

These are the best stopper bounds

Let S be any stopper.
$S \geq \text{tis}$ iff $S \geq 1$.
$S \leq \text{tis}$ iff $S \leq 0$.

Onside and offside notation

$$\text{tis} = 1\&0$$

$$\text{tisn} = 0\& -1$$

Comparison of Enders and Stoppers

Comparison of Enders

If G and H are **enders**, $G \geq H$ iff Left wins $G - H$ going second.

Comparison of Stoppers

If G and H are **stoppers**, $G \geq H$ iff Left survives $G - H$ going second.

Naive Generalization Fails

Left survives going second in both

$$G - \mathrm{dud}$$

and

$$\mathrm{dud} - H,$$

but not necessarily in

$$G - H.$$

Play can get "stuck" on dud.

Stuck

In the game $G - H$, a play **gets stuck on** G if the play is infinite and all but finitely many moves are on G.

[CGT] "concentrates on $-H$" = "does not get stuck on G"

Comparison Operators

Let G and H be loopy games.

Left and Right Biased Comparison

$G \mathrel{\hat{\gtrsim}} H$ (resp., $G \mathrel{\check{\gtrsim}} H$) iff Left can survive $G - H$ going second without getting stuck on $-H$ (resp., G).

Unbiased Comparison

$G \geq H$ iff $G \mathrel{\hat{\gtrsim}} H$ and $G \mathrel{\check{\gtrsim}} H$.

Comparison Operators 2

[WW] notations:

$$G^{+} \geq H^{+} \Leftrightarrow G \hat{\geq} H$$

$$G^{-} \geq H^{-} \Leftrightarrow G \check{\geq} H$$

Comparison with Zero

Let G be a loopy game.

- $G \hat{\geq} 0$ iff Left can survive G going second.
- $G \check{\geq} 0$ iff Left can win G going second.

Equality Operators

Let G and H be loopy games.

Biased Equality

$G \hat{=} H$ iff $G \hat{\gtrsim} H$ and $H \hat{\gtrsim} G$. (Similarly for $\check{=}$.)

Unbiased Equality

$G = H$ iff $G \hat{=} H$ and $G \check{=} H$.

Properties of Comparison Operators

Let G, H, K be loopy games.

- $G \mathrel{\hat{\gtrsim}} G$.
- If $G \mathrel{\hat{\gtrsim}} H$ and $H \mathrel{\hat{\gtrsim}} K$, then $G \mathrel{\hat{\gtrsim}} K$.
- If $G \mathrel{\hat{\gtrsim}} H$, then $G + K \mathrel{\hat{\gtrsim}} H + K$.

Similarly for $\check{\gtrsim}$, $\gtrsim$, $\hat{\eqsim}$, $\check{\eqsim}$, and $=$.

Comparison of Stoppers

Let S and T be stoppers, and let G be a loopy game.

You cannot get stuck on a stopper.

Comparing a stopper with a loopy game

$S \geq G$ iff $S \hat{\gtrsim} G$.
$G \geq S$ iff $G \check{\gtrsim} S$.

Comparing two stoppers

$S \geq T$ iff $S \hat{\gtrsim} T$ iff $S \check{\gtrsim} T$ iff Left survives $S - T$ going second.

Swivel Chair Argument

The **Swivel Chair Argument** is used to prove that $\hat{\gtrsim}$ is transitive.

Assumptions

Left survives $G - H$ going second without getting stuck on $-H$.
Left survives $H - K$ going second without getting stuck on $-K$.

Goal

Left survives $G - K$ going second without getting stuck on $-K$.

Swivel Chair Argument

Assumptions

Left survives $G - H$ going second without getting stuck on $-H$.
Left survives $H - K$ going second without getting stuck on $-K$.

Goal

Left survives $G - K$ going second without getting stuck on $-K$.

Initial setup

Right

$G - K$

Left

$-H + H$

right

Swivel Chair Argument

Assumptions

Left survives $G - H$ going second without getting stuck on $-H$.
Left survives $H - K$ going second without getting stuck on $-K$.

Goal

Left survives $G - K$ going second without getting stuck on $-K$.

Assume Right moves $G \to G^R$

Right

$$G^R - K$$

Left

$$-H + H$$

right

Swivel Chair Argument

Assumptions

Left survives $G - H$ going second without getting stuck on $-H$.
Left survives $H - K$ going second without getting stuck on $-K$.

Goal

Left survives $G - K$ going second without getting stuck on $-K$.

If Left has a response $G^R \to G^{RL}$, then we're done.

Right

$G^{RL} - K$

Left

$-H + H$

right

Swivel Chair Argument

Assumptions

Left survives $G - H$ going second without getting stuck on $-H$.
Left survives $H - K$ going second without getting stuck on $-K$.

Goal

Left survives $G - K$ going second without getting stuck on $-K$.

Otherwise, assume Left's response is $-H \to -H^R$.

Right

$$G^R - K$$

Left

$$-H^R + H$$

right

Swivel Chair Argument

Assumptions

Left survives $G - H$ going second without getting stuck on $-H$.
Left survives $H - K$ going second without getting stuck on $-K$.

Goal

Left survives $G - K$ going second without getting stuck on $-K$.

right copies the move $H \to H^R$.

Right

$$G^R - K$$

Left

$$-H^R + H^R$$

right

Swivel Chair Argument

Assumptions

Left survives $G - H$ going second without getting stuck on $-H$.
Left survives $H - K$ going second without getting stuck on $-K$.

Goal

Left survives $G - K$ going second without getting stuck on $-K$.

If Left has a response $-K \to -K^R$, we're done.

Right

$G^R - K^R$

Left

$-H^R + H^R$

right

Swivel Chair Argument

Assumptions

Left survives $G - H$ going second without getting stuck on $-H$.
Left survives $H - K$ going second without getting stuck on $-K$.

Goal

Left survives $G - K$ going second without getting stuck on $-K$.

If Left's response is $H^R \to H^{RL}$, then right copies it onto $-H^R$.

Right

$G^R - K$

Left

$-H^{RL} + H^{RL}$

right

Swivel Chair Argument

Assumptions

Left survives $G - H$ going second without getting stuck on $-H$.
Left survives $H - K$ going second without getting stuck on $-K$.

Goal

Left survives $G - K$ going second without getting stuck on $-K$.

If Left's response is $G^R \to G^{RL}$, we're done.

Right

$$G^{RL} - K$$

Left

$$-H^{RL} + H^{RL}$$

right

Swivel Chair Argument

Assumptions

Left survives $G - H$ going second without getting stuck on $-H$.
Left survives $H - K$ going second without getting stuck on $-K$.

Goal

Left survives $G - K$ going second without getting stuck on $-K$.

If Left's response is $-H^{RL} \to -H^{RLR}$, then right copies it again.

Right

$$G^R - K$$

Left

$$-H^{RLR} + H^{RLR}$$

right

Swivel Chair Argument

Assumptions

Left survives $G - H$ going second without getting stuck on $-H$.
Left survives $H - K$ going second without getting stuck on $-K$.

Goal

Left survives $G - K$ going second without getting stuck on $-K$.

Is it possible that we will get stuck playing on H forever?

Right

$$G^R - K$$

Left

$$-H^{RLR} + H^{RLR}$$

right

Swivel Chair Argument

Assumptions

Left survives $G - H$ going second without getting stuck on $-H$.
Left survives $H - K$ going second without getting stuck on $-K$.

Goal

Left survives $G - K$ going second without getting stuck on $-K$.

No, because Left's strategy on $G - H$ doesn't get stuck on $-H$.

Right

$G^R - K$

Left

$-H^{RLR} + H^{RLR}$

right

Swivel Chair Argument

Assumptions

Left survives $G - H$ going second without getting stuck on $-H$.
Left survives $H - K$ going second without getting stuck on $-K$.

Goal

Left survives $G - K$ going second without getting stuck on $-K$.

Therefore, Left will eventually find a response on $G^R - K$.

Right

$$G^R - K$$

Left

$$-H^{RLR} + H^{RLR}$$

right

Swivel Chair Argument

Assumptions

Left survives $G - H$ going second without getting stuck on $-H$.
Left survives $H - K$ going second without getting stuck on $-K$.

Goal

Left survives $G - K$ going second without getting stuck on $-K$.

The proof that Left will not get stuck on $-K$ uses the "stuck" part of both assumptions.

Right

$$G^R - K$$

Left

$$-H^{RLR} + H^{RLR}$$

right

Onside and Offside

Let G be a loopy game.

Onside

If $G \triangleq S$ for some stopper S, then S is the **onside** of G.

Offside

If $G \veeeq T$ for some stopper T, then T is the **offside** of G.

Stopper-Sided

G is **stopper-sided** if it has an onside and an offside. We write

$$G = S\&T.$$

Bounds for Stopper-Sided Games

Assume $G = S \& T$. Let S' and T' be any stoppers.

Onside and offside are the best stopper bounds

$S \geq G \geq T$.
$S' \geq G$ iff $S' \geq S$.
$G \geq T'$ iff $T \geq T'$.

Comparing and Adding Stopper-Sided Games

Assume $G = S_G \& T_G$ and $H = S_H \& T_H$.

Comparing Stopper-Sided Games

$G \geq H$ iff $S_G \geq S_H$ and $T_G \geq T_H$.

Adding Stopper-Sided Games

$G + H = (S_G + S_H)\&(T_G + T_H)$ **if** these sums are stoppers.

(The sum of two stoppers is not necessarily a stopper.)

Let G be a loopy game.

Sidling Theorem

If the sidling process applied to G converges, then the bounds obtained are the onside and offside of G.

See [WW] Ch 11 for many examples of sidling.

When sidling doesn't converge, see Aaron Siegel's paper in GONC3 "New results in loopy games" Section 5.

Also see his thesis for a general technique called *unraveling*.

Canonical Form for Finite Stoppers

Canonical Form for Finite Stoppers

Let S be a **finite** stopper.
There exists a unique simplest form stopper S' such that $S = S'$. The stopper S' has:

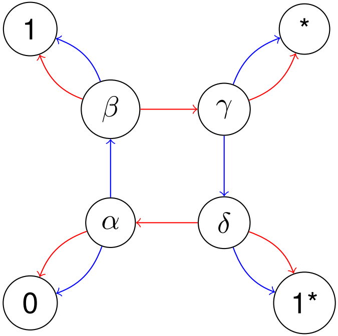
- no dominated options
- no reversible options
- no two distinct equal vertices.

S' is unique up to *graph isomorphism*.

(Not true in a general loopy game!)

Bach's Carousel

Discovered by Clive Bach, **Bach's Carousel** is an example of a non-stopper-sided game. See [WW] Ch 11 Extras.



For a proof that it is not stopper-sided, see [CGT] VI.4 p. 315–316.

Open Questions

From [CGT], p. 317

Is the sum of two finite stoppers always stopper-sided? (In [WW] p. 370, there is a counterexample due to Bach for infinite stoppers.)

If stoppers S and T satisfy $S \geq T$, does there exist a G such that $G = S\&T$? (Known for plumtrees, see [WW] p. 354, or [CGT] Exercise VI.4.7.)

Is there a canonical form for finite non-stopper-sided games?