ARTICLES

Dürer's Paradox or Why an Ellipse Is Not Egg-Shaped

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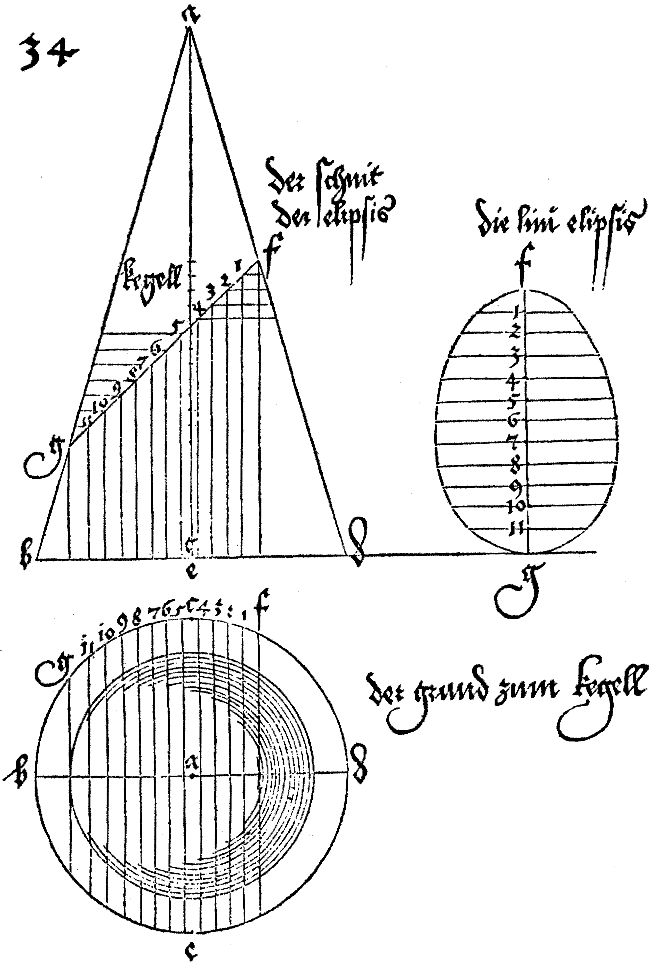
"The ellipse I will call an egg-curve because it is virtually equal to an egg"
—A. Dürer

The drawing of the cone and the ellipse is taken from the 1525 work *Treatise on Mensuration with the Compass and Ruler in Lines, Planes, and Whole Bodies* by the German artist and mathematician Albrecht Dürer (1471–1528). Two things are likely to attract the attention of the viewer: namely, the mass of lines and arcs and—to use Dürer's own expression—the "egg-curve." The purpose of this article is threefold: to describe Dürer's method, to explain why the use of the method might lead one to believe that the ellipse is egg-shaped, and to show how the analytic version of Dürer's method can be used to derive the Cartesian form of the ellipse. I have also included some historical material and suggestions for further reading in a separate section at the end of the article.

The method used by Dürer is essentially equivalent to what is now called descriptive geometry. At present it is only employed to obtain graphical solutions of complicated geometrical problems such as the intersection of surfaces, and as far as I know this branch of mathematics is now taught only in engineering drawing courses. However the French geometer Gaspard Monge (1746–1818), whose *Géométrie descriptive* provided the systematic development of the subject, wrote [**11**, p. 1]: "[One of the aims of descriptive geometry] is to give means of recognizing, based on an exact description, the forms of bodies and to deduce all the truths which are implied by their form and their respective positions."

Descriptive geometry is concerned with the representation of bodies and surfaces in space by means of two-dimensional orthogonal projections. Consider for example FIGURE 1 in which a point P is shown as being a units in front of a vertical plane and b units above a horizontal plane. If we project the point P onto these planes by means of perpendicular lines then the two projections P^V and P^H are both determined. To avoid constant awkward repetition and notation, a point and its projections will be referred to by a single letter without super and subscripts. If the horizontal plane is now folded back about the "folding line"— the line of intersection of the two planes —until the horizontal plane is also in a vertical plane, then we will have the situation of FIGURE 2. Here the point P is represented by two two-dimensional drawings which are such that the two projections lie on a line perpendicular to the folding line. Furthermore given the two-dimensional drawings, which are usually referred to as the front and top views, then the position in space—relative to the two planes—of the point P is completely determined. The folding line is not really necessary and if we are only interested in the relative position of various points with respect to one seytten/Also thů ich im durch die gantzen zal/so dann dise punckten zů rings herum gemacht sind/ alß dañ zeüch ich die eyer lini Elipsis von punckt zů punckt/wie ich sölchs hie bey hab aufgeryssen.



another then any convenient line can be used. This is the case in Dürer's drawing where the centre line of the circle is used as the reference line in the top view.

In the simple case just discussed we are given the distance of the point P from the planes, but in general life is not so simple. Sometimes, for example when one is dealing with a complicated intersection of two three-dimensional objects, the process of obtaining the two views is quite involved. Fortunately, in the case that interests us we are only dealing with a plane and a cone.

Thus suppose (FIGURE 3) that we have a cone that has been cut by a horizontal plane. Then in the front view the cone will appear as a triangle and the intersection of the cutting plane with the cone will appear as a straight line. On the other hand in the top view the intersection of the cutting plane and the cone will be a circle of some radius r. Let O be the centre of the circle and consider the horizontal centre line MON. Since the points M and N are at the extreme right and left respectively of the circle the same will be true of the front view of the centre line. Thus the line in the front view can also be thought of as representing the centre line of the circle, and so

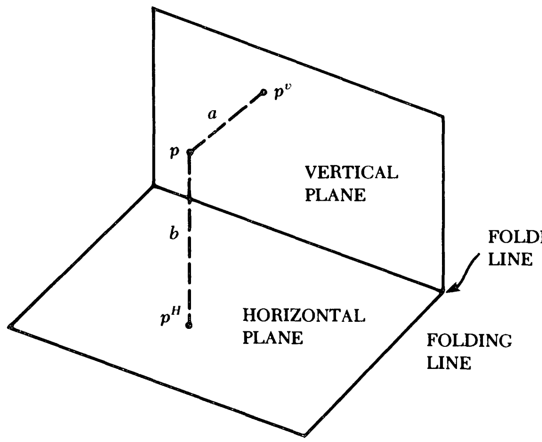


FIGURE 1

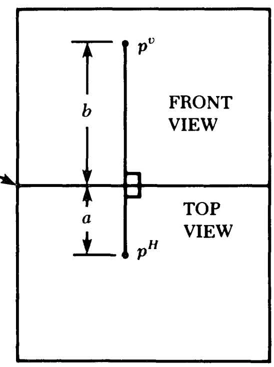


FIGURE 2

we also label this line as MN. Because the two projections of M lie above one another, they will both be at the same distance r from the centreline of the cone. This in turn means that in order to draw the circle in the top view we need only measure the distance r in the front view and then use this as the radius of a circle, about O, in the top view. If we have an arbitrary point P on the centre line MON then this will determine the points P' and P'' on the circumference and all three points will coincide in the front view.

Now let us cut the cone with an oblique plane (FIGURE 4). The intersection of the plane with the cone determines a curve which, because of previous knowledge, will be referred to as the ellipse. The front view of the plane is once again a straight line and by the symmetry of the cone this line FG also corresponds to the major axis of the ellipse. Because the cutting plane is at an angle with the horizontal the ellipse will appear to be distorted when looked at from directly above; it is only when we look at right angles to the cutting plane (FIGURE 4) that the ellipse appears in its true shape. It is for this reason that the top and front views must be used to obtain a new (auxiliary) view, such as the one that appears in Dürer's drawing, which shows the true shape of the ellipse.

In order to find points belonging to the top view of the ellipse we pass a horizontal cutting plane through the cone which intersects the given oblique plane at an arbitrary height (FIGURE 4). This brings us back to the situation of FIGURE 3 and given the radius—or equivalently the location of a point on the major axis FG—the circle determined by the horizontal plane can be drawn. Once again we designate the centre line of this circle determined by the horizontal plane as MN. But this circle is located on the surface of the cone, as is the ellipse, and so the circle and ellipse intersect in two points P' and P''. In the front view we will see these two points of intersection of the ellipse and the cone as the intersection of lines FG and MN. This observation in turn determines the location of P' and P'' in the top view, for these points must lie on both the circle and the vertical line drawn down from P', P'' in the front view. In particular the width ($2w$) of the ellipse at the point P on the major axis of the ellipse which corresponds to the points P' and P'' is now determined.

By repeating the above process we can find the width of the ellipse for as many points on the major axis as we wish. This process is illustrated in FIGURE 5 for two points 1 and 2 which are equidistant from the centre O of the centreline FG. In Dürer's drawing the major axis FG of the ellipse has been divided into twelve equal

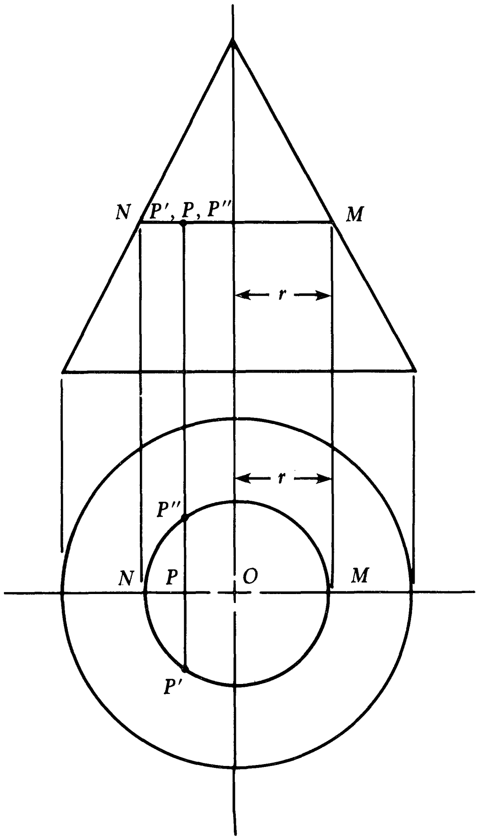


FIGURE 3

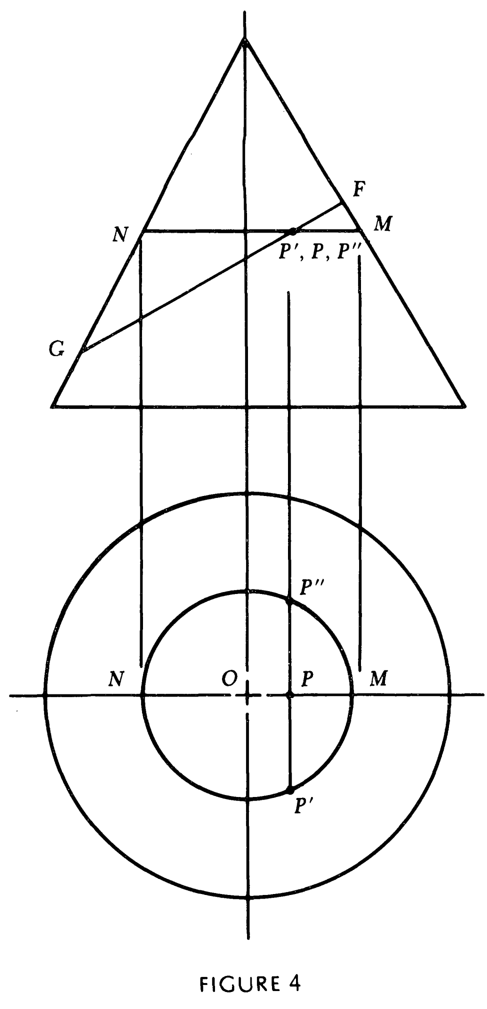


FIGURE 4

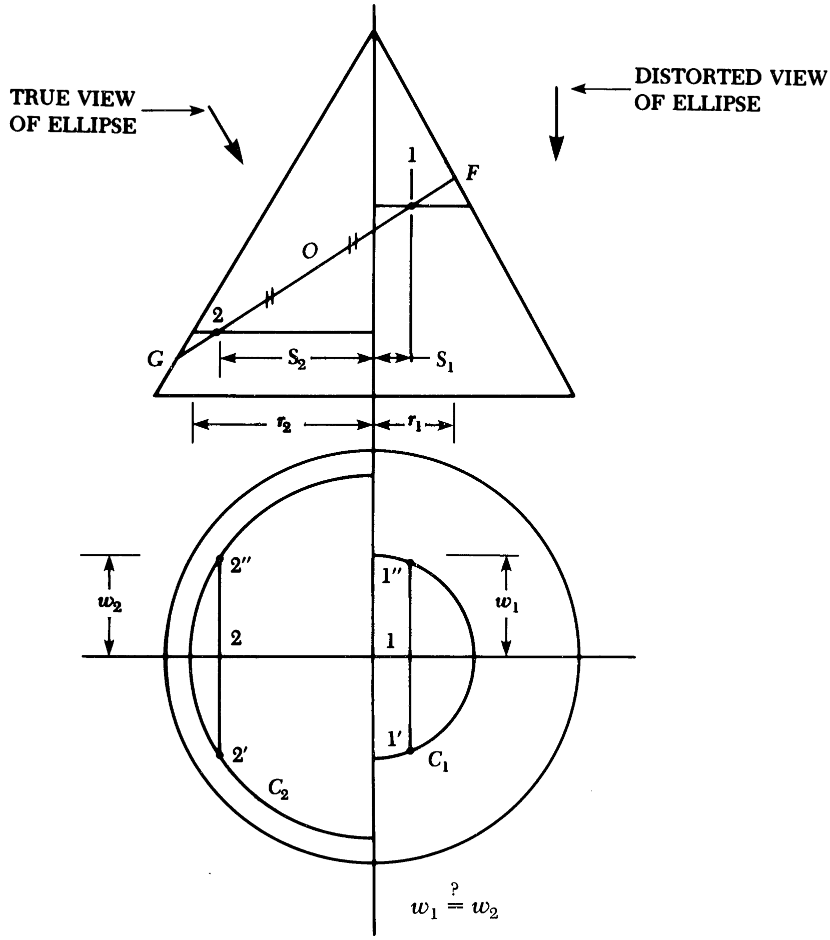


FIGURE 5

parts by means of the points labeled 1,...,11. A horizontal line drawn through each of these points in the front view corresponds to a horizontal slicing circle and these circles are then drawn in the top view as shown in Dürer's diagram. The downward projection of each of the eleven points on the corresponding circle now gives the width of the ellipse at that point.

Note now that even though points 1 and 11 are symmetrically located with respect to the centreline the corresponding constructions are not symmetric, for point 1 corresponds to the smallest of the eleven circles whereas point 11 corresponds to the largest. Thus it is not at all obvious from this approach that the ellipse is symmetric rather than egg-shaped, with the wider part corresponding to point 11. If one did not know the analytic form of an ellipse then it might seem more reasonable, using Dürer's approach to conclude that the ellipse is indeed egg-shaped. This not unreasonable conclusion is what I refer to in the title as Dürer's paradox.

Despite the apparent paradox we do know that the ellipse is *not* egg-shaped, and the key to understanding why is the fact that the centreline of the cone does not pass through the centre of the ellipse. With reference to FIGURE 5 let O be the centre of the line FG and let 1 and 2 be two points on FOG which are equidistant from O. Let r_1 and r_2 be the radii of the corresponding circles C_1 and C_2 and s_1 and s_2 the horizontal distances from the centreline of the cone. It is true that $r_1 < r_2$, but because 1 and 2 are equidistant from O we also have that $s_1 < s_2$, i.e., the point 1 is closer to the centre of the smaller circle C_1 than point 2 is to the centre of the larger circle C_2. The reason the ellipse is not egg-shaped is that the combination of shorter distance to the centre and smaller radius combine to make $w_1 = w_2$. Of course I—Dürer does not go through any reasoning of this nature—have flippantly stated that $w_1 = w_2$ because I know that this is the way things work out for an ellipse. What is needed is a proof of this fact.

Thus what I will now do is translate Dürer's construction, which involves a finite number of points, into an analytic argument and obtain expressions for w_1 and w_2 when points 1 and 2 are symmetrically located with respect to the centre O. It will then be shown that not only are the widths equal, but also that we can obtain the usual Cartesian equation of the ellipse from the expression for the width. Furthermore the relationships between the constants and the angles of the cone and cutting plane, as well as some other results, will fall out of the derivation. While the development is in principle straightforward, a nonjudicious choice of parameters can lead to some very messy algebra. After some trials the following seems to me to be the simplest approach.

We start with a cone whose cone angle is θ and then pass a cutting plane which makes an angle $\alpha > \theta$ with the centreline of the cone. Let FG be the major axis of the ellipse with centre O and length $2a$. We take arbitrary points 1 and 2 which are at a distance x from O and designate by s and t the horizontal and vertical displacements from O (FIGURE 6a). Thus:

$$s = x \sin\alpha; \qquad t = x \cos\alpha. \tag{1}$$

If d^* is the horizontal displacement of O from the centreline, then the points 1 and 2 are respectively at a horizontal distance $s - d^*$ and $s + d^*$ from the centreline. If r^*, r_1, r_2 are the radii corresponding to points O, 1, 2, then as indicated at the bottom of FIGURE 6a we have the relationships:

$$r_1 = r^* - t \cdot \tan\theta; \qquad r_2 = r^* + t \cdot \tan\theta. \tag{2}$$

Now looking at the top view (FIGURE 6b) and considering the circles of radii r_1 and r_2, we see that w_1 and w_2—the respective half-widths of the ellipse corresponding to points 1 and 2—satisfy:

$$w_1^2 = r_1^2 - (s - d^*)^2 = (r^* - t \cdot \tan\theta)^2 - (s - d^*)^2 \tag{3}$$

$$w_2^2 = r_2^2 - (s + d^*)^2 = (r^* + t \cdot \tan\theta)^2 - (s + d^*)^2. \tag{4}$$

What we need in order to continue are expressions involving r^* and d^*. These may be obtained by considering the triangles OFH and OJG (FIGURE 7) and then applying the law of sines to each. This gives the two relationships:

$$r^* + d^* = (a/\cos\theta) \cdot \sin(\alpha + \theta) \tag{5}$$

$$r^* - d^* = (a/\cos\theta) \cdot \sin(\alpha - \theta). \tag{6}$$

Adding and subtracting and applying trigonometric reduction formulae we obtain after simplification:

$$r^* = a \cdot \sin\alpha \tag{7}$$

$$d^* = a \cdot \cos\alpha \cdot \tan\theta. \tag{8}$$

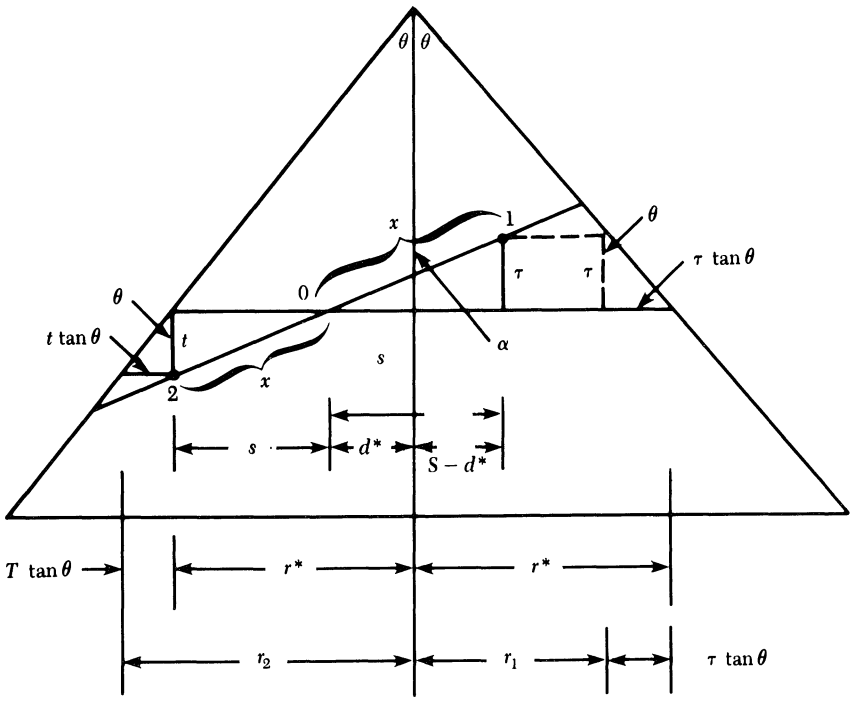


FIGURE 6a

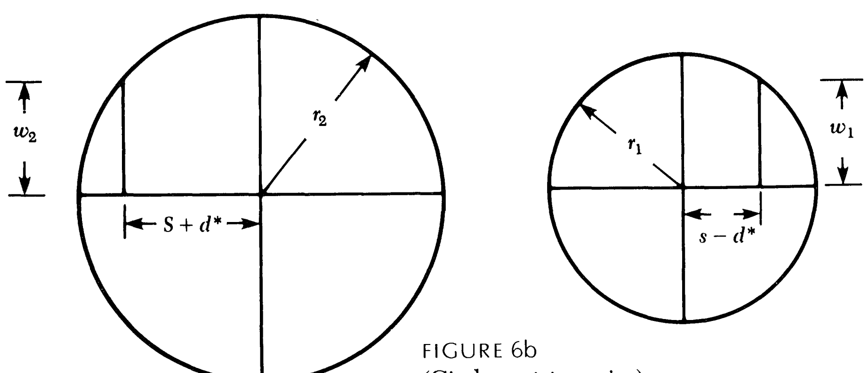


FIGURE 6b
(Circles not true size)

FIGURE 6

To obtain the equation of the ellipse—recall that we know by symmetry of the cone that the ellipse is symmetric about the major axis—we substitute (7) and (8) in (3) and (4) to obtain:

$$w_1^2 = (a \cdot \sin\alpha - x \cdot \cos\alpha \cdot \tan\theta)^2 - (x \cdot \sin\alpha - a \cdot \cos\alpha \cdot \tan\theta)^2 \tag{9}$$

$$w_2^2 = (a \cdot \sin\alpha + x \cdot \cos\alpha \cdot \tan\theta)^2 - (x \cdot \sin\alpha + a \cdot \cos\alpha \cdot \tan\theta)^2. \tag{10}$$

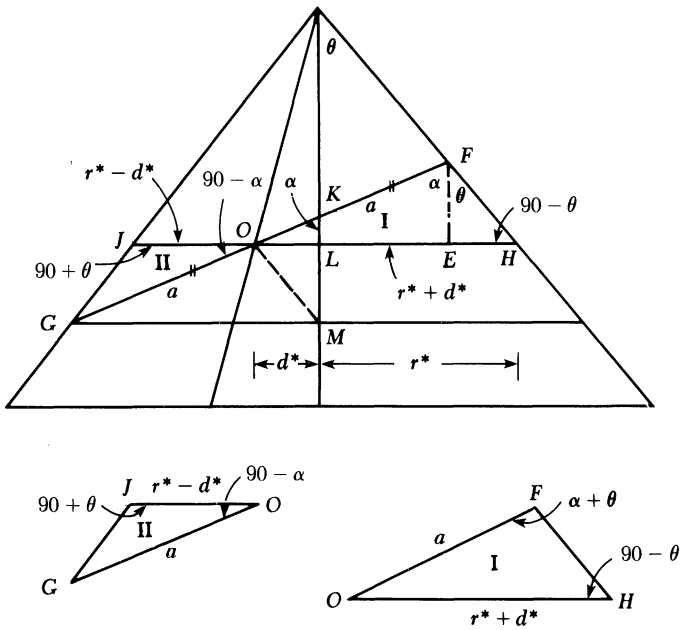


FIGURE 7

Since the expressions in (9) and (10) only differ by the signs inside the parentheses and since the inner terms in the expansions of the squared terms will cancel in each case, we have immediately—without any calculations—that $w_1 = w_2$, i.e., that the ellipse is indeed symmetric.

If we designate the common width at a distance x along the major axis by y and simplify we obtain:

$$y^2 = (a^2 - x^2)\left[\sin^2\alpha - \cos^2\alpha \cdot \tan^2\theta\right] \text{ or} \tag{11}$$

$$y^2/b^2 + x^2/a^2 = 1 \text{ where} \tag{12}$$

$$b^2 = a^2\left[\sin^2\alpha - \cos^2\alpha \cdot \tan^2\theta\right] = a^2\left[1 - (\cos\alpha/\cos\theta)^2\right]. \tag{13}$$

If the eccentricity is defined by $e = \cos\alpha/\cos\theta$, then (13) is just the usual relationship between the semi-axes of an ellipse so that (12) does indeed agree with the standard Cartesian equation of the ellipse. Here the eccentricity can be thought of directly as a measure of the distortion of the circle (corresponding to $\alpha = 90°$) due to the tilting of the slicing plane.

There is some more information that we can easily obtain from the above development. As an immediate consequence of (7) and (8) we have that if we draw the perpendicular FE in triangle OFH (FIGURE 7) then:

$$OE = r^*; \qquad EH = d^* \tag{14}$$

and further:

$$r^*/d^* = \tan\alpha/\tan\theta. \tag{15}$$

I was unable to obtain either (14) or (15) by more direct trigonometric or geometric means.

If K is the intersection of the centre line of the cone with the axis FG then by similar triangles we obtain:

$$OK/a = OL/OE = d^*/r^* = \tan\theta/\tan\alpha. \tag{16}$$

This latter relationship incidentally implies that the right focal point is to the right of point K, i.e., the centreline of the cone passes strictly between the centre of the ellipse and the "upper" focal point, for if $c = ae = a(\cos\alpha/\cos\theta)$ is the focal distance, then (16) implies that $c/OK = \sin\alpha/\sin\theta > 1$. This relationship between c and OK can also be obtained by drawing the line OM—whose length turns out to be equal to c—and then applying the law of sines to both triangles OMG and OKM.

Finally note that Dürer also constructed the parabola ("burn-curve") and the hyperbola ("fork curve"; the drawing is for the special case where the cutting plane is vertical) and the reader is invited to make the drawings—before looking at [5]—and to obtain the Cartesian equations for these conic sections.

Historical Notes and Further Reading

The diagram was taken from the first edition of [**5**] which has been reprinted twice in recent years, once with English translation and commentary. Just before the drawing (number 34 of Book I; this is on page 94 of the English edition) Dürer explains how the construction proceeds, but unfortunately he gives no historical information as to the origin of the method. He merely states at the beginning that the ancients, i.e., the Greeks, showed that three different curves are obtained when a cone is cut by different planes. Dürer then informs the reader that the learned names are "Elipsis," "Parabola" and "Hiperbole," but that he does not know the German names. He says, "We want to give them names which in themselves will serve for identification purposes." As Dürer states in the quotation given at the beginning of the article he calls the ellipse an egg-curve [eyer lini = eierlinie] because the ellipse is virtually [schyer = schier] equal to an egg.

As indicated by Dürer's remark about "the ancients" he was acquainted with the work of the Greek geometers. That he was well versed in the geometry of Euclid and others is evident from the various constructions throughout the book. There also exists other evidence relating to Dürer's mathematical studies and knowledge. The mathematical facet of Dürer's life is generally not known, even to people who are acquainted with his engraving "Melancolia" (reproduced in Boyer [**1**, p. 325]) which shows a magic square and a polyhedron. While a rarity in our day, many Renaissance artists had an advanced knowledge of mathematics. Another example of a great artist with a knowledge of mathematics is the Italian artist Piero de la Francesca (c. 1415 to 1492) who wrote several interesting treatises on perspective and mathematics (see [**9**, Section 31, C]). Staigmüller [**16**, p. 3] wrote that the lack of knowledge of Dürer's mathematical work among art historians was surprising in view of the emphasis that Dürer himself put on it and that when he had laid down his brush he wrote his theoretical works in order to pass on his knowledge. "Indeed he lived in the hope that through these works, even more than through his eternal creations with the brush, he would lay the foundation of the 'German art' because he regarded the lack of a theoretical, particularly mathematical, knowledge in his fellow artists as the main hinderance to a prosperous development of the arts in the fatherland."

On the life and work of Dürer, see [**17**, p. 35]; [**18**]; [**19**]; [**10**]; [**3**, p. 109]; [**4**, p. 61]; [**20**, p. 62]. The art of Dürer is discussed by Panofsky [**12**] who devotes his chapter 8 to Dürer as a theorist of art.

Of particular historical interest to us are Dürer's method of construction and the shape of his ellipse and I will briefly indicate what is known.

The history of the study of the conic sections in the Middle Ages and the Renaissance is examined in great detail by Clagett [2]. He suspects [p. 266] that Dürer was influenced by the 15th-century mathematician Regiomantus, but not by Dürer's contemporary Johann Werner who had published a book on the conics in 1522. Regarding Dürer's technique Clagett writes: "The method...has no counterpart in the medieval traditions of conic sections, nor indeed in the revived Apollonian tradition that was soon to follow."

The egg-shaped ellipse has been commented upon by several writers. Staigmüller [**16**, p. 16] believed, based on the drawing and the name egg-curve that he gave it, that Dürer thought that the ellipse only had one axis of symmetry and that it was wider at the end that was at the bottom of the cone. Steck [**17**, p. 35] also commented on this question and gave references to other authors (Doehlmann and Günther), but none of these has any precise information to give on this question. Hofmann [**10**, p. 119] mentions other authors who have an egg-shaped ellipse, but these are all in works published after Dürer except for Witelo (13th century; see [**2**, Chapter 3]). I checked [**20**], but did not find any sign of an egg-shaped ellipse. Typically Book 7, proposition 47 shows a cone and a crudely drawn ellipse in perspective. Coolidge [3] merely states "It is fair to state that Dürer's ellipse looks rather egg-shaped."

While reading Pottage's [**15**] treasure chest of geometrical and historical information my attention was drawn to Pedoe [**14**]. Pedoe seems to suggest that the egg shape can be accounted for on the basis of errors in the method of reproducing the drawing. He points out that, in terms of Figure 3 of this article, that $(PP'')^2=(PN)(PM)$ (because PP'' is the altitude of right triangle $NP''M$). He then says, without giving any further details, that the equations of the conic sections follow from Dürer's method if one uses a non-specified theorem on similar triangles. This method, says Pedoe, is the method of Apollonius [see **22**, p. 288 ff.]. Thus in view of Dürer's acquaintance with the ancients, Pedoe suggests that Dürer extracted his practical method for constructing the conic sections from the work of Apollonius.

The above survey suggests that we essentially do not know anything precise about the origin of Dürer's technique. It also seems to me that we must presume that Dürer thought that the ellipse was indeed egg-shaped. Whether this was for theoretical reasons or because of his drawing one cannot say. As regards the latter possibility I invite the reader to apply straightedge and dividers to Dürer's drawing and to note certain inaccuracies, particularly in the vertical projections. This seems very strange in the work of one of the master engravers of all time.

For descriptions and discussions of Monge's work see Taton [**20**, **21**]. Monge is responsible for systematizing and advancing earlier work involving graphical techniques and particularly in putting them on a sound mathematical basis. Taton [**21**] writes: "Monge viewed descriptive geometry as a powerful tool for discovery and demonstration in various branches of pure and infinitesimal geometry. His persuasive example rehabilitated the study and use of pure geometry, which had been partially abandoned because of the success of Cartesian geometry."

I have recently advocated ([7], [8]) a computer oriented, algorithmic approach to the teaching of certain types of three-dimensional shape, form and space problems, which is based in part on the descriptive geometry approach—although with the aim of obtaining a numerical answer and not a drawing—and which in a sense follows the spirit advocated by Monge. The methods of descriptive geometry—more or less explained—and in particular the construction of the ellipse can be found in various "technical drawing" or "descriptive geometry" books. See for example [13, chapter 21] and [6, section 337].

Acknowledgements. First of all, I must thank the anonymous student in my Mathematics for Architecture class—see [7]—who shook me out of my complacency by asking out loud, at the moment when I presented Dürer's drawing, why indeed he should not believe that the ellipse was egg-shaped. This forced me to think about the matter and this in turn led to the present analytic argument. I do not recall now if the student was convinced! Secondly, I wish to express my appreciation to one of the referees, who obviously devoted a great deal of time to my manuscript, for some very pertinent remarks and corrections and for providing me with the reference to the Strauss edition of Dürer. I would also like to thank Dr. Vera Huse now of Acadia University for help with the intricacies of the German language and to Dr. Gert Schubring of the Institut für Didaktik der Mathematik, Universität Bielefeld, for obtaining a copy of Staigmüller's 1891 "Schulprogramm" for me.

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