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A REMARK ON EUCLID II,11

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To The Memory of Albert Herz

SUMMARIES

Certain historical implications are drawn from the existence of a subproposition contained in Euclid II,11.

Nous mettons en évidence que la proposition II,11 d'Euclide contient une autre proposition, et nous en tirons les conséquences.

Although its announced intention is "to cut a given straight line [*AB* of Fig. 1] so that the rectangle [*ABDE*] contained by the whole [*AB*] and one of the segments [*CB* = *DB*] is equal to the square [*ACFG*] on the remaining segment" [Euclid, 1926], the proof of Euclid II,11 actually contains, as it stands, the proof of another result (Fig. 2):

II,11': *To extend a given line* [*AB*] *so that the rectangle* [*ACDE*] *contained on the extended line and the added segment* [*BC* = *DC*] *is equal to the square* [*ABFG*] *on the given line* [1].

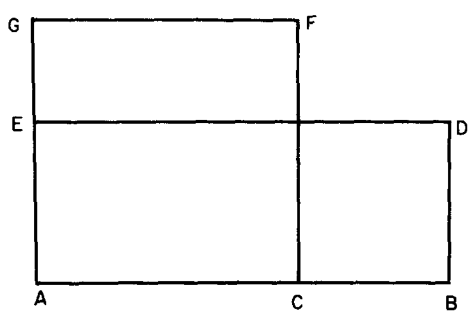

FIGURE 1

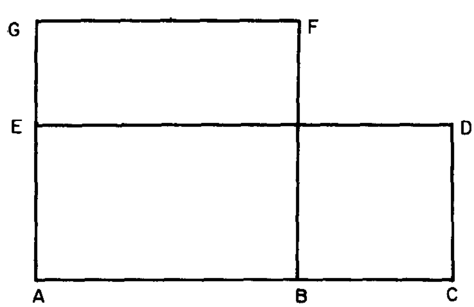

FIGURE 2

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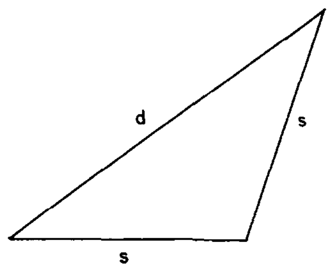

FIGURE 3

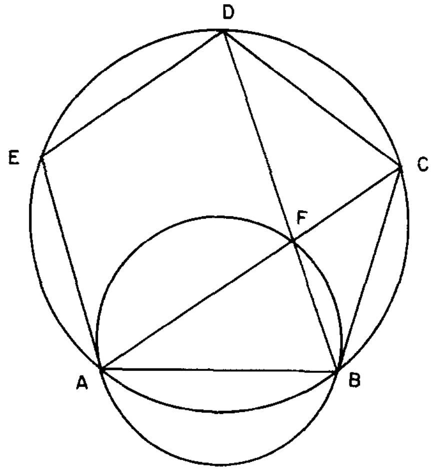

FIGURE 4

Using II,11' one can give a shorter proof of IV,11 (stated below) which bypasses IV,10 ("To construct the 36°, 72°, 72° triangle"), and uses only propositions already used in IV,10 and IV,11:

> IV:11: *In a given circle to inscribe an equilateral and equiangular pentagon.*

Proof. Draw (Fig. 3) the (36°, 36°, 108°) isosceles triangle whose sides are in the ratio *s, s, d,* where *s* is arbitrary, and *d* is obtained from II,11'

Next refer to Fig. 4 (this is based on Fig. 13 of Szabó [1974, 311], which differs from the diagrams accompanying IV,11 and XIII, 8 in Euclid). In the given circle draw triangle *ABC* similar to the isosceles triangle. Obtain *F* on *AC* so that *AF* = *AB*. Point *D* is obtained from the extension of *FB*, and *E* is obtained symmetrically. Now circumscribe a circle about triangle *AFB*. From the relationship between *s* and *d* and III,37, we have *CB* tangent to circle *AFB*. It follows from III,32 that $\sphericalangle$ *CAB* = $\sphericalangle$ *FBC*; this in turn implies that arcs *CB* and *DC* are equal. The equality of the remaining arcs, *DE, EA,* and *CB,* follows in the same way [2].

Now what is II,11'? It is precisely the construction of the diagonal of a pentagon whose side is the given segment. Indeed, we have:

> XIII,8: *The diagonals of a regular pentagon cut one another in extreme and mean ratio, and the greater segment is the side of the pentagon.*

Furthermore, XIII,8, formulated in the area terminology of II,11 rather than the ratio terminology of VI, Definition 3, and

VI,30, can also be proved using only propositions already used in IV,10 and IV,11 (Fig. 5).

Proof. Use the extant proof of XIII,8 (substituting III,27 for VI,33) until the statement that $\angle BAH = \angle BEA$. Now circumscribe a circle about triangle *EAH*. By the converse to III,32 (proved by *reductio ad absurdum*), AB must be tangent to the circle. Then by III,36, we have $EB \cdot BH = AB^2 = EA^2 = EH^2$.

Thus if XIII,8 were known, perhaps only intuitively, at the time IV,11 was proved, one could argue that a proof using only the "area" methods would have been possible at that time. But Euclid does not give such a proof.

In summary the inefficient path II,11 → IV,10 → IV,11 for the construction of the regular pentagon is followed instead of the path II,11' → IV,11 which could have been done using exactly the same mathematical setup.

We draw the following conclusions:

1. The author of II,11 was not aware of II,11'.

2. When II,11 was written, the result of XIII,8 was not known, even intuitively.

If the latter statement is true, it reverses the interpretation favoured by Heller [1958, 25], Szabó [1974, 310] and Junge [1948, 355], [3].

Furthermore, the above discussion would seem to support the following conjectures:

3. Euclid appears to be reproducing the original, or at least early, treatment in giving II,11; IV,10,11.

4. Originally, the concept of division in extreme and mean ratio in the earlier area form of II,11 was merely a by-product of III,36,37. It emerged essentially as presented by Euclid; that is, mathematicians wished to construct the regular polygons and, having arrived at the pentagon, noticed "the isosceles triangle having each of the angles at the base the double of the

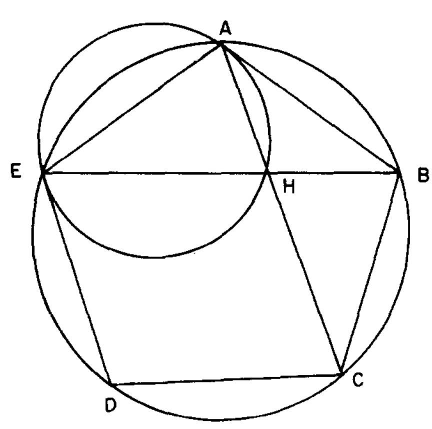

FIGURE 5

remaining one" (IV,10). In the course of an "analysis," use was made of III,36 to obtain (see Fig. 5) $EB \cdot HB = AB^2$. The formal proof of IV,10 then required the converse of III,36, namely, III,37, and as a preliminary the construction of II,11 was introduced [4].

NOTES

1. Using the ratio methods of Book VI, we could prove:

VI,30': *To construct a line on a given line so that the given line is the larger segment of the constructed line cut in mean and extreme ratio.*

Proof. (Refer to the diagram in Euclid [1926].) On *AB* given, describe the square *BC*. Let there be applied to *AB* the rectangle *AE* equal to the square *BC* and exceeding by a square (VI, 29).

Again the proof is simpler; in particular, one can apply the rectangle to the given line [*AB*] instead of the constructed line [*AC*].

2. Proposition II,11' can also be used to construct the isosceles triangle with sides *s*, *d*, and *d*. That this triangle is indeed the same as the 36°, 72°, 72° triangle is proved in IV,10.

3. Sachs [1917, 99] has already argued (on different grounds) for a late discovery of XIII,8. Knorr [1975, 199] argues that Book II owes its conception to Theodorus, but (209, n. 52) would also assign XIII,8 to the time of Theodorus. See also the discussion of the construction in II,11 on page 201 of Knorr.

4. Szabó [1974, 308], on the contrary, calls III,36 a special case of II,11. Junge [1948, 323] considers II,11 to be the special case which led to the more general "application of area theory." Our conjecture would imply the absence of any link between pentagon studies (in particular, division in extreme and mean ratio) and the pentagram, despite the various reports linking the latter to the "Pythagoreans."

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