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I - ARTICLES

THEOREM XIV, * * OF THE FIRST "SUPPLEMENT" TO THE *ELEMENTS*

ROGER HERZ-FISCHLER *

"Del suo 11. excellentissimo effecto"
Pacioli, *Divina proportione*, chap. XX

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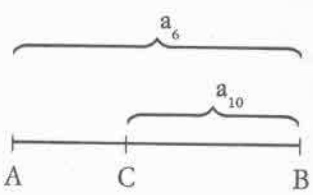
Introduction

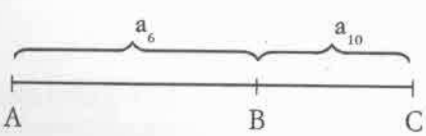
In the proofs of Theorems 2 and 7of the first "Supplement" to the *Elements* – which I shall refer to for convenience as Book XIV – the following result is applied without commentary or justification [1]:

XIV, **: Let a regular hexagon and a regular decagon be inscribed in the same circle. If we divide the side of the hexagon in extreme and mean ratio [*Elements* VI, def. 3; VI, 30] then the larger segment is the side of the decagon.



This result may be compared with:

XIII,9: Let a regular hexagon and a regular decagon be inscribed in the same circle. Then the line formed by adding together the sides of the hexagon and decagon is divided in extreme and mean ratio with the side of the hexagon being the larger segment.



The question that this paper addresses is whether XIV, ** existed as an independent result in one of the early Greek versions of the "Supplement" – either that of Hypsicles or those by Aristaeus and Apollonius mentioned in

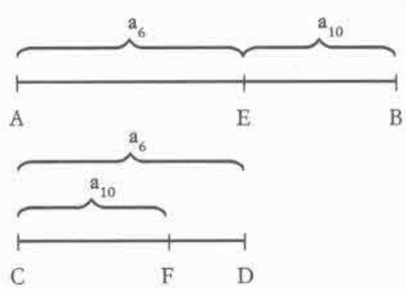
[1] All references to the *Elements* are based on [Euclid-Heiberg] unless otherwise indicated. Theorem 2 states that if a dodecahedron and icosahedron are inscribed in the same sphere then the circumscribing circles for the pentagon of the former and the triangle of the latter are identical. Theorem 7 states the relationship, involving division in extreme and mean ratio, between the edges of the cube and icosahedron inscribed in the same sphere. XIV, ** is used on page 12, line 2 and page 26, line 4 of volume 5. According to VI, def. 3, a line is divided in extreme and mean ratio if the ratio whole:larger segment = larger segment:smaller segment. For a detailed discussion of these and other theorems involving division in extreme and mean ratio one may consult my book *A Mathematical History of Division in Extreme and Mean Ratio*, Wilfred Laurier University Press, 1987.

Hypsicles' dedication to the "Supplement" – or if it was a result which, in view of XIII,9, did not merit a further separate consideration [2].

It will be seen in this paper that XIV, ** appears in many sources and indeed we will come across eleven extant distinct proofs of varying degrees of dissimilarity. Further we will find that in certain sources comments have been made about the location of XIV, ** in the text. Thus we will also be interested in the relationships among the various sources as well as where exactly XIV, ** might have been located.

For the purposes of this study I have divided the sources into four categories: Pappus, the Arabic tradition, the Arabic-Latin tradition and the Greek(?)-Latin tradition. In this last category (Section 5) we will be particularly interested in whether or not versions of XIV, ** that we find are based on Greek sources. In Section 6 I have brought together the various pieces of evidence and drawn conclusions where I thought that they were warranted. This latter section also contains a genealogical chart which exhibits, as much as seemed possible to me,

[2] There is a direct Euclidean proof of XIV, ** which uses only XIII,9 and the proportion theory of Book V in the form of V,19 and V,11. Theorem V,19 states that if AB:CD = AE:CF, then also AB:CD = (AB–AE):(CD–CF) = EB:FD and V,11 ("Ratios which are the same with the same ratio are also the same with one another") then implies that AE:CF = EB:FD.



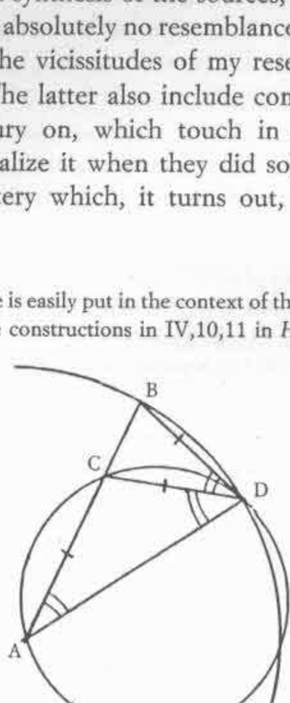
Theorem XIV, ** is now obtained from XIII,9 by putting $AB = a_6 + a_{10}$; $CD = a_6$; $AE = a_6$; $CF = a_{10}$ so that $EB = a_{10}$ and $FD = a_6 - a_{10}$. There is however no indication of this proof in any of the sources that I have seen. Of course its absence may simply mean that XIV, ** was indeed considered obvious when Book XIV was written. I note also that the proofs of Sections 3B and 5A, which I will argue are the best candidates for being an original Greek proof, both use V,17; *i.e.* the mathematician(s) who wrote them down were operating in the same proportion theory context as I was in giving the above straightforward proof. It may be that the lack of this elementary proof is somehow related to the existence of the 'Ratio Lemma' which will be discussed in Section 1; see also the discussion in fn. 54 of Section 4F.

Note also that XIV, ** follows immediately from the proof of IV,10, the construction of the 72°-72°-36° triangle. For in this proof C is picked so that AB, the radius of the larger circle and thus the side of the hexagon, is divided in extreme and mean ratio (in the proof this is stated in the area formulation of II,11) with AC being the larger segment. Further BD is chosen to be equal to AC. But since ∢ BAD in 36°, AC = BD is the side of the decagon. I have used modern the various relationships – or lack of relationships – among the various proofs of XIV, **.

In an appendix I have discussed the questions raised by the lettering of the diagram which accompanies the proof of XIV, ** in the version of the *Elements* attributed to Isḥāq ibn Ḥunayn and Ṯābit ibn Qurra. This will lead us to a discussion of who the editor(s) of Book XIV in this version might have been.

The requirements of academic publishing called for a presentation of the material which provided a synthesis of the sources, but I can assure the reader that my investigation bore absolutely no resemblance to the finished version. To give some indication of the vicissitudes of my research, I have made certain comments in the notes. The latter also include comments by various authors, from the sixteenth century on, which touch in some way upon XIV, **. Although they did not realize it when they did so, these commentators only added to the air of mystery which, it turns out, surrounds this apparently humble result.

terminology, but all the above is easily put in the context of the first four books of the *Elements*; see also the discussion of the constructions in IV,10,11 in *History of D.E.M.R.*, Section 2.



Heiberg [vol. 5, 13, fn. 1] justifies the use of XIV, ** by invoking XIII,9 and XIII,5 converse. No textual support is cited by Heiberg for this proof and none of the scholia that he published [vol. 5, 679-695] deal with this point. Now it so happens, as we shall see in Section 4F, that there does exist a mediaeval scholium, of which Heiberg was probably unaware, which proves XIV, ** in the manner suggested by Heiberg. It will be seen however that it is very unlikely that XIV, ** was originally justified in this way. I suspect that Heiberg's footnote is based on Commandino's 1572 edition of the *Elements*; see fn. 8, Section 2. Heath [vol. 3, 514] and Murdoch [a, 285; 301, fn. 116] follow Heiberg; Peyrard [vol. 3, 488] does not comment on this. I have presented a detailed discussion of the content of Book XIV and the possible historical layers suggested by the introduction and various proofs in Section 24 of *A Mathematical History of Division in Extreme and Mean Ratio.*

1. *Preliminaries*

An important role in this study will be played by a detailed mathematical comparison of the various proofs of XIV, ** that are found in different sources. For this reason I have presented the proofs in a numbered step by step form instead of merely reproducing the proofs verbatim. In order to facilitate the reading of the proofs, which often differ subtly from one another, I have used a modern idiom and employed symbolism where appropriate. I have attempted to remain as close as possible to the method and spirit of the originals be they purely geometrical, 'arithmetical' or a mixture of the two. The step numbers, which usually involve single mathematical statements, as well as the bracketed comments, are of course mine. For the sake of clarity I have added symbols to various diagrams.

The following abbreviations are used:

e.m.r. (d.e.m.r.) – (division in) extreme and mean ratio

S(AB) – area of the square of side AB

R(AB,CD) – area of the rectangle with sides AB and CD

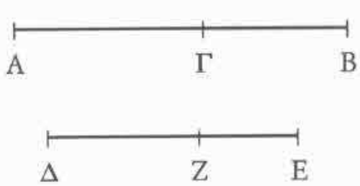
a_6 – side of the hexagon (understood to be inscribed in some circle)

a_{10} – side of the decagon (understood to be inscribed in the same circle as the hexagon in question)

a_5 – side of the pentagon (understood to be inscribed in the same circle as the hexagon or decagon in question).

In various proofs of XIV, ** use will be made of the following result which appears just after theorem 8 and before the concluding "Summary" of Book XIV:

"Ratio Lemma": If two lines are divided in e.m.r. [then the ratio of the entire line to the larger segment is the same in both cases] [3].



(after [Euclid-Heiberg, vol. 5, 33])

[3] The text [Heiberg, vol. 5, 32] does not call this result a lemma nor for that matter does it assign numbers to any of the theorems. I have adopted the numbers 1-8, and the terms 'Introduction', 'Ratio Lemma' and 'Summary' for convenience. The Summary starts on page 34, line 8 of [Euclid-Heiberg].

The reader should remark that this result only speaks about the ratio, entire line:larger segment and does not say anything about the ratio, larger segment: smaller segment [4]. We will see the consequences of this in Section 3B, step 6.

Not only is the Ratio Lemma used in various proofs of XIV, **, but it is also used implicitly in XIV,2 and XIV,7 which are the very propositions that use XIV, ** implicitly [5].

2. *Pappus of Alexandria (first half of the 4th century)*

It turns out that the earliest datable explicit statement and proof of XIV, ** appears in Book V of the *Collection* of Pappus. Since much of what I shall discuss in connection with Pappus is intertwined with material from Book XIV, and because Pappus has two proofs of both XIV, ** and XIV,2, I first present a 'flow chart' which at the same time will indicate the pages on which the results can be found in [Pappus-Hultsch, vol. 1] and [Euclid-Heiberg, vol. 5]. I have also indicated on the flowchart the various relationships among the results in both Pappus and Book XIV as well as the relationships, or possible relationships, between the two texts.

Whereas in the *Elements* theorem XIV, ** appears only *implicitly* (in the proofs of XIV,2,7), Pappus states XIV, ** *explicitly* in the form of his proposition 47. This explicit statement of XIV, ** in proposition 47 is then *explicitly* referred to in the *two* proofs of XIV,2 that Pappus gives in his proposition 48. The Ratio Lemma also appears in Pappus in the form of proposition 44. Further both the proof of the Ratio Lemma (proposition 44) and the second of the two proofs of XIV,2 that are given in proposition 48 are identical with the proofs of these results found in the *Elements*.

The first proof of XIV,2 that Pappus gives in proposition 48 is much more involved than the second proof, but as stated it too explicitly refers to XIV, **

[4] As one might suppose for example from the translation in [Euclid-Heath, vol. 3, 518] where one reads "... the segments of both are in one and the same ratio". As with so many results in the *Elements* the conclusion – in brackets – is only made clear at the beginning of the actual proof. An interesting question is why it was felt that there was a need to prove such a statement. The proof depends directly on the definition of division in extreme and mean ratio and one wonders what this implies about the theories of proportion and ratio or at least the attitude toward these theories when the Ratio Lemma was first written down; see also fn. 57, Section 4F.

[5] In XIV,2 the Ratio Lemma is used implicitly on p. 12, line 6 when it is stated that 3 S(ΔH): 3S(ΓH) = 5 S(MN):5 S(MΞ). In XIV,7 the Ratio Lemma is needed for the statement of the proposition "If *any* line is divided in e.m.r. ... ". Actually, more than the Ratio Lemma is needed because both propositions 2 and 7 involve the squares on the lines.

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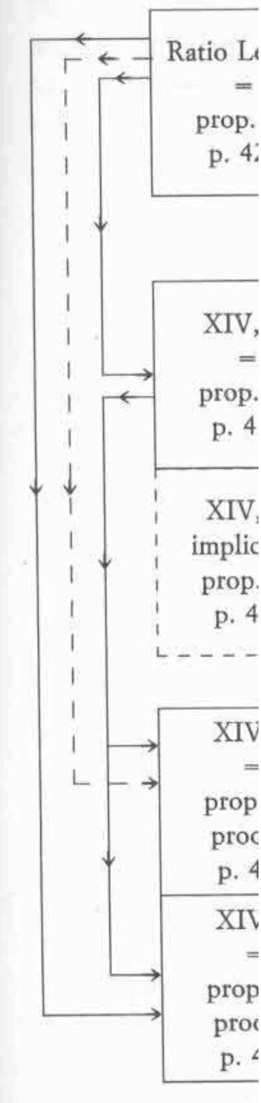

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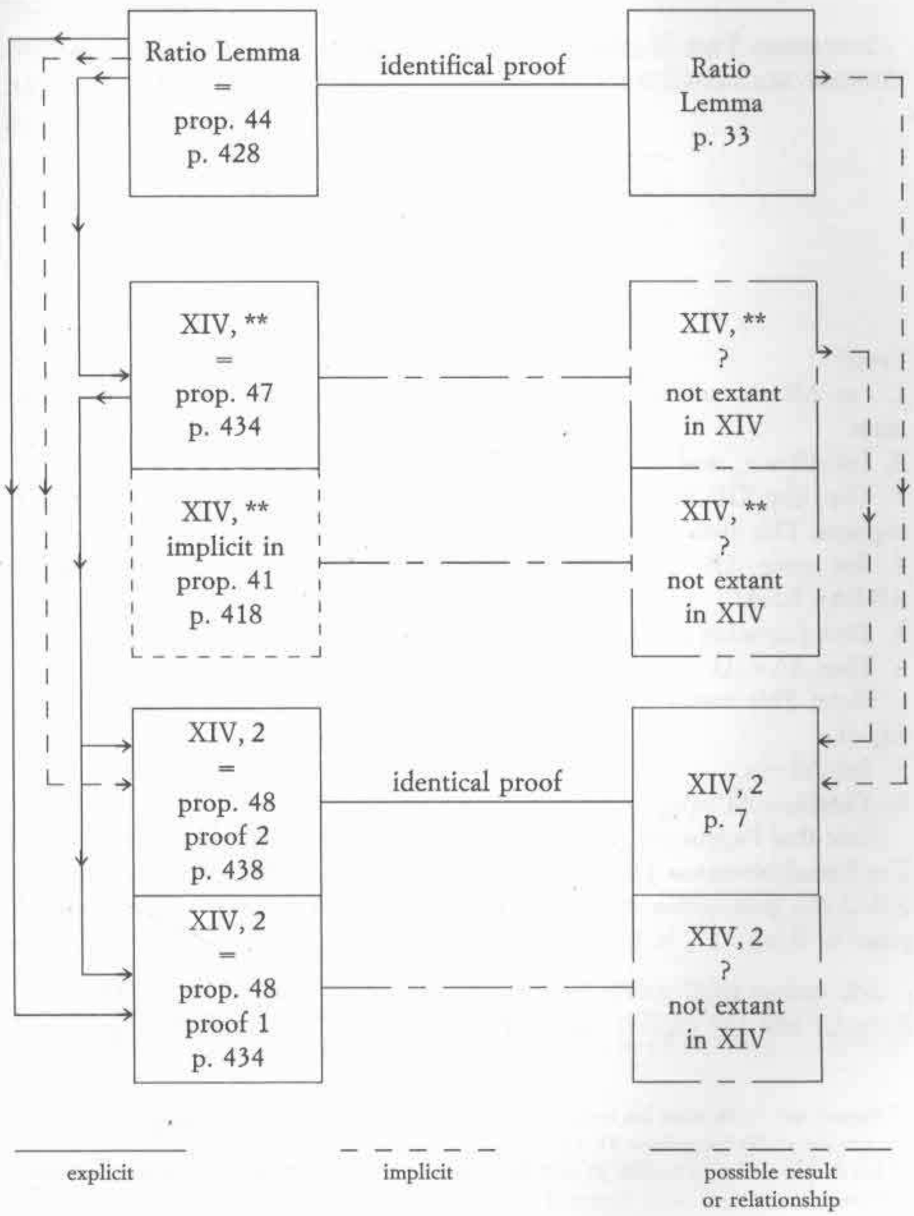
Pappus [Hultsch, vol. 1] Book XIV [Heiberg, vol. 5]

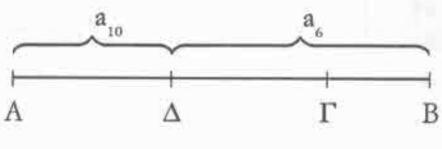


(*i.e.* to proposition 47).[6] Perhaps this more involved proof is an older demonstration that is essentially due to either Aristaeus or to Apollonius, with perhaps some editing by Pappus[7].

We are now ready to consider the proof of XIV, ** found in Pappus.

XIV, ** (Pappus, V, proposition 47 [Pappus-Hultsch, vol. 1, 434])

Statement: Τῆς δὲ τοῦ ἑξαγώνου πλευρᾶς ἄκρον καὶ μέσον λόγον τεμνομένης, τὸ μεῖζον τμῆμά ἐστιν ἡ τοῦ δεκαγώνου πλευρά.



(after [Pappus-Hultsch, 434])

Proof:
1. Let $\Delta B = a_6$ and divide it in e.m.r. at Γ with ΔΓ being the larger segment.
2. Let $A\Delta = a_{10}$ and join it to ΔB.
3. Then [by XIII,9] AB is divided in e.m.r. at Δ [with ΔB being the greater segment. This gives BΔ:AΔ = AB:BΔ].
4. But since ΔB is divided [in e.m.r.] at Γ, the Ratio Lemma gives AB:BΔ= BΔ:ΔΓ.
5. Thus [equating equals from 3 and 4] BΔ:AΔ = BΔ:ΔΓ.
6. Thus AΔ = ΔΓ.

Note: This would follow from V,9. I omit details such as these in the sequel.

7. But $A\Delta = a_{10}$.
8. Therefore $\Delta\Gamma = a_{10}$.

Note that Pappus explicitly uses the Ratio Lemma in his proof of XIV, **. The Ratio Lemma (*i.e.* proposition 44) is also used explicitly in the first proof of XIV,2 (*i.e.* proposition 48) but only implicitly, as is the case in the identical proof of Book XIV, in the second proof.

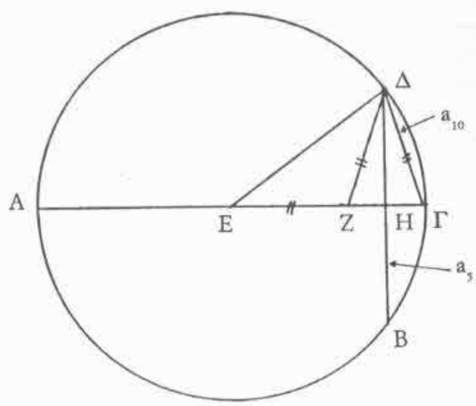
The various relationships between Book V of Pappus and Book XIV of the *Elements* and the explicit use by Pappus of XIV, ** of course suggest that

[6] Pappus says "... by what has been proved previously" which, from the context, is an obvious reference to his proposition 47, *i.e.* XIV, **.

[7] See the discussion in Section 24 of *A Mathematical History of Division in Extreme and Mean Ratio*, cit.

perhaps we have right here in Pappus a statement and proof of XIV, ** that was originally in Book XIV. However we must not jump to any conclusions because for all we know Pappus has merely filled in a lacuna that he found in Book XIV. Furthermore our story is far from over, even as far as Pappus is concerned, for Pappus provides us with another proof of XIV, **, but this time without explicitly telling us so. The result in question is:

Proposition 41 [Pappus-Hultsch, 418]: In a circle with centre E and diameter AΓ let ΔHB = a_5 and let ZH = HΓ. Then the point Z cuts EΓ in e.m.r. with EZ being the larger segment.



(after [Pappus-Hultsch, 418])

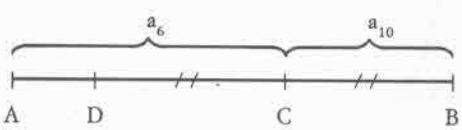
Proof:
1. Because ΔΓ = a_{10} we obtain ∢ ΔEF = 2/5 (90°); ∢ EΓΔ = ∢ EΔΓ = 4/5 (90°).
2. By construction ZΓΔ is isosceles which gives ∢ ΔZΓ = 4/5 (90°).
3. From 2. we have ∢ EΔZ = 2/5 (90°) = ∢ ΔEZ.
4. Thus EZ = ΔZ = ΔΓ.
5. Again comparison of angles shows that ΔEΓ ≈ ZΔΓ.
6. Thus ΓΔ:ΓZ = EΓ:ΓΔ and [by VI,16] R(EΓ, ΓZ) = S(ΓΔ) = S(EZ).
7. Since the statement R(EΓ, ΓZ) = S(EZ) is what is required for d.e.m.r. of E the proof is complete.

[Corollary: XIV, **] ("second proof" by Pappus):
Proof: There is nothing new to prove; EΓ being the radius it is also the side of the hexagon. In the course of the proof we learned that EZ, which as the

theorem tells us is the larger segment when E is divided in extreme and mean ratio, is equal to ΔΓ, the side of the decagon [8].

We cannot tell whether or not Pappus realized that his proposition 41 contained a proof of XIV, **. It is possible that proposition 41 represents an older proof. We have seen that proposition 48 had two proofs and this indicates that Pappus was not adverse to giving several proofs or using one, as is the case for the second proof (taken from Book XIV) that he found elsewhere.

[8] My attention was drawn to this fact by the remark in [Pappus-Ver Eecke, 322, fn. 1], which points out that this was already noticed by Commandino in his edition of Pappus. Indeed in the 1588 edition of [Pappus-Commandino, 101] the remark in question is made, without further elucidation, following commentary C to proposition 41. Furthermore it turns out that at the end of the proof of proposition 47 [p. 105], *i.e.* the explicit statement of XIV, **, Commandino states that he has given another proof in connection with theorem XIII,9 in his edition of the *Elements*. Turning thus to [Euclid-Commandino, 235], proposition 1 to XIII,9 we find that this other proof of XIV, ** goes as follows:



Let ACB be a line with AC and CB being the sides of the hexagon and decagon respectively. By XIII,9 point C divides ACB in extreme and mean ratio. Now let D on AC be such that CD = BC. Then, as Commandino had proved in connection with XIII,5, ADC is also divided in extreme and mean ratio with the larger segment being CD. Since D was chosen so that it divides AC = a_6 in e.m.r. and since CD = BC = a_{10}, the proof is complete. As I indicated in fn. 2 of the Introduction, I suspect that Commandino was the source of Heiberg's indication of why XIV, ** was true, with the result from Commandino's comments on XIII,5 being the converse that Heiberg is alluding to. I also suspect that Commandino in turn took his result from Campanus' comments on XIII,5; see the note to step 2 of the proof given in Section 4F. Note however that Campanus does not use this converse in the proofs of XIV, ** that he gives; see Section 4E. Incidentally we are not finished with Heiberg's comments on XIV, **; see fn. 11, Section 3B. Note also that in his remark C to XIV,2 [Euclid-Commandino, 245] Commandino states that XIV, ** is needed in the proof and refers back to his proposition 1 to XIII,9. Petrus Ramus (or more probably Schoner the editor), who was a contemporary of Commandino also commented on XIV, ** in his "Book XXX on Book XIV [of the *Elements*]" in the *Scholarum mathematicarum* [p. 306]. He states "These two lemmas [*i.e.* the lemma to XIV,2 and XIV, **] are (I say) here mixed together, which, if they ought to have been proposed at all, should have had separate propositions in geometry. The second is retained by us because of its usefulness in inscribing the pentagon [but not by Euclid!; see *Mathematical History of D.E.M.R.*, Section 2A]. But nonetheless each [lemma] demonstrates the logic of Apollonius, by so mixing and confusing the subject-matter of the propositions; and I do not know of a much greater aberration of this kind, because the fallacy of Theon or Euclid is extraordinary ..."(!)

3A. *Versions of al-Ḥaj*

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3. *Arabic tradition*

3A. *Versions of al-Ḥajjāj (fl. end of 8th, beginning of 9th centuries)*

For discussions and bibliographic references concerning what seems to be known or conjectured concerning either the pure or mixed al-Ḥajjāj versions see [De Young a, b; Murdoch]. According to al-Ṭūsī there were fifteen books, but there are apparently no surviving versions of Books XIV and XV, either pure or commentated, that can be attributed to him [9]. Parts of various Arabic or Arabic-Latin versions of the *Elements* have been attributed to al-Ḥajjāj, but it does not seem possible to make any definite statements about Books XIV and XV from these attributions [10].

3B. *The versions of Books XIV and XV in the so-called Isḥāq ibn Ḥunayn-Ṯābit ibn Qurra edition (9th century)*

In contrast to the Greek critical edition, the result that we have labelled XIV, ** is stated explicitly and proved as a separate theorem in this Arabic edition. However it is not found in Book XIV but rather as the first theorem of Book XV. [11] Since Book XV of the critical edition does not contain any

[9] For the statement of al-Ṭūsī in Munich 848, see [Gerard of Cremona-Busard, 12]. Sezgin states in connection with Hypsicles [p. 144] that the al-Ḥajjāj version did not contain Books XIV and XV. However under the main entry for al-Ḥajjāj [p. 225] nothing is said.

[10] On Adelard and al-Ḥajjāj, see [Clagett a, especially 18, fn. 7; Adelard I-Busard, 5]. On Gerard of Cremona and al-Ḥajjāj, see [Gerard of Cremona-Busard, xii] and [Adelard I-Busard, 4 and addendum 4] which would appear to replace [Clagett a, 27 and Murdoch c, 445]. Because of a comment in the introduction to Book XI, Copenhagen LXXXI was sometimes considered as being a mixture of al-Ḥajjāj and Isḥāq-Ṯābit material. This is now rejected by De Young [b] who considers Escurial arabe 907 and Leningrad, Akademia Nauk, C 2145 as the only two manuscripts that he has examined that can possibly be mixed. The latter as we shall see has a proof of XIV, ** identical to that of Thurston 11.

[11] My translator Dr Idris found the result in XV, after looking in vain for it in XIV, only because of a slip of the hand and microfilm – at that point in time I only suspected that XIV, ** might be in the Arabic Euclid; see fn. 73, Section 5C. Such are the vagaries of historical research! Only later did I notice that Klamroth [276, 280] had pointed out that XV,1 (*i.e.* XIV, **) was in the Arabic text but not in the Greek text. Heiberg [p. 18] on the other hand, as part of his argument against the Klamroth's contentions concerning the validity of the Arabic text, says that XV,1 was one of the lemmas that the Arabic writers made up. Presumably Heiberg bases this statement on the fact that XV,1 does not appear in the Greek text. Neither author gives the statement nor mentions the relationship with Book XIV, so fortunately I had not consulted these articles first.

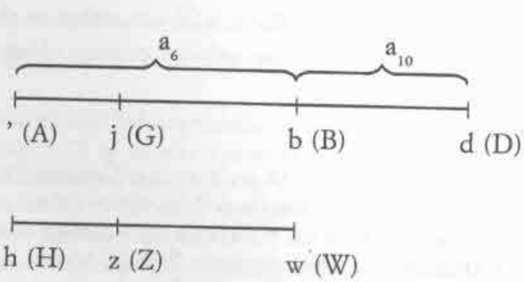
references, explicit or implicit to d.e.m.r. and since XIV, ** is needed in Book XIV, one must ask if this placement is not due to an error. I shall return to this point later on for even more important to us than the location of the result is the proof itself. For whereas the proof found in Pappus only involves *one* line, the proof found in the Arabic text is more involved and involves *two* lines, although it too uses the Ratio Lemma.

Because I shall discuss the lettering of the diagrams in the Appendix to this article I have given the Arabic letters on the diagram by means of a precise transliteration, but, in order to facilitate the reading of the proof itself, I have used the Latin characters indicated in parentheses.

XIV, ** (Thurston 11, Akademia Nauk C 2145) [12]

Statement:

ان قسم ضلع المسدس على نسبة ذات وسط و طرفين فان قسمة الاعظم هو ضلع المعشر الذي تحيط به الدايرة التى تحيط بالمسدس •



(after Bodleian, Thurston 11)

[12] Oxford, Bodleian Library, Thurston 11 was my source for the various statements made in this article. The manuscript is dated 635 (= 1238). Professor Greg De Young kindly sent me a photostat of XV,1 in Leningrad, Akademia Nauk, Ms. C 2145. The texts of XV,1 in these two manuscripts are virtually the same word for word; the diagrams are however reversed. In a letter Professor De Young has informed me that XIV, ** does appear as XV,1 in all Arabic manuscripts of the *Elements* in his possession that do indeed contain Books XIV, XV. Thurston 11 belongs to what De Young [1984 a,b] calls family B of the manuscripts of the *Elements*. Akademia Nauk C 2145 is not assigned to either of the families and De Young has given evidence which indicates that there is a relationship between Books VII-IX of this manuscript and the work of al-Ḥajjāj, but see Section 3A. For the information of future researchers, I point out that due to a binding error in Thurston 11 theorems XIV,7,8 and part of the Ratio Lemma are to be found between XIII,16 and XIII,17.

Proof:

1. Let AB = a_6 and divide it in e.m.r. at G with the larger segment being BG. The claim is that BG = a_{10}.

2a. We saw in Book XIII [*i.e.* XIII,9] that if a_{10} is added to a_6 then we have a line divided in e.m.r.

2b. Thus if we let BD = a_{10} and join it to AB then AD will be a line divided in e.m.r. with the larger segment being AB.

3. Let HW = AB and divide HW in e.m.r. with the larger segment being WZ.

4. Then WZ = BG.

Note: No reason is given but this does not mean that the result was assumed to be obvious; a Latin version of this proof (see Section 5A, step 4) implicitly uses the Ratio Lemma at this point and further the Ratio Lemma is also used implicitly in the next line of the present proof. It thus does not seem unreasonable to suppose that this step was originally justified by use of the Ratio Lemma. Whether the details were written down is of course another question.

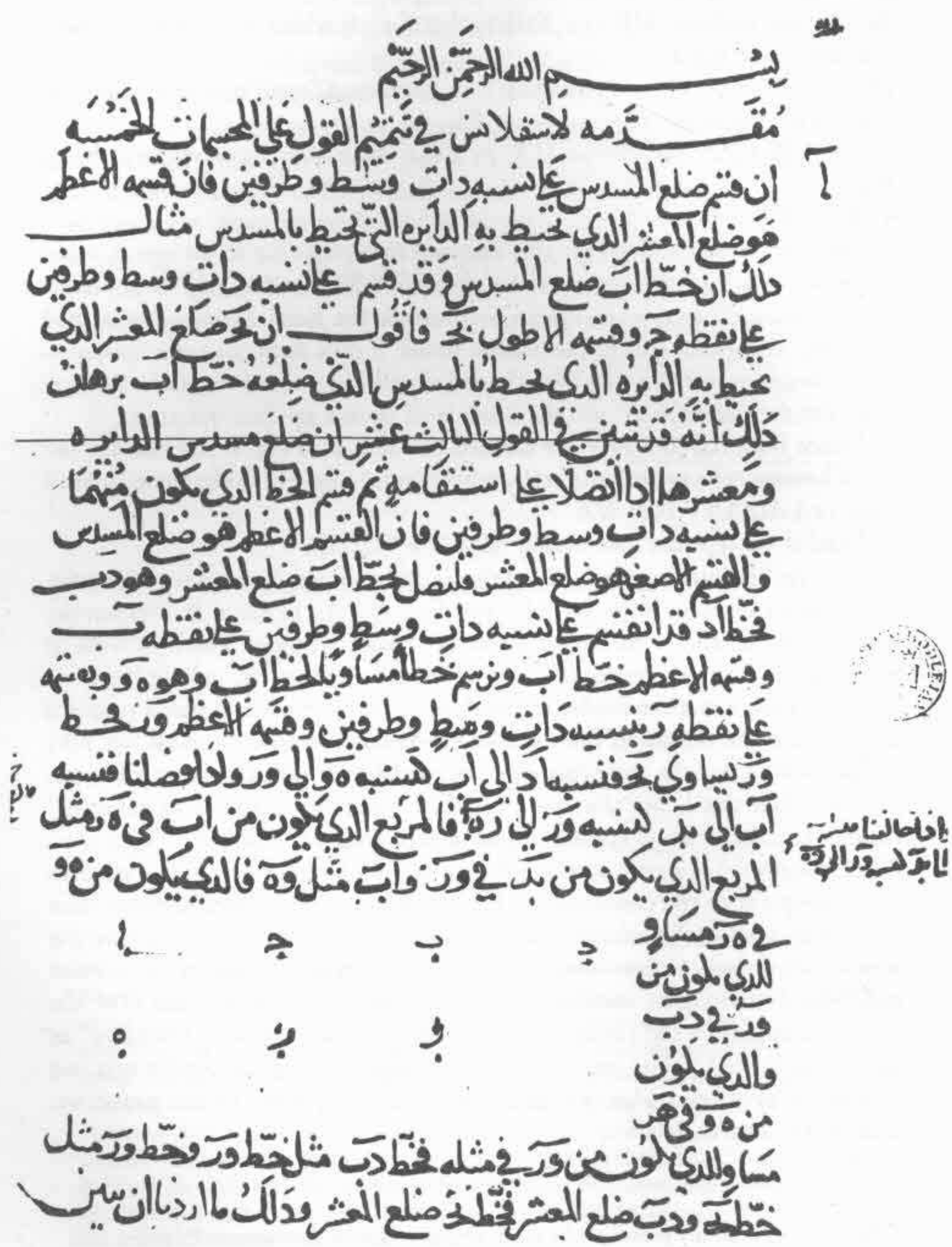
5. [Since both AD and HW are divided in e.m.r. at B and Z respectively the Ratio Lemma tells us that the ratio, whole:larger segment is the same in both cases *i.e.*] AD:AB = HW:WZ.

6. "And if we separate" we obtain AB:BD = WZ:HZ.

Note: As I pointed out in Section I the Ratio Lemma only states that the ratio, whole:larger segment is the same for two lines divided in e.m.r. It does not say anything about the ratio, larger segment:smaller segment which is what is involved in line 6. Since the definition of d.e.m.r. is, whole:larger = larger:smaller, a modern reader might think, if he even noticed that a problem existed, that the transition from the "Ratio Lemma" would be made via V,11 "Ratios which are the same with the same ratio are also the same with one another". But this is not the case here. The expression "And if we separate" (Arabic: و اذا فصلنا) appears to be a reference to V,def.15: "Separation [Greek: *diairesis*] of a ratio is taking the excess by which the antecedent exceeds the consequent to the consequent". If a = whole and b = larger segment then separation would take a:b into (a – b):b. Theorem V,17 now states that the separation of each of two equal ratios will still result in equality *i.e.* we can conclude that the ratio, smaller:larger is the same for both AD and HW. To make the switch to the ratios, larger:smaller one needs to use "inversion" as defined in V,def.13. The use of inversion is missing from the Arabic text but appears to have been what was in mind in a Latin version of this proof; see Section 5A, step 6. [13]

[13] Euclid never explicitly proves results about inversion; see the discussion in [Mueller, 129].

XIV, ** (=XV,1)-Thurston 11


بسم الله الرحمن الرحيم
مقدمه لاسقلاس في تتميم القول على المجسمات الخمسه
ان قسم ضلع المسدس على نسبه ذات وسط وطرفين فان قسمه الاعظم
هو ضلع المعشر الذي تحيط به الدايره التي تحيط بالمسدس مثال
ذلك ان خط اب ضلع المسدس وقد قسم على نسبه ذات وسط وطرفين
على نقطه ج وقسمه الاطول

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3C. *Epitome of Ibn*

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[15] Professor Greg De
XIII,9´ which follo
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7. [Using VI,16 in connection with line 6] we have AB·HZ = BD·WZ.
8. Since AB = HW we obtain HW·HZ = BD·WZ.
9. [Since HW is divided in e.m.r. at Z we have] HW·HZ = WZ·WZ.

Note: This is the second time that the proof uses the fact that HW is divided in e.m.r., but whereas in step 5 the division in e.m.r. of HW is used to compare the ratio of segments of HW with the corresponding ones of the other line AD, here the division is used to compare the segments of HW among themselves.

10. Thus, [equating the expressions for HW·HZ in 8 and 9 to obtain BD·WZ = WZ·WZ, we have] BD = WZ.
11. But [by 4] WZ = BG and [from step 2b] BD = WZ.
12. Thus BG = [WZ =] BD = a_{10}.

This version of the *Elements* presents certain difficulties as to the source and editorship of Books XIV and XV and I have placed the discussion of this in the appendix of this article.

3C. *Epitome of Ibn Sīnā (979-1037)*

This has been edited by Sabra and Lofti [Ibn Sīnā-Sabra, Lofti] and discussed by [al-Daffa, Stroyls [14], 85]. An examination of the texts of Books XIV and XV shows that XIV, ** does not appear in either of those two books and from the information available to me it does not appear to be in Book XIII either[15]. However after XIII,9 there appears the following result:

XIII,9´: If to the side of the hexagon we add a line shorter than it such that the extended line is divided in e.m.r. then the shorter segment is the side of the decagon.

It turns out this result is needed in the proof of XIV, ** given by al-Maghribī (Section 3G). Furthermore it appears in the Campanus *Euclid* (Section 4E).

[14] In [al-Daffa, Stroyls, 86], table 6, it is indicated that the order of the results in Book XV is 3, 2, 1, 4, 5. In fact what has happened is that the diagrams for XV, 1, 2 somehow appear in the text of Book XIV [Ibn Sīnā, Sabra, Lofti, 439, 440] which has a definite explicit [p. 441]. However the first two theorems stated in Book XV are XV, 1, 2.

[15] Professor Greg De Young informed me of the edition of Ibn-Sīnā and mentioned the result XIII,9´ which follows. When Interlibrary loans at Carleton University was unable to find any North American locations, I wrote to professor De Young requesting photocopies of Books XIV and XV which he kindly sent. After learning about the al-Maghribī material of Section 3G I would have liked to examine Book XIII, but by this point Professor De Young was on a research leave in India.

3D. *Epitome of Muẓaffar al-Asfuzārī (12th century)*

Only Book XIV has been published [al-Asfuzārī-Sédillot, 146-148] and does not contain XIV, **. I do not know however if Book XV even ever existed. A result which may be related to the Ratio Lemma, but which I do not understand, is the 11th and last result [16].

3E. *"Taḥrīr" of Nasir al-Din al-Ṭūsī (13th century)* [17]

Despite some differences in phrasing and terminology it can be said that al-Ṭūsī 's proof of XIV, ** is the same as that of Section 3B. [18] Once again we find XIV, ** placed at the beginning of Book XIV. Al-Ṭūsī realized that something was wrong for he says: "I would say and I believe that this result should be at the beginning of the previous chapter. It is placed here because the author forgot that some of the rules of that chapter [*i.e.* XIV] depend on it, but it is not needed here".

Since al-Ṭūsī had access to one of the al-Ḥajjāj versions [19] we can probably infer from this statement that either the al-Ḥajjāj version did not contain XIV and XV or else XIV, ** appeared at the beginning of XV in that version.

[16] The epitome does not list what is in effect the fifth proposition of Book XIV [Heiberg, vol. 5, 17] dealing with the ratios of the surfaces of the dodecahedron and icosahedron. This result appears as proposition 6 in Thurston 11. What Sédillot has translated as "virtuellement" is really "strength", *i.e.* the Greek *dynamis*, *i.e.* the square or in the case of propositions 9 (also defective as stated or translated) and 11, the sum of the squares of the segments involved. proposition 11 may possibly just be mixing up the beginning of the statement of the Ratio Lemma and part of the 'Summary' at the end of XIV. Little seems to be known about the author, much less his sources; see [Suter, 119, Section 268, 226, Murdoch c, 439; Hall, 345].

[17] I have used [al-Ṭūsī-Constantinople]. Note that the so called pseudo-Ṭūsī [Murdoch-*DSB*, 440, 453 no. 5] has only 13 books. I do not know if the existence of this and other 13 book versions implies that some early versions were also limited to thirteen books. Theorem XIV, ** is on page 213. There are no diagrams in the text.

[18] Whereas the Thurston 11 version uses the terminology "multiplication", the al-Ṭūsī version uses an "area" terminology and speaks of "squares". I do not know if this implies that al-Ṭūsī used an older manuscript with more Greek influence. Dr Hogendijk has pointed out to me that there is a certain fluidity in Arabic texts concerning the use of 'rectangle' vs 'multiplication' so perhaps the actual vocabulary used is not of any significance.

[19] See [De Young, 1984, 153].

3F. *Oxford Bodley 2773 (Hebrew translation 1309)*

This manuscript, which was translated from Arabic into Hebrew, is presently being edited by Professor Y. Langermann [20] and will be published together with the manuscript discussed in section 3G; see [Langermann, Hogendijk]. From the introductory material and the parenthetical remarks of the author, Langermann believes that the underlying arabic version is ultimately based on a Greek original.

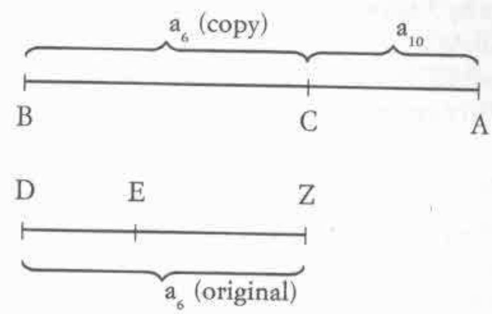
Of interest to us is the order of the results for we find that propositions 13, 14, 15, and 16 are respectively the Ratio Lemma, XIV, **, the lemma to XIV,2 and XIV,2 *i.e.* the Ratio Lemma and XIV, ** are placed where a logically correct order would demand.

XIV, ** (Oxford, Bodley 2773)

Statement:

נרצה לבאר שצלע המשׁושה כשנחלק על יחס בעל אמצע ושתי קצוות הגדול צלע מוגשם.

Note: The text has מוגשם "solid" instead of מעושׂר "decagon".



(not in text)

Proof:

1. Let $DZ = a_6$ and divide it in e.m.r. at E with the larger segment being EZ. The claim is that EZ is a_{10}.
2. Let $CB = a_6 = DZ$ and $AC = a_{10}$.
3. [By XIII,9] AB is divided in e.m.r. with CB being the greater segment.
4. Then AB:DZ = C[B]:ZE as was shown in the preceeding problem.

Note: The preceeding proposition (number 13) referred to is an extended version of the Ratio Lemma in which it is shown that the ratio, smaller:smaller is

[20] Professor Langermann kindly provided me with the text and translation of XIV, ** as well as that of the Ratio Lemma and XIV,2.

equal to the ratios, whole:whole and larger:larger. This bringing in of the smaller segments is precisely what is involved in step 6 of the proof of Section 3B; but in the present proof of XIV, ** this fact is not used. The proof of this version of the Ratio Lemma uses XIII,4 and similar triangles and is quite different from the proof found in the critical edition which in turn is the only other proof that I have ever seen in those sources which actually give a proof. I also remark that in the usual version of the Ratio Lemma the ratios involved, *i.e.* whole:larger, involve each line separately, whereas in this version the ratios involve the corresponding quantities for the two lines.

5. "When we convert" this last ratio we obtain AB:CB = DZ:ZE.

Note: To change the ratio of step 4 to that of step 5 it suffices to use V,16 which deals with alternate proportions. I do not know why the text speaks of "conversion". If this is reference to V, def. 16, where the conversion of ratio A:B is defined to be the ratio A:A–B, then there is an error. It turns out that a Latin text of a proof of XIV, **, which is close to that of Section 3B, uses the word 'converso' which may also be a mistaken reference to V, def. 16 (see the discussion on the proof of Section 5A, step 6). Whether or not there is any relationship between the two proofs and statements I cannot say; see also the note to step 6 of section 3B.

6. However [since by 3 the new, extended line is divided in extreme and mean ratio] AB:CB = CB:AC.

7. [Therefore equating terms in 5 and 6 we have] CB:AC = DZ:ZE.

8. Since CB = [DZ] [we have DZ:CA = DZ:ZE or CB:AC = CB:ZE and thus] AC = ZE.

9. But AC = a_{10} so that also EZ = a_{10}.

At first glance this proof seems to be similar to that of Section 3B but there is a difference. In the proof of the Isḥāq-Ṯābit version the segment equal to a_{10} is added onto the original line whereas here it is added onto the copy of segment BC of the original line. Because of this there is no need for a step in the present proof that corresponds to step 6 in the proof of Section 3B. From a strictly mathematical viewpoint I would describe this proof as being someplace between the proof of Section 3B and that found in Pappus.

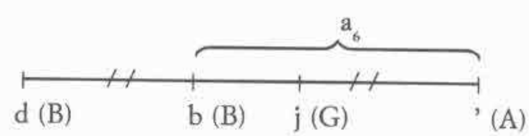
3G. *The version of the* Elements *by Muḥyi al-Dīn al-Maghribī (fl. 2nd half 13th century)*

This version is being studied by J. Hogendijk, in particular with regard to what is called Book XV in the extant manuscripts. The content of this book is related to, but not identical with those of the Hebrew manuscript discussed in

Section 3F; [21] see [Langermann, Hogendijk]. Theorem XIV, ** is found not in Book XV, as was the case with the Arabic Euclid of Section 3B, but rather as the 14th proposition of Book XIII. The latter in turn corresponds to a large extent to Book XIII of the critical edition.

The Ratio Lemma also appears in Book XIII as the 10th proposition. The statement involves the extended form, involving the small segments, that appears in the Hebrew version of Section 3F. However, the main portion of the proof is essentially the same as that found in the critical edition. The smaller segments are brought in via an appeal to V,19.

XIV, ** (Version of the *Elements* of al-Maghribī)



(after Utrecht 1440)

Statement:

اذا قسم ضلع المسدس بنسبة ذات وسط و طرفين فان قسمه الاعظم ضلع المعشر .

Proof:

1. Let AB = a_6 and divide it in e.m.r. at G with AG being the larger segment.
2. Let BD = AG and join it to AB.
3. By XIII,5 [= XIII,7 of the text] AD is divided in e.m.r. with the larger segment being AB.
4. But AB = a_6 and thus by XIII,9 [= XIII,13 of the text] BD = a_{10}.
5. Therefore AG = BD = a_{10}.

The key step here is number 4 where XIII,9 is invoked. But XIII,9 says that $a_6 + a_{10}$ is a line divided in e.m.r.. What is really needed is:

[21] The manuscript in question is Utrecht 1440. Dr Hogendijk not only provided me with the text of some of the manuscript material, but also other material and he supplied several important references and made some very pertinent commentaries. On al-Maghribī, see [Sabra b,14]. The present author is not to be confused with al-Samawa'al al-Maghribī who lived *ca* 1126-1175 and who was also a mathematician; see [al-Maghribī-Pearlman]. Some of al-Maghribī 's sources are identified in [Sabra b,15]. These sources include Ibn Sīnā; see fn. 22.

XIII,9´: Let AB = a_6 and add on BC smaller than AB. Suppose that AC is divided in e.m.r. Then BC = a_{10}.

In my humble opinion if one is ready to accept XIII,9´ from XIII,9, then one should – as the above proof shows – be ready to accept XIV,** itself from XIII,9.

As noted in Section 3C, XIII,9´ appears as a corollary to XIII,9 in the version of the *Elements* of Ibn Sīnā. Furthermore, al-Maghribī himself states that he had seen the Ibn-Sīnā version and indeed traces of the changed ordering of results – with some results missing – as found in Book XIII of Ibn Sīnā are to be found in Book XIII of al-Maghribī [22]. Essentially the same proof of XIV, ** is to be found in the *Elements* of Campanus (Section 4E).

4. *Arabic-Latin tradition*

4A. *"Adelard I" (ca 1130)* [23]

This version does not exist as such in one manuscript, but has been pieced together by Clagett [1953, 18] and a critical edition has been published by Busard [Adelard I–Busard] [24]. Clagett has assigned, on the basis of an attribution in one of the manuscripts, this version to Adelard of Bath (fl. 1116-1142) but Busard [p. 16] has cast some doubt on this. The Latin text is full of Arabicisms and both Clagett [p. 19] and Busard [p. 19] point out that there is a close relationship between Adelard I and the partial, commentated version of the al-Ḥajjāj text.

What is of particular interest is that there are two statements and proofs of XIV, **. [25]

[22] See [al-Daffa, Stroyls, 86, 88] and the forthcoming article by Langermann and Hogendijk.

[23] Busard [Adelard I-Busard, 20] suggests that this version was written between 1126 and 1130. Clagett [a, 20] points out that Adelard mentions an edition of Euclid in his treatise on the astrolabe which, it is suggested, was written between 1142 and 1146; see [Clagett d] for a discussion of Adelard.

[24] The texts of XIV and XV are based on Bodleian D'Orville 70 (13th or 14th centuries) which is the only one of the manuscripts to contain these books in the "Adelard I" version. For the statements of the results and the proofs I have used [Adelard I-Busard] but in addition I asked Dr Curchin to check certain features of the manuscript that I will discuss presently.

[25] From the fact that Busard [p. 27] points this out I assume that this is the only example of a repetition in this version.

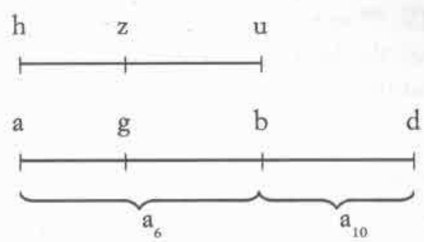
XIV, ** ("Adelard I", proof 1) [26]

Statement: Diviso latere exagoni secundum proportionem habentem medium et duas extremitates erit eius maior pars latus decagoni contenti a circulo continente ipsum exagonum [27].

The proof corresponds statement by statement to that of the Arabic proof of Section 3B, the w of the latter being transcribed here by the long vowel u which it also represents. It is to be noted however that in the manuscript two parts are bracketed. The first corresponds to what I have labelled statement 2a in the Arabic proof. The second bracket includes statement 5 and part of statement 6 *i.e.* the Latin text has "[we have that ad:ab = hu:uz and if we separate] we obtain ab:bd = uz:zh".

XIV, ** ("Adelard I", proof 2) [28]

Statement: Si latus exagoni secundum proportionem habentem medium duasque extremitates dividatur, erit dividens maior latus decagoni contenti in circulo continente exagonum.



(after Bodleian, D'Orville 70 [the position of z has been corrected, so as make zu = gb])

Once again we find a proof that corresponds statement by statement to that of the Arabic proof of Section 3B. However the actual wording of the proof, as well as that of the statement, are different from those of proof 1. [29]

This difference in language seems to preclude the possibility that the translator (copiest) simply translated (copied) the same theorem twice. It is possible

[26] [Adelard I-Busard, 386, < XIV,13 >].

[27] There is no diagram with version 1. Presumably one was to refer to the diagram accompanying version 2.

[28] [Adelard I-Busard, 386, < XIV,13 bis >].

[29] I note that Busard [p. 27] considers these proofs different, for he writes "The first form is very similar to XIV,9 of [Adelard] II, whereas the second is similar to XIV,3 of Gerard's version although the proof in the latter is longer". As I point out in Section 4D, I consider Gerard's proof different.

that an editor used two Latin manuscripts [30]. Proof 2 uses the word 'dividens' in the statement rather than the usual 'pars' – as in proof 1 – or 'portio', although proof 2 uses 'dividens' in the text [31]. Perhaps more significant is the fact that in the statement of both proofs one finds the word 'extremitates' instead of the usual 'extrema' which is employed in both of the proofs themselves.

Let us now consider the placement of XIV, ** in Adelard I. Busard has labelled the two statements and proofs as < XIV,13 > and < XIV,13 bis > , but since Book XV does not have a formal incipit, although there is one for Book XIV, [32] the question arises as to where the copiest considered XIV, ** to have been located.

If we turn to the manuscript of D'Orville 70 we note that at the end of the proof of each theorem of Book XIV, the next theorem is announced by what appears to be the original hand. Thus at the end of the proof of theorem 11 we read "I(ncipit) XII th.", referring to the Ratio Lemma [33] and the Summary is introduced as "I(ncipit) XIII th.". [34] At the end of the Summary we read "I(ncipit) XIIII th." which announces the first statement of XIV, **, but there is nothing at the end of the first proof. It would appear then that at least the copiest considered XIV, ** as being the 14th theorem of Book XIV. However, given the double proof, the later (13th or 14th century) date of this manuscript and the incomplete nature of Book XV, some reservation is called for [35].

[30] This might help explain the brackets in the first version. There are brackets elsewhere in the manuscript. I note that Adelard I is characterized by two different proof endings and both Latin and Arabic terms for the words sphere and cube, see [Adelard I-Busard, addendum 2].

[31] These and the following remarks about the language of the text are based on comments by Dr Curchin. The usual sense of 'dividens' in mediaeval mathematical Latin seems to have been 'divisor'; see [Latham, 154].

[32] Only Books VII and XIV are identified by a number at the beginning. Since Book XIII has the formal explicit "Dei gratia eiusque adiutorio." marking the end of the first thirteen books, it is not surprising to find an incipit for XIV.

[33] The Ratio Lemma is labelled < XIV,12 > by Busard [Adelard I-Busard, 384].

[34] What I have called the Summary corresponds to lines 346-365 in [Adelard I-Busard, 385]. Busard does not give a number to the Summary which is why XIV, ** is labelled < XIV,13, 13 bis > .

[35] Only XV,1 and XV,2 are present with the latter ending in the middle of 71v. This means that we cannot account for the missing three theorems on the basis of a lost page. Further there is nothing said at the end of the second proof of XIV, ** nor in connection with either XV,1 or XV,2. Thus we can only assume that the latter two were meant to be in the XVth book. For all we know this might be a translation of an Arabic text, in which what we call Books XIV and XV were treated as one; this possibility is considered in the Conclusion. Confusion reigned among various users of this manuscript. One hand has written "vacat" next to the first statement of XIV, **, possibly indicating that the result was not in another manuscript (this hand uses 'v' as opposed to 'u' of the main scribe). Still another hand has numbered the

4B. *"Adelard II" (12th century)*

For discussions of this version of the *Elements*, see [Clagett a, 20; Adelard I-Busard, 16; Goldat, chapter 3, Gerard of Cremona-Busard, x]. Contrary to the case of Adelard I there are many manuscripts containing Books XIV and XV [Clagett a, 22]. I have only checked two of them.

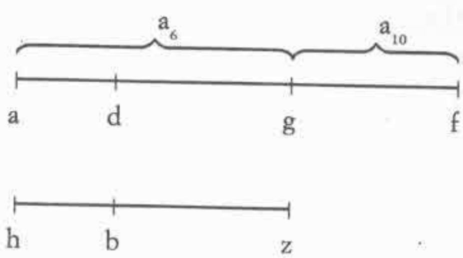
XIV, ** (Oxford, Trinity 47, 12th century) [36]

Statement: Diviso latere exagoni secundum proportionem habentem medium duoque extrema maior eius portio erit latus decagoni circumscripti a circulo ipsum exagonum circumscribente.

Even though there is no proof or diagram associated with XIV, ** this manuscript is of special interest because Book XV has a definite incipit and the location of XIV, ** can thus unquestionably be said to lie at the end of Book XIV, following immediately after the Ratio Lemma and the Summary [37].

XIV, ** (Oxford, Bodleian, Auct. F. 5,28) [38]

Statement: Same as that of Trinity 47.



(no diagram in text)

theorems in Arabic numerals. This hand did not label the first version of XIV, ** but has written 14 next to the second statement of it. At the end of the first proof of XIV, ** the statement "Incip. lib. 15" is written (again with Arabic numerals) but this is crossed out and rewritten at the end of the second proof. A further marginal notation next to the last incipit says "L(iber) 15. th. 5", so this person knew how many theorems there should be.

36 Edited by Goldat. There are no proofs for Books XII-XV. Theorem XIV, ** is result < x > on page 399. There is a brief introduction to XIV, which mentions, in addition to Aristaeus and Apollonius, a Necnon (corrected to Nennon by [Goldat, 389]). I have been unable to identify this person but wonder if it is not the same person as the Seneclunus (or Mesenclunus) of the manuscript Oxford, Bodley Heb. d. 4 discussed in Section 3F, see [Langermann, Hogendijk, 326].

37 Book XIV has an incipit but no explicit. Book XV has five theorems.

38 The manuscript was read and translated for me by Dr Curchin who also commented on the

1. Let ag = a_6 and divide it in e.m.r. [at g] with the larger segment being gd. The claim is that gd = a_{10} of the circle whose radius is ag.
2. Let gf = a_{10}. Then by XIII,9 af is divided in e.m.r. with the larger segment being ag.
3. Let hz = ag and divide hz in e.m.r. with the larger segment being zb.
4. By the Ratio Lemma (text: "[theorem] 8 of the present [book]") ag:gf = zb:bh.

Note: Here the Ratio Lemma is used to directly involve the ratio, larger:smaller as compared with the proof of Section 3B (steps 5 and 6) where the text uses the Ratio Lemma to consider the ratios, whole:larger.

5. By VI,16 [text: VI,15] ag·bh = gf·zb.
6. Since ag = hz we have hz·bh = gf·zb.
7. Also since hz is divided in e.m.r. we have hz·bh = zb·zb.
8. Thus [equating the expressions for hz·bh in 6 and 7 to obtain gf·zb = zb·zb we have] zb = gf.
9. But zb = gd.

Note: This step corresponds to step 4 of the Arabic proof of Section 3B.

10. Therefore [from 8 and 9] gd = gf.
11. But gf = a_{10}.
12. Thus gd = [gf =] a_{10}.

This proof is very similar to the Arabic proof of Section 3B but there are certain differences. The most noticeable is, as discussed in the note to step 4, that in this proof the ratio larger:smaller is obtained directly. Two other differences are the mention in step 1 of the circle whose radius is ag the side of the hexagon, and the moving of step 4 of the Arabic text to step 10 here. Furthermore in this text the references to the theorems used in steps 2, 4 and 5 are very precise. In particular note the explicit reference to the Ratio Lemma in step 4.

The proof that we find here in Adelard II is thus more detailed than that found in Adelard I, which seems to be at variance with the usual situation [39]. I cannot say if the proof that we have here was based, with details added by the editor/translator, directly on the proof of Section 3B or if it was based on another Latin text (the letters do not 'correspond' to the Arabic). In any case the

text. Theorem XIV, ** is found on what is marked as 14ᵛ, but is really 54ᵛ. Dr Busard kindly sent me his transcription of the text. I also now note that one of the manuscripts of the Gerard of Cremona version of the *Elements* has, at the end of Book XIV, a variant reading which was taken from an Adelard II manuscript. This appears in printed form in [Gerard of Cremona-Busard, 500]. The statement is the same as the one given here except that it has 'duo' instead of 'duoque'. The proof itself contains the statement "by [theorem] 8 of the present" as in step 4 of the present proof.

[39] See [Clagett a, 20].

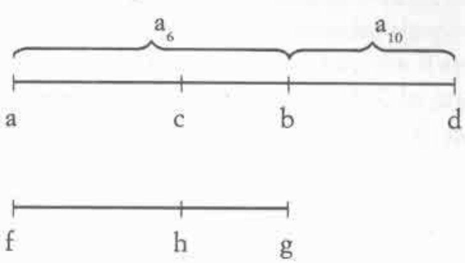
statement of step 4 makes it clear that the editor definitely considered XIV, ** to be in Book XIV. [40]

4C. *"Adelard III" (12^{th} century)*

According to [Clagett a, 23] this version generally uses the statements of Adelard II but with more detailed proofs. We saw in Section 4B that for XIV, ** Adelard II had a more detailed proof than that of Adelard I and here we find a proof which is different from the other two and which also shows signs of confusion.

XIV, ** (British Library, Burney 275) [41]

Statement: Diviso latere exagoni secundum proportionem habentem medium duoque extrema maior eius latus erit proportio [42] decagoni circumscripti a circulo ipsum exagonum circumscribente.



(no diagram in text)

[40] The Ratio Lemma – as step 4 indicates – is XIV,8. This is followed by what I call the Summary (which apparently is never given a number) and then XIV, ** which, because it is stated to be in the same book as XIV,8, is thus XIV,9. The manuscript Auct. F 5,28 has a 9 between the proof and the statement of XV,1, which follows, is labelled 1. However this may be due to a later hand because some of the other numbering is irregular and off to the side (*e.g.* XV,2). But even though it is presumably possible that XV,1 was considered to be in Book XIV, all indications, in particular the evidence of Trinity 47, oppose this possibility. Busard [Gerard of Cremona-Busard, xi] considers Adelard II to have been translated from the Arabic whereas Clagett [a, 20] does not think so, although he does consider it a possibility.

[41] This manuscript contains several other works apart from the *Elements*. XIV, ** is found on fol. 335^r = page 665. The text was read and translated for me by Dr Curchin.

[42] Adelard II reads "portio erit latus" at this point.

1. Let ab = a_6 and divide it in e.m.r. [Latin: proportionaliter divisum [43]] at c with the larger segment being ac.
2. Let bd = a_{10}. Then by XIII,9 ad is divided in e.m.r. with the larger segment being ab.
3. Let fg = ab and divide fg in e.m.r. at h.
4. Then ac = fh and bc = hg "because since ab and fg are both divided in e.m.r. the same thing will be true for both lines".

Note: The Latin is unclear and confusing at this point and the statement in quotations represents what I believe is the approximate meaning of this portion of the text. I suspect that this is a reference to the Ratio Lemma; see the note on step 4 of the Arabic proof of Section 3B.

5. [Then by the Ratio Lemma] [ad:ab] = fg:fh.

Note: The text has ab:ad instead of the ratio in the brackets.

6. "Similarly", ab:fg = ac:fh.

Note: Since by 4 the ratios on both sides are 1:1 the meaning of this step, if any, escapes me. Perhaps it should read:
ab:ac = fg:fh (by the Ratio Lemma). The Latin here and elsewhere says "ser" which appears to be an abbreviation for s(imilit)er.

7. Therefore with equals substituted, ad:ab = ab:ac.

Note: This follows if we use fg = ab from 3 and fh = ac from 4 and substitute in the corrected form of 5. See also the discussion below.

8. "Similarly" [since from step 3 the extended line ad is divided in e.m.r.] ad:ab = ab:bd.
9. This step is a repeat of step 7 with the expression "and ab:ad" stuck on.
10. Thus [using V,11 to equate the common ratio ad:ab in 7 and 8] ab:ac = ab:bd.
11. "Since ac and bd are equally proportional to the same segment ab they are [by V,9] equal".
12. Therefore ac = bd = a_{10}.

Ignoring for the time being the errors, repetitions and strange expressions, there is something which distinguishes this proof from the Arabic proof of Section 3B. For in that proof both the original line and the copy are used until the end whereas here at line 7 one has resubstituted so that we are only dealing with the new line. In fact line 7 could have been obtained by applying the Ratio Lemma involving, whole:larger to the lines ad and ab. Note that this is precisely what is done at step 4 of Pappus' proof of Section 2 with step 8 here being the same as step 3 of that proof.

[43] The proof always uses "proportionally divided" instead of d.e.m.r. but I will use the latter for greater clarity.

Two possibilities suggest themselves. First, step 7 may just be the result of incompetence on the part of the editor who became fouled up, as witness the signs of confusion, with the resemblance to the Pappus proof being a coincidence. Or this text may be based on an Arabic text somewhat different from that of Section 3B with the mistakes being due to copying errors and the like. In the latter case the Arabic may in turn reflect a Greek text, possibly one used by Pappus.

I note that there is no separation between Books XIV and XV so that we cannot find where XIV, ** was located in the text on which this version was based. We do find the Ratio Lemma followed by the Summary, followed by XIV, **. There are only three theorems from Book XV but one manuscript of this version does contain the five theorems from XV. [44]

4D. *Gerard of Cremona (1114-1187)*

Gerard of Cremona was part of the group of translators, usually working in teams of two – Arabic to the vernacular to Latin – centred around Toledo [45], who were responsible for the transmission of much of the knowledge of the Arabic lands into Europe.

In this version we find XIV, ** as the third theorem of Book XIV between the lemma to XIV,2 and XIV,2 itself, *i.e.* in a logically correct position. However, the Ratio Lemma is still found at the end of Book XIV [46] just before the Summary. The proof that we find of XIV, ** contains, with respect to the proof that we found in Section 3B, several elements of surprise.

XIV, ** (Paris, Bibliothèque Nationale, Lat. 7216) [47]

Statement: Si latus exagoni secundum proportionem habentem medium et

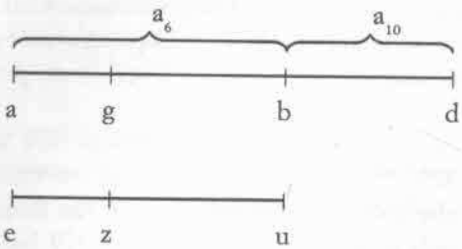
[44] See [Clagett a, 25], especially manuscript 4 (BN 16648).

[45] According to [Glick, 258], who discusses this school of translators, Gerard worked with a Mozarab (*i.e.* an arabized Christian from Islamic Spain) named Galippus. Another member of this group was Hermann of Carinthia, who also produced a version of the *Elements*. However, Books XIII-XV are not extant; see [Hermann of Carinthia-Busard]. For a discussion of Gerard's version of the *Elements*, see [Clagett a, 27; Gerard of Cremona-Busard].

[46] In fact that the Ratio Lemma appears twice (105^{r}, 105^{v}) as does the 'Summary'. There is an explicit for XIV and an incipit for XV so that XIV and XV are clearly separated. Book XV contains XV, 1-5.

[47] Dr Curchin translated this text from the manuscript before I was aware of the critical edition [Gerard of Cremona-Busard, 416,498] which I subsequently also used. Dr Busard also kindly provided me with some comments on the text.

duo extrema dividatur maior eius sectio erit latus decagoni qui a circulo continetur.



(after Paris, Bib. Nat., Lat. 7216)

1. Let ab = a_6 and divide it in e.m.r. at g with the larger segment being bg. The claim is that bg = a_{10}.
2. Let bd = a_{10} then by XIII,9 [text: by what is shown in Book XIII] ad is divided in e.m.r. with the larger segment being ab.
3. Let eu = ab and divide it according to the ratio, db:ba with the larger segment being uz.

Note: In the text of Section 3B, step 3, the copy of the original line is divided in e.m.r. but this is not what is happening here; confer step 10 where it is *shown* that eu is divided in e.m.r.

4. Thus [because eu is divided as in 3] db:ba = ez:zu.
5. Then by 'composition' [V, def. 14; V, 18] ad:ba = eu:zu.

Note: Step 4 involves the ratio smaller:larger and in step 5 the composition of ratios [Greek: *sunthesis*], which transforms the ratio A:B into the ratio (A + B):B, is used in order to obtain the ratio, whole:larger. But in the Arabic text of Section 3B, step 6, one starts with the ratio, whole:larger to obtain, via 'separation', the ratio, larger:smaller.

6. [By step 2 and the definition of d.e.m.r.] ad:ba = ab:bd.

Note: This step involving d.e.m.r. for the extended segment does not appear in the proof of Section 3B.

7. Thus, [equating the expressions for ad:ba in steps 5 and 6 we have] ab:bd = eu:uz.
8. But [by step 4 and 'inversion', V, def. 13] ad:ba = uz:ze.
9. Thus [equating the expressions for ad:ba in 7 and 8] eu:uz = uz:ze.
10. Therefore line eu is divided in e.m.r. at z with the larger segment being uz.

Note: In the proof of Section 3B, step 3 this is assumed; see step 3 above.

11. Since eu = ba we have that uz = bg.
Note: See on step 4 of the proof of Section 3B.
12. [As already stated in 8] ad:ba = uz:ze.
13. Thus [by VI, 16] ab·ze = bd·uz.
Note: We have now arrived at statement 8 of the proof of Section 3B and from this point on the two proofs coincide, except that step 16 here is only implicit in Section 3B.
14. Since ba = eu we have eu·ze = bd·uz.
15. But [since eu is, by step 10, divided in e.m.r.] eu·ze = uz·uz.
16. Thus [equating the expressions for eu·ze from 14 and 15] bd·uz = uz·uz.
17. Therefore bd = uz.
18. But uz = bg so that bg = db.
19. Therefore bg = db = a_{10}.

I find it difficult to arrive at any conclusions about this proof. It seems of course at first to be based on a text similar to that used for Section 3B, but the appearance of steps 3 and 6, especially the former, would seem to rule out a simple expansion of the next. Note that because of step 3 and the way that the rest of the proof proceeds, the Ratio Lemma is not used either explicitly or implicitly as was the case for the proof of Section 3B. [48]

4E. *Campanus of Novara (died 1296)*

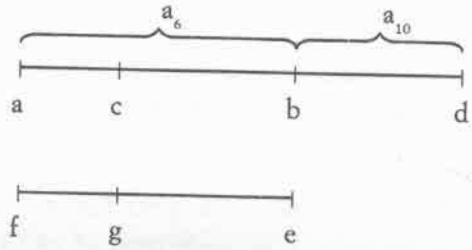
Theorem XIV, ** is the third theorem in Campanus' edition where it is placed between the Ratio Lemma and the lemma to theorem 2. Thus, as was the case in the Gerard of Cremona version, XIV, ** is found in the most mathematically satisfying location [49].

Campanus appears to have particularly researched this result for we find three different proofs. These correspond to the proofs of Pappus and al-Maghribī and that of Section 3B, although in the latter two cases there are some differences.

[48] In a private communication Dr Busard suggested that the fact that XIV, ** is the third theorem in Gerard's version – as opposed to being at the end of Book XIV in Adelard II – is related to Gerard's proof not using the Ratio Lemma. This may be correct, but we must remember that XIV,2 uses the Ratio Lemma, albeit implicitly, and yet the latter is at the end of Book XIV here and in the critical edition. Campanus, as we shall see in the next section, has the Ratio Lemma as the second theorem and XIV, ** as the third theorem (as per Gerard).

[49] The sequence: Ratio Lemma, XIV, **, lemma to theorem 2, theorem 2 is, aside from in the Campanus edition, only found in the manuscript of Section 3F. In the Gerard of Cremona version the Ratio Lemma is at the end of Book XIV.

XIV, ** (Campanus of Novara) [50]



(after [Euclid-Paris])

Statement: same as Adelard II.

Proof 1. Essentially the Pappus proof of Section 2.
Proof 2. This proof is the same as the al-Maghribī proof. However, whereas al-Maghribī (step 4) simply invokes XIII,9 to justify the conclusion that if we have a line divided in e.m.r. with the larger segment being a_6 then the smaller segment is a_{10}, Campanus specifically refers to his XIII,9 converse – what I have called theorem XIII,9´ in Section 3G. Turning to XIII,9 in the Campanus edition we find that he states that this converse is easy to prove by working backwards. Then he very perceptively remarks that Ptolemy assumed XIII,9´ in

[50] I have used the 1516 Paris edition [Euclid-Paris, 245] which contains both the Campanus and Zamberti editions. For discussions of the Campanus version see [Clagett a, 29; Murdoch b]. Campanus can be considered not only as representing the end of the history of XIV, ** – by his giving of three proofs of his predecessors – but also in a sense the spiritual godfather of the 'golden numberists' who followed him. For in his text of XIV,10 [p. 251] Campanus waxes eloquently about d.e.m.r.: "Mirabilis itaque est potentia linee secundum proportionem habentem medium duoque extrema divise ..."; see *Mathematical History of Division in Extreme and Mean Ratio*, Appendix II, for the texts of Campanus and his successors. In particular Pacioli's *Divina proportione* (1509) was inspired by this remark, and the quotation at the beginning of this article is the 'title' of Chapter XX in which Pacioli states XIV, ** and gives a numerical example [Pacioli-Winterberg, 56]. Another example from Piero della Francesca's *De quinque corporibus regularibus* [Piero della Francesca-Mancini] shows not only the influence of the Campanus edition but also the confusion that sometimes existed – it still does (!) – concerning XIII,9 and XIV, **. For in connection with problem 3 of Part IV, Piero della Francesca says: "And by XIII,8 [of the Campanus edition = XIII,9] of Euclid [if we] divide the side of the exagon in e.m.r. the larger segment will be the side of the decagon inscribed in the same circle".

Book I, chapter 10 of the *Almagest* when constructing the chords (a_5 and a_{10}) of the circle [51].

Note that Campanus and al-Maghribī were contemporaries so that it can be argued that Campanus may have had access to the al-Maghribī text [52]. Possibly Campanus many have used a prototype of the al-Maghribī text or may perhaps have given his own proof. We noted in Section 3C that XIII,9´ appears in connection with XIII,9 in the Ibn Sīnā edition.

Proof 3. This proof is close to the proof of Section 3B but contains the following differences (which I shall indicate using the step numbers and lettering of the latter [53]):

Step 4: In addition to saying that the larger segments (WZ, BG) are equal, Campanus also says that the smaller segments (HW, AG) are equal. This is also found in Adelard III (step 4) but does not seem to be used by Campanus.

Steps 5, 6: Campanus states right away that the ratio larger:smaller is the same for the extended line and the new line. This corresponds to step 4 in Adelard II.

4F. *A scholium from manuscripts of the* Elements *of Gerard of Cremona*

Busard has published a number of scholia that appear in manuscripts of Gerard of Cremona's version of the *Elements* and scholium IX [54] turns out to be XIV, **:

XIV, ** (Scholium to the *Elements* of Gerard of Cremona) [55]

Statement: [XIII,9] et declaratur ex illo quia quando dividitur latus exagoni

[51] See *Mathematical History of D.E.M.R.*, Section 26A. Campanus also discusses the relationship of XIII,9 to the construction of the 72°-72°-36° triangle in IV,10, see *D.E.M.R.*, Section 2A. One can safely state that, assuming that these are his own comments, Campanus was no mathematical slouch.

[52] Toomer [a, 24] states that Campanus did not have the linguistic ability to translate directly from the Arabic; using various indicators he suggests the period 1255-1259 for the composition of Campanus's *Euclid*. [Tekli] takes 1260-1265 as the central period of activity of al-Maghribī and Sezgin, [p. 144] takes 1281-1291 as the period of his death.

[53] Campanus has completely latinized the lettering – presuming that this was not done in the editing of the printed edition – and uses the seven letters a through f. He also indicates on three occasions which theorem is to be used: VI,16 at step 6; VI,17 at step 8 and "prima sexti" (?, an area form of V,9 is needed) at step 10.

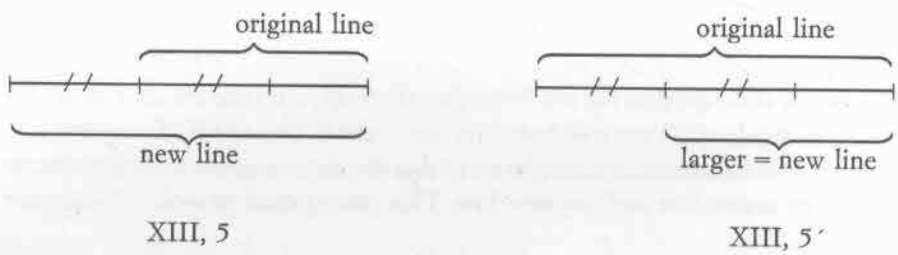
[54] Busard [p. 100] speaks of this as a corollary to XIII,9 which the scholium refers to, but I am not sure where exactly in the manuscripts this scholium is located, *i.e.* whether with XIII,9 or, what would be of perhaps greater interest, in Book XIV.

[55] Busard [p. 115]. This is reprinted in [Gerard-Busard, 443].

circuli secundum proporcionem habentem medium et duo extrema, tunc sectio maior est latus decagoni ... .

Proof:

1. By XIII,9 $a_6 + a_{10}$ forms a line divided in e.m.r.
2. [XIII,5′] If a line is divided in e.m.r. and the smaller segment is subtracted from the larger then the larger segment is itself divided in e.m.r. and its larger segment is the smaller segment of the original line.



Note: This result, that I have called XIII,5′, can be considered as a converse to XIII,5. No proof is offered in the scholium. The same result, although stated slightly differently, is found in Campanus' commentary on XIII,5. [56]

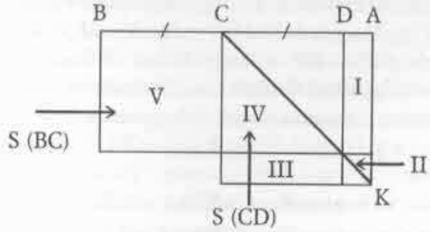
3. And this is what we wanted to prove.

Note: This is the medieval equivalent of the modern "It is now clear that ... " that haunted so many of my student and professional days [57].

[56] The Campanus text is given in Busard [p. 100].

[57] Although our scholiast thinks he is done I myself do not consider the matter so easily disposed of. Since Heiberg and Heath (see fn. 2 of the Introduction) also glibly invoke the 'converse' to XIII,5 let us see how a proof of XIV, **, using this result, looks like when we restrict ourselves entirely to the methods of Book XIII:

Lemma (XIII,5′): Let AB be a line which is divided in e.m.r. at C with AC being the larger segment. Let D be the point of AC such that DC is equal to BC. Then the line AC is divided in e.m.r. at D with DC being the larger segment.



Proof: From the given, I + IV + V = R(AB,BC) = S(AC) = I + II + III + IV. But by construction V = IV and by I,43 I = III. Thus the above becomes III + IV + IV = I + II + III + IV

The manuscripts in question apparently all date from the fourteenth and fifteenth centuries [58], *i.e.* later than Campanus, although since XIV, ** is in all of them it was some earlier prototype. I have thus placed this version after that of Campanus. We of course do not know if the scholiast took the result of step 2 from Campanus. Recall that Campanus did not use this 'converse' to XIII,5 for the proof given in the last section.

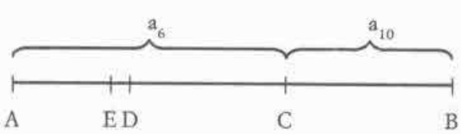
For the benefit of the reader I point out that despite what may at first reading appear to be a close relationship between the present scholium and the proof of XIV, ** in the al-Maghribī text of Section 3G, they are in my opinion to be considered as quite distinct. In the present proof XIII,9 is applied first and then the 'converse' to XIII,5 is used whereas in the al-Maghribī proof it is rather XIII,5 which is applied first to the original line divided in e.m.r. and this is followed by an implicit appeal to the 'converse' of XIII,9.

5. *Greek(?)-Latin tradition*

While none of the existing Greek manuscripts has, to my knowledge, the explicit statement of XIV, ** it turns out that the statement does appear in two

and by subtracting III + IV from both sides we have IV = I + II *i.e.* S(CD) = R(AK,AD) = R(AC,AD) as required.

Proof of XIV, ** (in the spirit of Book XIII): Let AC be the side of the hexagon and CB the side of the decagon. Then by XIII,9 the conditions of XIII,5′ are met so that if from AC we subtract off CD equal to the side of the decagon, we have that AC is divided at D in e.m.r. with CD being the greater segment.



One would now like to complete the proof as follows: divide AC in e.m.r. at E with CE being the larger segment. Since D also divides AC in e.m.r. it must be that D and E coincide, *i.e.* CE = CD = side of decagon. Q.E.D.

I write Q.E.D. but I am not that sure that this proof would have been acceptable to whoever did the original proof(s) of XIV, **. Why not? Because when I write "it must be" I am thinking of V,9: "... things to which the same thing has the same ratio are equal", while on the other hand the extant early proofs of XIV, ** all use what I have called the Ratio Lemma. It may however be that it was felt necessary to use the Ratio Lemma because the definition of d.e.m.r. does not involve ratios as such but rather proportions. For a discussion of all the historical references and problems involved with the concepts of ratio and proportion, see [Fowler] and his various articles on this topic.

[58] See [Clagett a, 28].

Latin manuscripts of the *Elements* that are apparently based, at least to some extent, on Greek versions [59].

Before discussing these specific manuscripts in detail I wish to point out a distinguishing vocabulary feature which while, as we shall see, not being absolutely conclusive, lends weight by its presence to the conclusion that a statement, involving extreme and mean ratio, is based ultimately on a Greek source.

An examination of various Arabic and Latin texts of the *Elements* shows that those Latin texts known to have been translated from a Greek text (for example the manuscript to be discussed in Section 5A [60]) use a variation of the Greek expression "extreme and mean ratio" (*akros kai mesos logos*) whereas Arabic texts and those Latin texts based on the Arabic (*e.g.* those mentioned in Section 4) use a variation of the Arabic expression "proportion having a middle and two ends" (*nisbah ḏāt wasaṭ wa-ṭarafayin*). Now it is extremely unlikely that anybody translating from Arabic to Latin without a Greek text at hand would translate "middle and two ends" as "extreme and mean ratio", but it certainly is possible that someone having seen the Greek term in say *Elements* VI,30 would then proceed to 'correct' a Latin translation of XIV, **. An example of a 'correction' is given in Fibonacci's statement of XIV, ** itself in his reworking of problem 17 of Abū Kāmil's *On the Pentagon and Decagon* [61]. This example thus shows that the appearance of the expression 'extreme and mean ratio' cannot by itself be taken as a sure sign of Greek origin.

[59] While making the final revisions for this article, I came across a reference to [Busard b], *The Latin Translation of Euclid's* Elements *Made Directly from the Greek*. Given the title and Dr Busard's mastery of the subject, this book undoubtedly contains much of interest for this study, but unfortunately I was unable to obtain a copy before completing the revisions. I also note that Arrighi [b] has claimed that Bibliothèque Nationale (Paris), Ms. Lat. 16648 was translated from the Greek. Since this manuscript contains Book XIV it would be of potential interest (I can not tell from Arrighi's article if XIV, ** appears). However the appearance of the Arabic word *alkaidan* (= base) [p. 208] would tend to argue against this; see [Arrighi a] and the review of it [Busard c] as well as [Clagett b, 376].

[60] Paris, Bibliothèque Nationale, Lat. 7373, fol. 53r (VI,30). Books I-XIII and XV of this manuscript are a literal translation of a Greek text. Other examples are the later Renaissance versions (*e.g.* Zamberti; see [Euclid-Paris]).

[61] [Fibonacci-Boncompagni, II, 215]. The part of the text involving XIV, ** reads: "Potes etiam lineam ab, cum sit latus exagoni, aliter invenire, videlicet cum dividitur latus exagonicum media et extrema proportione, tunce maior pars eius erit latus decagonicum, ut in EUCLIDE habetur". Problem 17 of Abū Kāmil involves finding the side of a decagon whose area is 100; see *History of D.E.M.R.*, Section 28B,i, and the Appendix to this paper. On Fibonacci's reworking of the problem, see Section 31B,iii. Fibonacci consistently uses the Greek expression throughout his version of Abū Kāmil's text.

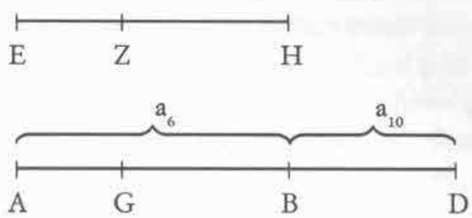
5A. *Paris, Bibliothèque Nationale, Lat. 7373 (12th century)*

This is a very interesting manuscript in general and is of particular interest as far as XIV, ** is concerned. The manuscript, which has been discussed in [Murdoch,a] contains Books I-XIII, XV in a very literal translation from a Greek text and a combined version of Books XIV and XV in place of the missing Book XIV. [62]

Theorem XIV, ** appears in the combined XIV-XV and not only is the proof somewhat different from that of the Arabic version, but, also for a reason that we shall discuss presently, is now to be found as the first theorem of what corresponds to Book XIV.

XIV, ** (Paris, Bibliothèque Nationale, Ms. Lat. 7373) [63]

Statement: Latus exagonici si <pro>portione medii et extremorum dividatur pars maior est latus decagonici.



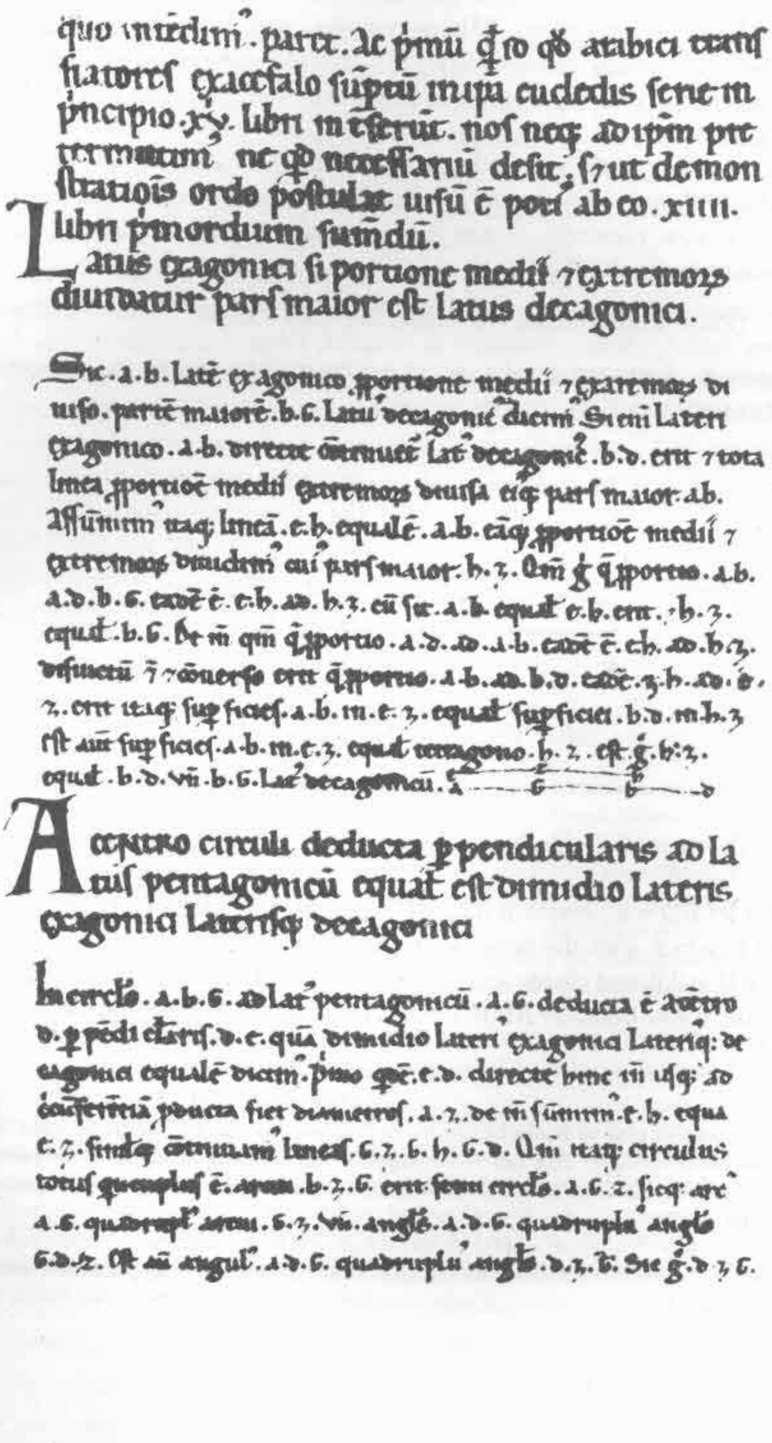
(after Paris, Bib. Nat., Lat. 7373)

Proof:
1. Let AB = a_6 and divide it in e.m.r. with the larger segment being BG. The claim is that BG = a_{10}.
2. If we let BD = a_{10} and join it to AB = a_6 then [by XIII,9] AD will be a line divided in e.m.r. with the larger segment being AB.
3. Let EH = AB and divide EH in e.m.r. with the larger segment being HZ.
4. [By the Ratio Lemma] AB:BG = EH:HZ and since AB = EH we have that HZ = BG.

[62] There is another copy of Books I-XIII, XV in Florence, but the manuscript does not contain the combined XIV-XV. This fact adds weight to Murdoch's opinion, based on vocabulary differences, that the combined XIV-XV was not translated by the same person who translated the main portion.

[63] The text of XIV, ** is given in [Murdoch a, 301, fn. 114]. Dr Curchin and I have checked the diagram and some other points from the manuscript in Paris and from photostats in Ottawa [Euclid-Ms. 7373].

XIV, ** (= XIV,1); Part of Introduction
B.N. (Paris)-Lat. 7373


quo in medium . pareat. Ac primū quidem quod arabici trans
latores exacfalo super un ipsa euclidis fere in
principio .xv. libri inserunt. nos neque ad ipsum pre
termittimus nec quod necessarium desit, sed ut demon
strationis ordo postulat visum est potius ab eo .xiiii.
libri primordium sumendum.
Latus exagonici si portione medii et extremorum
dividatur pars maior est latus decagonici.

A centro circuli deducta perpendicularis ad la
tus pentagonicum equalis est dimidio lateris
exagonici laterisque decagonici

5. [By the Ratio L
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9. Thus [equating
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5. [By the Ratio Lemma] AD:AB = EH:HZ.
6. Therefore by 'separation' and 'inversion' AB:BD = ZH:EZ.

Note: The Latin text reads " ... proportio AD ad AB eadem est EH ad HZ, disiunctum etiam et converso [64] erit que proportio AB ad BD eadem est eadem ZH ad EZ ... ". For the interpretation, see the note to step 6 of Section 3B.

7. Therefore [by VI,16] R(AB, EZ) = R(BD, HZ).
8. Also, [since EH is divided in e.m.r. at E we have R(EH, EZ) = S(HZ) and since EH = AB] R(AB, EZ) = S(HZ).
9. Thus [equating the expressions for R(AB, EZ) in 7 and 8 to obtain R(BD, HZ) = S(HZ)] we have HZ = BD.
10. Thus BC = [HZ = BD =] a_{10}.

When we compare this proof with the Arabic version of Section 2B we notice the following differences:

(i) The argument of step 4, which shows via an implicit use of the Ratio Theorem that the larger segments of the given line AB and its image EH are equal, is missing from the Arabic text.

(ii) The Arabic text is missing the 'inversion' operation of step 6.

(iii) In step 8 the equality EH = AB is used to replace EH (the new line) so as to match up with the term involving AB in line 7. In the Arabic version it is rather the term AB (original line) which is replaced – in step 8 – by the term (HW) which corresponds to EH here.

(iv) This manuscript uses the area terminology of rectangles and squares ("superficies", "tetragono") which is typical of the Greek Euclid. The Arabic manuscript on the other hand speaks of multiplication [65].

To continue our discussion we must leave the proof of XIV, ** and turn to the introduction to the combined XIV-XV for it is this which makes the manuscript so intriguing for us.

After discussing Hypsicles and Apollonius, but in a text which deviates considerably from that found in the critical edition, the introduction ends with the following statement [66]:

[64] We have a problem here for 'converso' could be the linguistic equivalent of the Greek 'anastrophe' in V, Def. 16. Conversion takes the ratio A:B into the ratio A:(A–B), but this does not give the desired end result. But 'converso' could also mean invert, *i.e.* the reference could be to the operation of inversion defined in V, def. 13 and this, as explained in my note to step 6 of 3B, is what is needed from a mathematical viewpoint. I have chosen the mathematically correct interpretation and translated the Latin as 'separation' and 'inversion'.

[65] See however the note of caution in fn. 18, Section 3E.

[66] The Latin text is given in [Murdoch a, 285].

> And first, indeed, what the Arab translators [67] took from Acefalus [*i.e.* Hypsicles] in the Euclidean series itself and inserted at the beginning of Book XV we too do not omit in regard to it [Book XV?; the result in question?] lest anything be lacking; but, as the order of demonstration demands, the beginning [of XV] seems rather to be taken from that [*i.e.* the beginning] of XIV.

Since we have seen that XIV, **, which mathematically belongs in Book XIV, does appear in Book XV in certain Arabic texts and since the result does appear at the beginning of what corresponds to Book XIV in the combined text it seems safe to conclude that the person who wrote the introduction is referring to XIV, **. Further it would appear that this editor had an Arabic text or Latin text known to have been based on an Arabic text in front of him, perhaps along with other texts.

The problem with which we are faced is that of determining whether or not the proof of XIV, ** given in the text comes from a Greek text, either directly or via an Arabic and/or Latin text. There are arguments for and against a Greek prototype.

From a linguistic viewpoint this manuscript uses the Greek term "extreme and mean ratio" and we remarked in (iv) that the operational terminology is 'Greek', but these might just represent 'corrections' as discussed in the introduction to this section. Again, even though Murdoch [p. 282] has suggested on the basis of Graecisms that the combined XIV-XV is of Greek origin, we cannot use this in our discussion of one particular theorem; especially since it does not itself contain any Graecisms. If we turn to the diagram, we again run into difficulties. The letters do indeed follow the Greek alphabet but we do not have a natural Greek order since the Z is the division point and H the endpoint. These letters could possibly have come from an Arabic text based on a Greek version with E coming from h, Z from z, and H being used as the replacement for the ḥ which is the seventh letter in the 'normal Greek-Arabic' ordering (see the Appendix). This of course still does not do away with the possible problem raised by the failure of the letters to follow the natural Greek order.

[67] Murdoch [p. 283], who did not check any of the Arabic texts, believed that it is rather a Latin translation from the Arabic which is being referred to and in particular suggests that the compilor meant to correct an error found in Adelard's translation. I do not think that his arguments are valid. First of all the compilor clearly says "Arab translators" and despite his difficulties with Latin we would expect the text to have said "translators from the Arabic" if this was what was meant. Secondly whereas XIV, ** is indeed found at the beginning of XV in Arabic texts it is not, to my knowledge, found there in any Latin text that clearly delineates XIV and XV. Compare also the "Adelard" statements and texts with the present one.

It is the tantalizingly vague introduction that seems to me to argue most strongly against the editor having a Greek text before him. For if I am correct in asserting above that an Arabic text is being referred to why would the editor explain the changing of the position of XIV, ** by saying " ... Arab translators took ... and inserted ... seems rather to be taken ... " and not say anything about the Greek text that he also had before him.

Even if the editor did not have a Greek manuscript, it is still possible that there was a Latin text which had been translated from the Greek. In view of this possibility and what I have said above it seems to me that there are two possibilities that we should consider, the second of which in turn leads to two further possibilities:

1. The editor had a Latin text based on a Greek version that he compared with an Arabic or Arabic based Latin text. This Greek based Latin text had XIV, ** after the Ratio Lemma and Summary, but did not separate Books XIV and XV, as perhaps indicated by the present text [68], and had XIV, ** with the more complete proof that we see in the manuscript. The editor, who obviously understood the mathematics of the text (witness his " ... as the order of demonstration demands ... ") realized that this was not the mathematically correct position and that the position at the beginning of XV in the Arabic or Arabic based Latin text was even worse. So taking editorial matters in hand he placed XIV, ** in the more mathematically satisfying position at the head of Book XIV.

2. Whether or not the editor had other texts – Greek based or not – only the Arabic or Arabic based text had XIV, **.

Either:

a. XIV, ** appeared with the more complete proof of this manuscript,

or

b. The editor was a very sharp mathematician who not only could provide the

[68] This of course may be due to a copiest, but recall that there is no separation in Adelard III and possibly none in Adelard I. A check of the manuscript (fol. 171^v) shows that no space was left between what appears to be a shortened version of the Summary at the end of the critical edition [Heiberg, vol. 5, 34] and what corresponds to XV,1. Murdoch [p. 285] has mistakenly treated the Summary as part of the Ratio Lemma. The copiest has in fact left a small space on the ninth line from the bottom, presumably to mark the difference between the two. The combined XIV-XV starts on fol. 167^v and is indicated by "explicit XIII – incipit XIV". It ends on the bottom of fol. 172^v with the inscription of the dodecahedron in the icosahedron (XV,5). On the top of 173^r we find explicit XIV, incipit XV. The incipits and explicits for the combined XIV, XV are in the same hand as the incipits and explicits for the rest of the manuscript and since according to Murdoch [302, fn. 121] these are due to a later rubricator, the designations of the books may not provide us with a sure indication of the original intention. Indeed the mention in the introduction to the combined XIV-XV of both Books XIV and XV would indicate that the editor considered the two as being distinct.

details of i and ii above, but would also for some reason change, as discussed in (iii), a perfectly good step into another step, of the same nature; this change being one that cannot be accounted for by "scribal reasons".

In view of the absence of a more complete proof in surviving Arabic manuscripts or Latin manuscripts I would tend to discount possibility 2a. In view of the lack of other mathematical changes or statements in the introduction and the nature of (iii) above, coupled with the seemingly Greek flavor of the vocabulary, I tend to favour possibility 1. I however consider the situation as being far from clear or certain for, as we have seen, many of the manuscripts, although apparently based on the Arabic text of Section 3B, deviate from it in various mathematical details [69].

5B. *Paris, Bibliothèque Nationale, Lat. 10257 (12th century)*

This manuscript had been edited and discussed by Goldat [1957] who concludes, despite numerous Graecisms, that the manuscript does not represent a direct translation from the Greek, but rather a rendition based on various Latin sources including a work containing Greek-Latin geometrical terminology [70].

Theorem XIV, ** appears after the Ratio Lemma and before XV, 1-5. Since no separation of books is indicated, we cannot tell where XIV, ** was in the prototype [71].

XIV, ** (Paris, Bibliothèque Nationale, Ms. Lat. 10257) [72]

Statement: Diviso latere exagoni alicuius circuli secundum proportionem medium et extremam continentem maior pars latus est decagoni eiusdem circuli.

Note that the Greek term "extreme and mean ratio" is used; see the discussion in the introduction to this section.

[69] The referee of this article pointed out that the combined XIV-XV omits material that is in the definitely Greek-Latin part of the manuscript and contains only XV,1-5 as is the case with the Arabic translators. It is the opinion of the referee that the combined XIV-XV follows the Arabic-Latin tradition (presumably without their being a Greek-Latin manuscript available).

[70] [Goldat, 126]. Clagett [b, 376] writes " ... this is the only book known where propositions from Book V onward are, at least in part, translated from Greek before [the fifteenth century]". This manuscript has also been discussed and compared to other manuscripts in [Folkerts].

[71] There is no introduction at the beginning and no Summary at the end. While some of the early books have incipits this is not so for Books XII-XV. The manuscript only contains, except for some theorems in Book I, statements.

[72] [Goldat, 398, < XII >].

5C. *Fibonacci*–Practica geometriae *(1220)*

In his *Practica geometriae* [Fibonacci-Boncompagni, II, 162] Fibonacci lists the theorems of what he labels Book XIV and we find XIV, ** as the first theorem [73]. As a result of a comparison of Fibonacci's text with the manuscript of BN Latin 7373 (discussed in 5A) I am in a position to state that not only is the statement of XIV, ** virtually identical to that of the manuscript, but also that the same is true of all of Fibonacci's statements [74]. Fibonacci however does not include propositions 8 and 9 of the manuscript.

Because of the above we can safely say that Fibonacci did not take XIV, ** from an Arabic text, but rather from either BN 7373 or the prototype of the XIVth book [75].

6. *Conclusions*

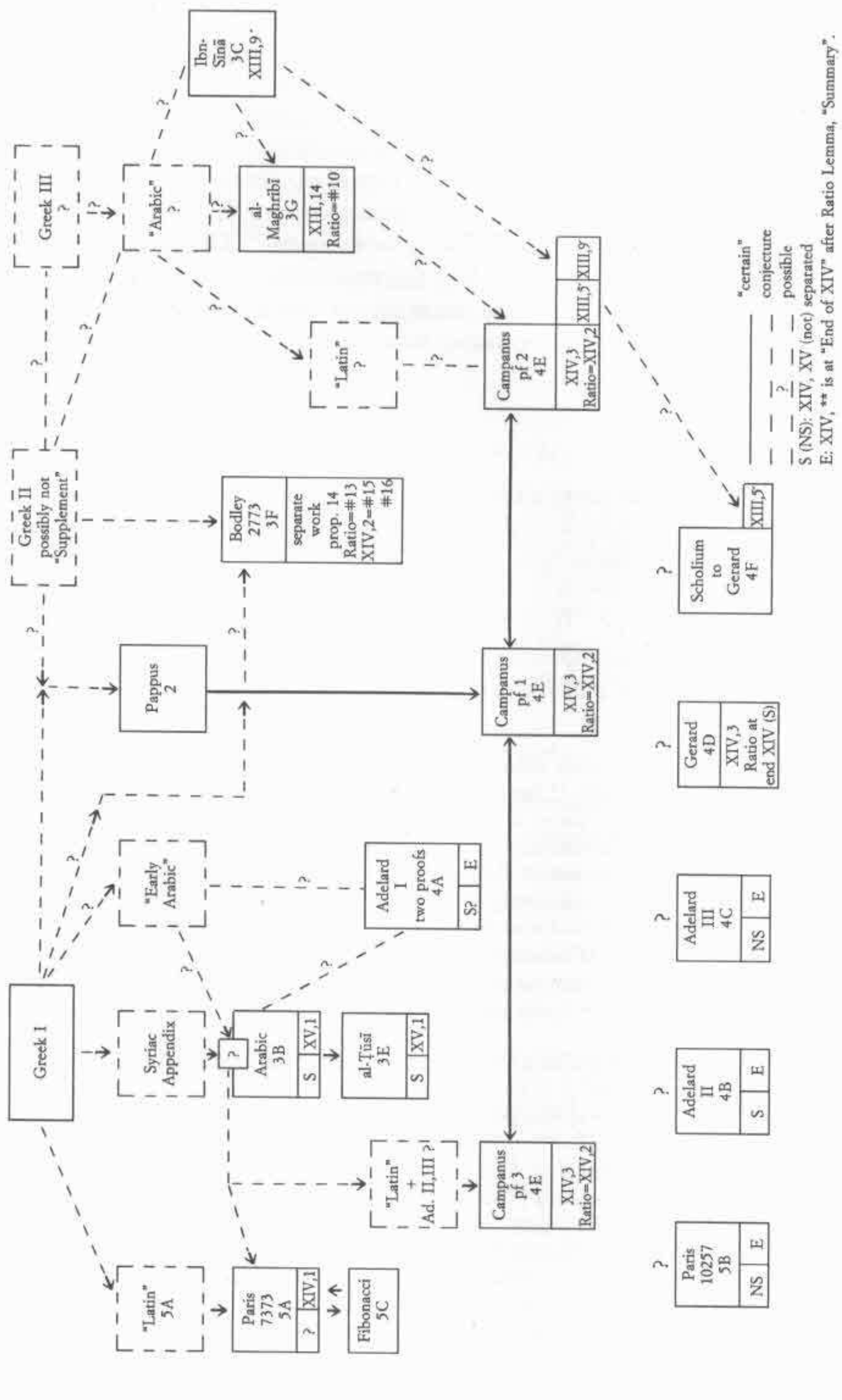
I started this paper off by posing the question of whether or not XIV, ** was originally in a Greek version. However the multitude of proofs and locations in the various manuscripts shows that in reality neither the question nor the answer can be simple or straight-forward. The schema below not only indicates the relationships – fairly certain, conjectural or possible – between versions of XIV, ** but also indicates the location of XIV, ** in the texts and whether or not Books XIV and XV are separated.

[73] I had remarked while doing research for my book that several Arab mathematicians were implicitly using XIV, **, but it was only when I found the statement in Fibonacci that I realized that it must have existed as an explicit theorem at some point. That was the genesis of this article. I only accidentally came across [Murdoch a] much later while sitting in the Bibliothèque Nationale in Paris with the manuscript one floor above me. While checking [Murdoch c, 450] concerning Commandino (see fn. 8, Section 2) I was referred to page 444 and out of curiosity looked at the article. Other pieces of information dribbled in, this being the fifth or so version of the article, so that the finished article in no way reflects how the pieces of the detective story came together. Indeed the original title was to be "A Missing Theorem from ... " but as Greg De Young pointed out it was not missing at all; quite on the contrary as we have now seen!

[74] For XIV, ** Fibonacci has 'exagonicum' and 'decagonicum' instead of 'exagonici' and 'decagonici'. Vogel [p. 611] has suggested that BN 7373 was one of the sources of knowledge of Greek geometry in Fibonacci's time, but he does not specifically suggest that Fibonacci had used this manuscript.

[75] Recall (fn. 62, Section 5A) that one of the manuscripts does not contain the combined XIV, XV. Thus for all we know Fibonacci took his theorems from another Latin source. The referee has pointed out to me additional pieces of evidence that Fibonacci was acquainted with the combined XIV-XV. Further the referee cites two examples – one from *Flos* and the other from *Practica geometriae* – which indicate an acquaintance with VI,14 and VI,20 of the Greek-Latin portion of the manuscript. The referee writes: "I would not be surprised if Fibonacci had something to do with the [combined XIV-XV] ... ".

Genealogy of XIV, **



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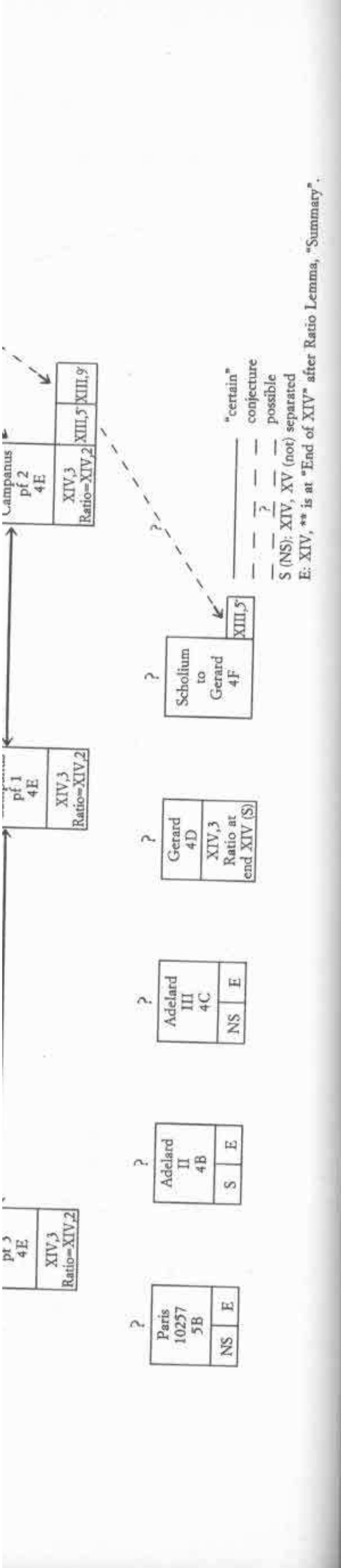


First of all I feel certain that XIV, ** did appear in one of the Greek versions of Book XIV. My principal reason for this belief is based on the appearance of XIV, ** in both the text of Pappus and in the Isḥāq-Ṯābit version.

As far as Pappus is concerned, we have seen that not only does XIV, ** appear as a separate result, but it also is explicitly referred to in both his proofs of XIV,2; namely in the second proof which is identical to that found in the critical edition and in the first, more involved, proof which is presumably older [76].

In the Isḥāq-Ṯābit version on the other hand we find a longer proof of XIV, ** and it is very unlikely that this proof was based on the one found in Pappus. For this would imply that the Arab mathematicians transformed a short elegant proof into a longer one that used the same key result – namely the Ratio Lemma.

Thus we have two ancient sources, Pappus and the Arabic Euclid, both of which have the explicit statement of XIV, ** and apparently independent proofs of this result. It would be a great coincidence indeed if both these sources, and in particular the Arabic mathematicians who added at most five theorems [77] to the entire fifteen books of the *Elements*, felt that XIV, ** was not sufficiently evident from XIII,9 and that the statement and proof of XIV, ** had to be added to the text.

Secondary support, although for the reasons indicated in Section 5A I do not consider the situation as being certain, is given by the Paris 7373. For not only does the proof supply details missing in the Arabic text and display a variation

[76] I cannot imagine anyone who had seen the second proof setting out to give a longer, less elegant proof. Pappus gives another, absolutely marvellous, proof, but starting from a different context, in Book III proposition 58 – see *History of D.E.M.R.*, Section 27A. I also point out that in the Arabic text of Section 3B the proof of XIV,2 does not contain an explicit reference to XIV, **.

[77] Klamroth pointed out that aside from XV,1 (= XIV, **) only VIII,24,25; IX,30,31 are not in the surviving Greek texts – see the table of correspondences in [De Young a, 158]. Based on a remark by Ṯābit to the effect that he had not found IX,30,31 in the Greek manuscripts that he had consulted, Klamroth [p. 279] concludes that these four results were in some Greek manuscript known to Isḥāq. Heiberg is convinced that these were made up by Isḥāq, but just why he is convinced is unclear to me. It may be just that as he states [p. 3], that he cannot see why – and nor can I for that matter – Klamroth is so sure that they were in the manuscript used by Isḥāq. Or it may simply be that these results are not in the preserved Greek texts. I myself find it hard to believe that the Arabic mathematicians would add only a few results. It would appear more likely that they would either not add any new theorems or introduce them throughout the text. I would not however wish to base an argument on such a 'likelihood'. This is also as good a place as any to wonder out loud about certain 'anomalies' in the Arabic text; in particular the missing second batch of theorems in Book XV and the drawing with five lines in Arabic XIV,10 (critical XIV,7) as opposed to three lines in the Greek text. These may also be signs of another Greek tradition.

on the proof, but also the manuscript of Books XIV, XV shows signs of having been based on a Latin text which in turn was based on a Greek version.

Additional, although certainly circumstantial, support is given by the appearance of XIV, ** – although with no proof in the first case and a somewhat different proof in the second – in Paris 10257 of Section 5B and Bodley 2773 of Section 3F, both of which display internal signs of being at least in part of Greek origin.

As indicated in the schema it is possible that there was more than one early Greek version of XIV, **, perhaps corresponding to the other versions of the "Supplement" to which the historical introduction to that work alludes. However, the best candidate for an early Greek version is one related to the long proofs found in the Arabic text of Section 3B and Paris 7373 of Section 5A. Of the two the choice must go to that of Paris 7373 for here we find a detailed, if implicit, use of the Ratio Lemma at step 4 and the necessary 'inversion' operation of step 6 whereas both are missing from the Arabic text.

As far as the Pappus proof is concerned, I suspect, because the main idea involving the use of the Ratio Lemma is the same in both cases, that it is due to Pappus himself who realized that there was no need to work with two lines as in the longer version. It is possible, however, that Pappus based his proof on a proof which was the same or close to that found in Bodley 2773 (see the discussion in Section 3F) which may have come from a second Greek source. This text in fact may not be the "Supplement" to the *Elements* as such, but rather may be related to the Greek work being edited by Langermann and Hogendijk which contains more than just the "Supplement" [78]. It would not be suprising to learn that Pappus had a text of the Supplement different from those we know, for this situation holds for Euclid's *Data*, as is shown by Pappus' description, in Book XII of the *Collection*, of the version that he had before him [79].

A third possible Greek source may lie behind the al-Maghribī and Campanus proofs but it is also possible that some Arabic source developed this simpler proof.

Since Paris 10257 (Section 5B) has no proof and the Greek 'connection' is not

[78] The analysis by Langermann and Hogendijk was not completed as of this writing and I only have the proofs of the Ratio Lemma and XIV, ** in addition to some general comments that the authors kindly provided. Thus I cannot state at this point if all of Book XIV of the critical edition is included in the manuscripts and if the proofs are the same. It is possible that the treatise in question had simply absorbed an older version of the "Supplement".

[79] [Pappus-Hultsch, vol. 2, 639]. For a translation and discussion as well as a reconstruction of the plan of the version known to Pappus, see [Marinus-Michaux, 49, 53]. The existing Greek and Arabic texts (one of which is also an Isḥāq-Ṯābit translation) as well as the Pappus texts are compared in [Thaer]. Both Arabic 39 and 64 are additional proofs that are not in the Greek and some of the Greek propositions are not in the Arabic texts.

sure, nothing can be said about it. The lack of other sources with the same proof and the fact that the proof is in a scholium leads me to suspect that the proof of the scholium to the Gerard text (Section 4D) is a medieval addition. Finally, one would suppose, given the Arabic background of the texts in general, that the Adelard II, Adelard III and Gerard versions come from an Arabic tradition of XIV, ** related to that of Section 3B or some 'early Arabic' (al-Ḥajjāj? Isḥāq ibn Ḥunayn?; see [De Young a,b]) text, but the garbled nature and deviations, and the location as XIV,3 in the case of Gerard, in these texts make me hesitate.

Finally one must ask about the location(s) of XIV, ** in the earliest version(s). If we consider only those sources for which the location presents no difficulties we see that there are three groups:

I. XV,1: Isḥāq-Ṯābit and al-Ṭūsī.

II. At the end of XIV following the Ratio Lemma and Summary: Adelard II.

III. "Elsewhere": Bodley 2773 – proposition 14 (Ratio Lemma = 13; XIV,2 = 15,16); al-Maghribī – XIII,14 (Ratio Lemma = XIII,10); Gerard – XIV,3 (Ratio Lemma at end; lemma for XIV,2 = XIV,2; XIV,2 = XIV,4); Campanus – XIV,3 (Ratio Lemma = XIV,2; XIV,2 = XIV,4,5).

From a strictly mathematical viewpoint one would expect to find the Ratio Lemma and XIV, ** which employs the Ratio Lemma in its proof, located just before the main proof of XIV,2 which uses both of these results. Among the strictly Euclidean texts we only find this situation in Campanus, unfortunately a late witness, who apparently used three older versions. In al-Maghribī the Ratio Lemma and XIV, ** are for some reason in Book XIII and in Bodley 2773 we are dealing with a work that cannot be called Book XIV as such.

If we now consider what we find in the Arabic text of Section 3B there are many possible scenarios, none of which is completely satisfactory in my eyes. We could thus assume that both the Ratio Lemma and XIV, ** were at the end of some Greek manuscript of the first "Supplement", perhaps put there as finishing touches by an early editor, for some reason. Now even though there are two distinct supplements, only the first of which is really attributable in some part to Hypsicles, we know from the Arabic title of Book XV that at some point in time both supplements were attributed to Hypsicles. Thus it would not be surprising to learn that in fact both supplements were once considered to be one work attributable to Hypsicles and with no internal separation. Then later on they were separated again with the new incipit attributing Book XV to Hypsicles being placed before XIV, ** which thus became XV,1. Presumably this separation took place before Qusṭā ibn Lūqā 's name was associated with Book XIV. [80] Indeed it may have even taken place when the manuscript(s) was (were) still in Greek form.

[80] See on fn. 88 of the Appendix.

It may be that Adelard II is a witness to an Arabic tradition in which XIV, ** was still in Book XIV, but on the other hand the editor may have realised that Book XV was not the correct placement and simply moved it to the end of XIV. Indeed the fact that Gerard has the Ratio Lemma at the end of Book XIV, but XIV, ** as XIV,3 may be due to his moving XIV, ** from its place as XV,1 to a more mathematically suitable position.

One thing that really puzzles me though is how XIV, ** arrived after the Summary. We would expect the Ratio Lemma and XIV, ** to follow one another, both being after the Summary if they are late additions or both before the Summary if they were part of an 'original' version. Perhaps a copiest accidentally omitted XIV, ** and then simply stuck it on at the end thus adding one more turn to that Gordian knot that I have called XIV, ** and which is, I am afraid, not yet unraveled.

Appendix

The lettering of the diagrams in Books XIV and XV of the Isḥāq ibn Ḥunayn-Ṯābit ibn Qurra version. The question of sources and editorship

This Appendix examines the possible significance of certain distinguishing features of the diagrams of Books XIV and XV of the manuscript of Section 3B; in particular I will discuss the question of who edited these books.

Our first task is to ascertain what the 'normal' usage is in the Arabic version of the *Elements*. This question has been studied by Klamroth [p. 288] who gave the following correspondence between the letters that appear on the diagrams in the Greek Euclid and those of the Arabic Euclid:

Α	Β	Γ	Δ	Ε	Ζ	Η	Θ	Κ	Λ	Μ	Ν	Ξ	Ο	Π	Ρ	Σ	Τ	Υ	Φ	Χ	Ψ
'	b	j	d	h	z	ḥ	ṭ	k	l	m	n	s	'	f	q	r	t	ṯ	ḫ	ḏ	ḍ

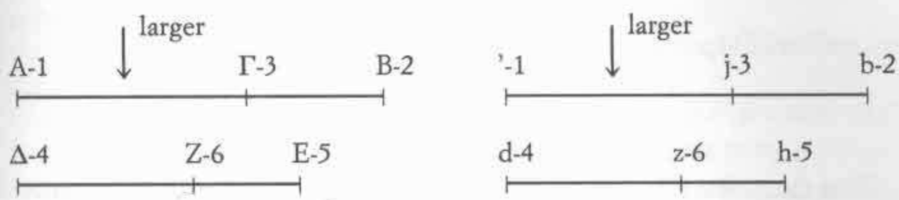
'normal Greek-Arabic' order

I will refer to this ordering of the 22 Arabic letters as the 'normal Greek-Arabic' ordering of the letters even though this correspondence does not always hold in the first thirteen books [81].

[81] For example in VI,30. Klamroth too used Thurston 11 (his 0 – formerly number 279) and also Copenhagen LXXXI. For a discussion of other literature dealing with the correspondence between Greek and Arabic lettering on diagrams, see fn. 108.

Now referring back to the diagram of XIV, ** we notice the appearance of the w (waw) which is missing from the above correspondence and which according to Klamroth is preferably avoided (with the apparent sense from the sentence being that it is never used) [82].

The other aspect of the diagram which does not correspond to what we might expect if the Arabic text were a direct translation from a Greek text is the order in which the letters are used. Consider first the Ratio Lemma. If we compare the Greek and Arabic texts [83] we find the following situation:



Greek [Heiberg, vol. 5, 33] Arabic [Thurston 11]

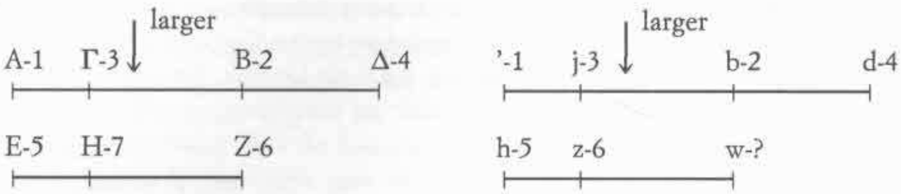
Ratio Lemma

Thus we find the letter pairs 1,2 and 4,5 used to represent the end points of the segments with the succeeding letters 3 and 6 used to represent the division points. Furthermore in both cases the segments 1-3 and 4-6 are used to denote the larger segments. Now it is not true, as the diagram of V,18 of [Euclid-Heiberg] shows, that the analogous situation always holds, but for lack of an actual model let us compare a hypothetical Greek diagram of XIV, **, based on

[82] Although my translator Dr Idris and I did not check all of Books I-XIII, we did not find a single waw in those theorems we did examine. Of the three 'vowel letters' h, w, y, which are found at the end of the Arabic alphabet, only the h is found in the table of correspondence. The y, which can represent the long vowel i, is missing – although Klamroth mentions one occurrence – just as its Greek homologue I is missing from the Greek text. We will see later on that even when the w is present in the texts discussed the y is omitted.

[83] In Arabic the z is distinguished from the r by the addition of a dot on top. In Thurston 11 it is not clear whether the letter is meant to have a dot and indeed in some of the diagrams of Books XIV and XV there is no dot. In Akademia Nauk C 2145, however, the dot is clearly indicated in XV,1. Furthermore the Ratio Lemma in Thurston 11 definitely has a z, and since the r is the seventeenth letter in the normal Greek-Arabic order and would seem to be out of place I presume that z and not r is always meant. The table in [Diringer, vol. 2, plate 15.21] indicates that in early Naski at least, the dot did not appear on the z. See also fn. 103.

the pattern that we have just seen in connection with the Ratio Lemma, with what we find in the Arabic text:



Greek [hypothetical] Arabic [Thurston 11]

XIV, **

Thus the Arabic text does not seem to follow the normal Greek-Arabic order for the z is not at the endpoint of the lower line, as it would be if it followed the hypothetical Greek model, but rather is used to indicate the midpoint. It is the waw – which is missing from the normal Greek-Arabic order – that has been used to indicate the right endpoint.

Of course we might dismiss all the above as being an aberration but the reappearance of the waw in six of the theorems of XIV and all five theorems of XV, together with a certain consistency in ordering, which I will discuss later on in this Appendix, obliges one to ask if there was not some definite system behind these appearances. The occurrence of the waw and the ordering of the letters on the diagram itself immediately suggests that we possibly are dealing with an ordering based, at least in part, on the old Northwest Semitic alphabet [84]. Since we are dealing with a text written in Arabic, this possibility in turn suggests three hypotheses. These three hypotheses all correspond to the following order of the Arabic alphabet:

' b j d h [w] z ḥ ṭ [y] k l m n s ʿ f [ṣ] q r [š] t ...

'abjad/Syriac' ordering of the Arabic alphabet

[84] For the filiation between the Northwest Semitic alphabet (22 letters) and the Arabic alphabet, see [Diringer vol. 1, chap. 9; vol. 2, plates 15.1,15.21]. According to [Gaudefroy-Demombynes, Blachère, 18] " ... the present order [of the Arabic alphabet] seems to have been adopted for pedagogical reasons". I presume that it is because of their special nature that the 'vowel letters' h, w, y were put at the end of the Arabic alphabet.

For convenience I will simply refer to this ordering, in cases where I am not specifically talking about one of the hypotheses, as 'abjad/Syriac' order with the understanding that the actual number of letters (28 or 22) involved depends on which one of the hypotheses we are dealing with.

If we compare this sequence with 'normal Greek-Arabic' we see that the latter is missing the w, y, ṣ, š and has four of the 'new' letters tacked on at the end. In particular for short sequences only the w or w and y distinguish the two lists [85].

The three hypotheses, which I shall first list and then discuss are:

(i) An Arabic transcription of Syriac; in this case only the 22 letters shown form the list.

(ii) An Arabic transcription of Hebrew; in this case only the 22 letters shown form the list.

(iii) The abjad ordering of the Arabic alphabet; in this case the list continues as indicated by the dots (the six other letters are not of interest to us).

Hypothesis i. Transcription of a Syriac text. Specifically this hypothesis states that the lettering that we find in the Arabic text of XIV, **, and those of some theorems from XIV and XV that will be listed further on, is due to the fact that the Arabic text that we have is a translation of a Syriac version of one or more Greek texts.

Unfortunately almost nothing is known about Syriac intermediaries for Greek mathematical texts [86] and the only Euclidean manuscript discussed in the litera-

[85] Klamroth [p. 288] makes no mention of the relationship between the orders. He explains the omission of the y by the fact that I is not used in the Greek diagrams and he seems to be saying that the w is not used because there is no corresponding sound in Greek. He points out that the f (for π) is followed by the non-Greek q, presumably suggesting that the next letter P should be transliterated with its equivalent r. But we see that the r does show up one letter later in its correct place according to the abjad order. I do not know of a reason why the ṣ and š were omitted. It certainly is not to avoid confusion because the omission of ṣ causes the f and q, which only differ by a dot in the Arabic, to be juxtaposed.

[86] For general discussions of Syriac language, Christianity and the transmission of Greek knowledge, see [McCullough, O'Leary]. Sezgin [86, chapter II, c] also discusses the question of transmission. For a discussion of translations from the Greek via Syriac done by Ḥunayn ibn Isḥāq and his school see [Shehaby, Anawati, Iskandar, Diophantus-Rashed, xxiv, fn. 44 (on the search for manuscripts)]. Duval [p. 9] is of the opinion that most Arabic translations were made via a Syriac intermediary; Sezgin [p. 211] is not willing to go as far as Duval. I note that translation via an intermediary language occurred in the transmission of Arabic knowledge into Latin Europe; see [Glick, 257; Haskins, 18], who state that the usual modus operandi in twelfth- and thirteenth-century Spain was for two scholars to work in tandem, one translating aloud from the Arabic to the vernacular and the other from the vernacular into Latin. Duval [p. 283] only mentions Bar Hebraeus (13th century) in his discussion of mathematics by Christian-Syriac writers. I have examined the mathematical section of Bar Hebraeus' *Quadrivium* [p. 57],

ture has been the subject of much controversy. The manuscript in question is a fragment of the *Elements* and contains I,1-23, 37-40. It was published by Furlani who argues [p. 230] that this text is simply a Syriac paraphrase of an al-Ḥajjāj text. The Syriac contains Greek technical terms but Furlani dismisses this as not being an uncommon occurrence even among translators who did not know Greek. Baudoux [p. 74] on the other hand states that these Greek words are rare in Syriac and missing in the Arabic. She argues that if the translator into Syriac had had an Arabic translation in front of him, he would have used the Syriac word corresponding to the Arabic word and not a Greek word. Thus, claims Baudoux, the Syriac version is taken directly from the Greek.

Busard [Adelard I-Busard, 19; Gerard of Cremona-Busard, xvii] agrees with Furlani that the presence of Greek technical words does not prove a Greek origin of the manuscript, but feels that with our present lack of knowledge no conclusion can be reached.

Thaer [a, 117] does not mention the above fragment but also comes to the conclusion that there was a Syriac version (although he does not say so explicitly, I presume that Thaer meant that it predated the Arabic texts). He bases this on the al-Ṭūsī text of VI,12 which is not listed as a separate proposition, but rather as a porism to VI,11. Following the proof al-Ṭūsī writes: "Ṯābit has this porism as a separate proposition but it is not a separate proposition in either the Greek or Syriac" [87]. Sezgin [86,72,95,211], basing his conclusion in part on Thaer's remark, also concludes that there was a Syriac intermediary.

Returning to the Syriac text itself, it is unfortunate for our discussion that the fragment does not contain Books XIV or XV. I have checked the lettering on the diagrams and typically the ordering goes', b, g, d, h, z and in I,22 the series continues ḥ, ṭ, k in agreement with either a Syriac transcription of a 'normal Greek-Arabic' order or what would presumably be 'normal Greek-Syriac' order. The only departure from the correspondences occurs in I,2 where the Greek H is transcribed as h not ḥ and two of the letters are missing.

Although there is a shortage of Syriac material with which to work we have further information which adds weight to any argument in favour of a Syriac version containing XIV, **. This involves two mathematicians who appear to have been connected with Books XIV and XV.

but this only involves very low level geometry and the diagrams are not labelled beyond a, b, g, d. I shall discuss other mathematical writings in Syriac, both translations and original work, presently.

87 The remark about the existence of a Syriac Euclid is only secondary for Thaer and confined to a footnote. The subject of his discussion is Klamroth's remark [p. 277] that VI,12 (mistakenly written VII,12) was not in al-Ṭūsī 's version. Thaer points out that it is indeed in the text; al-Ṭūsī is justifying his reason for joining VI,12 to VI,11.

The first is the mathematician whose name appears in the manuscript of Book XIV as the translator, namely Qusṭā ibn Lūqā who lived sometime in the period 825 to 913.[88] Gabrieli [p. 361] has collected various biographical notices concerning Qusṭā ibn Lūqā from early Arabic sources, and from these it appears that Qusṭā ibn Lūqā was a native of Heliopolis (Baalbek), of Greek origin and a Christian. Furthermore he knew Greek, Arabic and Syriac, travelled to the Byzantine empire and brought back many Greek works. In Baghdad he both translated, and had translated, various works into Arabic and also corrected manuscripts.

Given this information we would not be surprised if indeed Qusṭā ibn Lūqā 's announced relationship to Book XIV is due to his having brought back a Greek manuscript. The appearance of XIV, ** in Qusṭā ibn Lūqā 's Greek manuscript, but not in one of the manuscripts known to us would certainly not be an unheard of occurrence; in fact Books IV-VII of Diophantus' *Arithmetics* exist only in the Arabic translation of Qusṭā ibn Lūqā [Diophantus-Sesiano; Diophantus-Rashed; Hogendijk][89]. As far as Syriac intermediaries are con-

[88] Both the incipit and explicit of Book XIV in Thurston 11 ascribe the translation to Qusṭā ibn Lūqā, but absolutely nothing is said about him or any other translator in Book XV although it has its own incipit and explicit. In various modern commentators one finds the statement that Qusṭā ibn Lūqā translated both XIV and XV, but there is never any justification for this claim. The earliest reference that I have come across is [Klamroth, 271] where this is mentioned not only in connection with Thurston 11, the manuscript that I used, but also with Copenhagen LXXXI which I have not seen. In [Steinschneider a, 504] the above is stated in the section on Euclid. However in the section [p. 522] on Qusṭā ibn Lūqā 's mathematical works there is no mention of this. In [Steinschneider b, 172, section 101.3], Qusṭā ibn Lūqā is not even mentioned in connection with Hypsicles and Books XIV, XV, even though he is mentioned in connection with *On The Ascension of Stars*. Steinschneider in turn is used as a source by [Gabrieli, 354]. Suter [p. 39] relegates the statement concerning XIV, XV to a footnote to the discussion of the translation of Euclid, but again we find no mention of this in the section [p. 40] dealing with Qusṭā ibn Lūqā. Suter in turn is cited by Kapp in the section on Isḥāq ibn Ḥunayn [vol. 23, 58] but he does not say anything in the section [vol. 24, 38] dealing with Qusṭā ibn Lūqā. Both Plooj [p. 5] and Sezgin [p. 96; again there is no mention in the section on p. 144 on Hypsicles] not only state that Books XIV and XV were translated by Qusṭā ibn Lūqā, but also that this translation was improved by Ṯābit ibn Qurra. I do not know how this conclusion was arrived at either, but even if it were arrived at by simply assuming that Ṯābit ibn Qurra's known improvements of Books I-XIII of the Isḥāq ibn Ḥunayn version simply (see fn. 92) carried over to XIV and XV, it is still something to be considered and I will return to this point later on. For lists of known works by Qusṭā ibn Lūqā, see [Sezgin, 285; Gabrieli; Kapp, 38]. I have followed Gabrieli [p. 328] who suggests the dates 220 H. and 300 H. as limits for the dates of birth and death; this problem has been discussed recently by Rashed [Diophantus, vol. 3, xvi].

[89] Of course this is a case where entire books are missing in the Greek. For a recent example of an Arabic text which preserves something not in the surviving Greek manuscripts, see [Wilkie and

cerned, Rashed [Diophantus-Rashed, xxx] states that there was one for Qusṭā ibn Lūqā 's Arabic translation of Archimede's *On the Sphere and Cylinder* [90].

As far as Qusṭā ibn Lūqā 's geometrical works are concerned I have only seen his treatment of the method of double false position. This was edited in [Suter] but the method of transcription is such that I am unable to determine what, if any, system was used in labelling the diagrams. [91]

The second personage of interest is Ṯābit ibn Qurra (836-901) who is best known in the Euclidean context for his revision of the translation of the *Elements* of Isḥāq ibn Ḥunayn [92]. According to the *Chronography* of Bar Hebraeus (13th century) [Bar Hebraeus-Budge, 152], Ṯābit ibn Qurra

> ... was adequately acquainted with three languages – Greek, Syriac and Arabic. He composed in Arabic about one hundred and fifty books on logic; and mathematics and astrology, and medicine. And in Syriac he compiled about sixteen books, the greater number of which we [*i.e.* Bar Hebraeus] have seen and possess ... . [93]

So far we only have generalities, but if we continue on and read the titles of Ṯābit's books written in Syriac in the possession of Bar Hebraeus we come to one of particular interest: " ... A book on the statement 'two straight lines being extended diminishingly from two straight angles, meet together' ... ". Now this

Lloyd] who show that Ḥunayn ibn Isḥāq's Arabic version of Galen's *Ars parva* contains a preface not in Greek manuscripts. For a survey of lost Greek mathematical works in Arabic translation, see [Toomer b].

[90] Rashed does not indicate why he believes that there was a Syriac intermediary; on this translation, see also [Gabrieli, 352; Sezgin, 128].

[91] Figure 1 has, in addition to a, b, g, d along the base, h, o, u (which I presume cannot both represent the waw) z, f, i, c, k, l, m, n, s. This may be an Arabic abjad/Syriac order. Figures 2 and 3 have o, u, w, p, q... . I have also consulted the discussion of the treatise *On the Use of the Celestial Globe* in [Worrell] but there are no diagrams.

[92] Ṯābit ibn Qurra's name appears on the title page of Thurston 11. There is an explicit at the end of Book XIII and Books XIV and XV have their own incipit and explicit. It is not sure however if whoever wrote the title page meant to attribute only the revision of I-XIII to Ṯābit ibn Qurra. This question was already broached in fn. 88 and will be considered again later on in this Appendix. For the text from the *Fihrist* discussing this, see [Murdoch c, 438]. Wiedemann [p. 190] suggests that the correct birthdate is rather 219 H., *i.e.* 834/835.

[93] I found a reference to this quote in [O'Leary, 173] although the section reference there is not correct. I have not found the Bar Hebraeus material mentioned nor these details given in any mathematical discussion of Ṯābit ibn Qurra. O'Leary also states: "Ṯābit had many pupils, one of whom a Christian named Isa ibn Asd translated into Arabic various works which Ṯābit had composed in Syriac". I could not find a reference to this person in the *Chronography*. No mention of him is made in [Sezgin].

of course is a reference to postulate 5 of Book I, the 'parallels postulate', and it turns out that two different 'proofs' of this result, both written in Arabic and attributed to Ṯābit ibn Qurra, are extant [Sabra a] [94]. Further if we look at the 'titles' of the proofs it appears that it is the one that Sabra has labelled Proof I that corresponds to the one mentionned by Bar Hebraeus [95]. In addition, although unable to provide documentary or stylistic evidence, Sabra [p. 17] suspects that Proof I was written before Proof II. If we now consider the lettering on the diagrams of the two proofs we observe the following (with respect to the abjad/Syriac order):

Proof I [96]		Proof II	
I.1	'to l except y	II.1	'to z except w
I.2	only 4 letters	II.2	'to t except w
I.3	only 5 letters	II.3	only 4 letters
I.4	'to m except y	II.4	'to z except w
I.5	'to h	II.5	'to m except w, y
I.6	'to n except y		
I.7	'to s except y		

If we combine the appearance of the waw in the diagram of Proof I with the evidence provided by Bar Hebraeus that this proof is indeed based on a Syriac original, then it is certainly not unreasonable to suppose that the presence of the waw in the Arabic text is precisely due to the presence of a waw in the original Syriac. As to the absence of the ya in Proof I or the lack of a waw in proof II one can only speculate.

[94] Sabra [a, 17] cites al-Qifṭī for evidence that Ṯābit ibn Qurra actually composed two treatises on this subject.

[95] Sabra [a, 19] translates the Arabic text as: "The Treatise of Ṯābit ibn Qurra on (the fact) that if Two Lines are Drawn at Less Than Two Right Angles, They Meet". The 'title' of the second proof reads [Sabra a, 28]: "The Treatise of Ṯābit on the Fact that if a Straight Line Falling on Two Straight Lines Makes the Two Angles on One Side Less than Two Right Angles, then the Two Lines, if Produced on that Side, Meet".

[96] Sabra uses E for h and h for ḥ; compare the Arabic and English diagrams in [Sabra b, 9,19].

In addition to the 'parallels postulate' proofs just discussed, several other Arabic mathematical texts attributed to T̲ābit ibn Qurra have been published. The letters mentioned below are with respect to the abjad/Syriac order.

a. *Geometric Proofs for the Correctness of Algebraic Procedures* [T̲ābit ibn Qurra-Luckey]

In all three diagrams we find the first six letters ' to w.

b. *On the Inscription of a [semiregular] Polyhedron with Fourteen Faces* [T̲ābit ibn Qurra-Bessel-Hagen, Spies]

There is one diagram and we find the ten letters ' to k with the y missing.

c. *A Generalization of the Pythagorean Theorem* [T̲ābit ibn Qurra-Sayili; Sayili]

Figure 2: a to ṭ.

Figure 3: a to k except y.

d. *On the Qarastun* [T̲ābit ibn Qurra-Jaouiche; T̲ābit ibn Qurra-Wiedemann; Clagett c]

The arabic text of this work has been published by Jaouiche using one (probably 13th or 14th century) of the three manuscripts available to Wiedemann (who only gives a transcription of the letters). When we examine the letters on the diagrams we find the following [97]:

Proposition 2: a, b, j, d, h, r (z?).

Proposition 3: a through k except y.

Proposition 4, first figure: a through t except y and r.

Proposition 4, second figure: a through ṯ except y.

Proposition 5: a through k except y.

From this list we see that abjad/Syriac order is followed in the last three diagrams, but not in the second.

In one of the manuscripts (now lost) published by Wiedemann, the main part of the text, which ends with a statement attributing it to T̲ābit, is followed by an addition and from the lettering of the diagrams [Wiedemann, figs 7, 8, 9] the waw is not used in this addition.

There has been a discussion in the literature concerning the ultimate origin of this text. Clagett [c, 31], who has published a Latin version based on an Arabic text, says that the text is ultimately based on a Greek original. Jaouiche [14, 31] on the other hand claims that there never was a Greek text and that the text is due to T̲ābit ibn Qurra himself.

[97] In propositions 2, 3 and the first figure of proposition 4 ' is written instead of d and in proposition 3 d̲ is written instead of z. I have corrected these according to the text.

e. *On the Measurement of Parabolic Bodies* [Ṯābit ibn Qurra-Suter]
I am unable to tell from the transcription what the original Arabic letters were [98].

I will return to my discussion of Qusṭā ibn Lūqā and Ṯābit ibn Qurra after considering the two other hypotheses.

Hypothesis ii. Transcription of a Hebrew text. The earliest translation mentioned by Steinschneider [a, 504] dates to the thirteenth century although there may have been earlier ones in existence. It would thus seem that we can eliminate this hypothesis [99].

Hypothesis iii. Abjad ordering of the Arabic alphabet. The abjad order was formerly employed to express numbers and indeed we find it so used in Books I-XIII as well as in Books XIV and XV. [100] However as far as I have been able to determine this is the only known occurrence, aside from serving as a mnemonic device. Nevertheless one cannot dismiss this possibility out of hand particularly in view of what we find in the geometrical text *On the Pentagon and Decagon* by Abū Kāmil (*ca* 850-*ca* 930), who was a contemporary of both Qusṭā ibn Lūqā and Ṯābit ibn Qurra [101]. As far as is known this is an original text [102] and Abū Kāmil did not know Syriac.

When we examine these problems among 1-16 which have diagrams with a sufficient number of points marked to be of interest we find that most of the time ', b, d, h are used. The rest of the letters are chosen from z (? [103]), ḥ, ṭ, k, l, m, s, ', ṣ.

[98] The problems here are the same as those discussed in fn. 91.

[99] Gandz [1932] had argued that a Hebrew geometrical work, the *Mishnat ha-Middot*, dates to the 2nd century. Sarfatti [1968, 58] has rejected this claim on the basis of a study of the terminology used which he says shows an Arabic influence. On the role of Arabic speaking Jews in the transmission of science to Spain, see [Glick, 258; Hawkins, 17].

[100] See [Weil, Wright, 28]. In this system the letters ' to y express the numbers 1 to 10 then 11 ... 19 are expressed as 10 + 1, ... , *i.e.* 'y, ... ; k = 20, l = 30 (see *e.g.* VI,30) etc. The equivalent of this system is still used in printed versions of the Hebrew Bible except that for theological reasons 15 and 16 are expressed as wṭ and zṭ (*i.e.* 9 + 6 and 9 + 7) instead of hy (which would begin to form the divine name) and wy. To see that this is not only numerically confusing, but probably theologically incorrect see *Mishnah*, "Yadaim", 4, 6; [Mishnah-Danby, 784].

[101] Edited in [Abū Kāmil-Yadegari, Levy]. For a discussion and some corrections, see Herz-Fischler, *Mathematical History of D.E.M.R.*, Section 28B.

[102] In the introduction Abū Kāmil says "We describe ... . We determine ... . We find ... " which may just be rhetorical, but he also says, "Other subjects are touched upon in this book most of the discovery of which was difficult for mathematicians of our time. Thus I hope that Allah will ease the path for me to achieve what was not possible for others".

[103] The printed text shows an r in problems 3, 13, 15, 20. A check of the manuscript shows that there is indeed no dot on the top of the letter. However whereas nothing can be deduced from

Thus we find the ṣ but neither the w or y of the abjad order nor the n and f which appear in both the normal Greek-Arabic and abjad orderings. When we arrive at problem 17 we do indeed find a waw used with the sequence continuing z, ḥ, ṭ, l, m, s; *i.e.*, letters 1 through 15 of the abjad order except y, k, n. It also turns out that problem 17 is very special. For not only is this the first problem that uses d.e.m.r., something which of course appears strange in a text on the pentagon and decagon, but it also shows some confusion at a critical point. For after a perfectly valid development, obtained without the use of d.e.m.r., Abū Kāmil starts another, but garbled, derivation which implicitly uses XIV, **, explicitly mentions Euclid and may just possibly be trying to emulate lines 5 and 6 of the proof of XIV, ** given in Section 3B. Because of this confusion, the mention of Euclid and the period of composition, we cannot be sure that this particular problem did not have its lettering influenced by the very result that interests us [104].

Problems 18 and 19, which also involve d.e.m.r. but nothing from Book XIV, do not have a waw. But then the last problem number 20, once more has a waw with the letters going ', b, j, d, h, w, z, ḥ, ṭ, k, m, *i.e.*, the first thirteen letters of the abjad order except for the y and l [105].

Thus the evidence of Abū Kāmil would seem to suggest the possibility that some sort of partial abjad order was sometimes in use at the time of Qusṭā ibn Lūqā and Ṯābit ibn Qurra for the purpose of labelling diagrams and that the waw was indeed sometimes used.

Having considered three hypotheses related to the assumption that the lettering of the diagram in XIV, ** was related to a system derived from the old Northwest Semitic order, let us now consider the other theorems found in Books XIV and XV:

problems 3, 13, 15, problem 20 displays the letter in question between the w and the ḥ, *i.e.* in exactly the correct position for the z in a sequence of the first nine letters of the abjad order. I suspect therefore that we should read z for r in all four problems. A dot does appear in problem 10; see also fn. 83 above.

104 If I wished to speculate I would boldly suggest in the text that the combination of introduction, the double solution in problem 17, the delay in use of d.e.m.r. until the last four problems, the separation of the text by problems 12-15, which are of a different nature, all perhaps hint at a possibly Greek source for part of the text. However the evidence is very circumstantial and I note that Hero considers, using different methods, the opposite problem of finding the area given the side; see *Mathematical History of D.E.M.R.*, Section 25. I thus bury the mention of the possibility of a Greek source here in the footnotes where only the intrepids dare to (t)read.

105 The printed text omits the diagram of problem 20 which appears in the manuscript. The diagram of footnote 52 has its r (= z) placed incorrectly between the k and '; see fn. 103. The corrected diagram is given in figure viii-5 of *Mathematical History of D.E.M.R.* (p. 127).

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r 20, once more has a waw
the first thirteen letters of

suggest the possibility that
e at the time of Qusṭā ibn
ing diagrams and that the

assumption that the letter-
tem derived from the old
other theorems found in

between the w and the ḥ, *i.e.* in
nine letters of the abjad order. I
s. A dot does appear in problem

he combination of introduction,
until the last four problems, the
ent nature, all perhaps hint at a
nce is very circumstantial and I
ite problem of finding the area
. I thus bury the mention of the
y the intrepids dare to (t)read.
ppears in the manuscript. The
en the k and ’; see fn. 103. The
tory of D.E.M.R. (p. 127).

Book XIV

Arabic	Heiberg	normal Greek-Arabic	waw	letters, comments (abjad/Syriac order)
1	1	yes		
2	2, lemma	yes		
3	2		yes	’to n except ḥ
4	3	yes		
5	4	?		only 5 letters
6	5	–		no diagram
7	6 pf. 1		yes	’to ṭ except ḥ
8 [106]	6 pf. 2, lemma		yes	’to l except ḥ, y
9	6 pf. 2		yes	’, b, j, d, h, w, z, ṭ, s, f
10	7		yes	’to l except h, y, k (’corresponds to the circle)
11	8	–		no diagram
12	ratio lemma	yes		

On the other hand in Book XV there is no mixture. The first theorem is XIV, ** which, as stated, follows the abjad/Syriac order. Theorems 2-6 correspond to the five 'basic' constructions of Heiberg, vol. 5, 40-49, *i.e.* tetrahedron in cube, octahedron in tetrahedron, octahedron in cube, cube in octahedron, dodecahedron in icosahedron. The five diagrams all follow the abjad/Syriac order, in particular theorems 4 and 5 use the first 19 and 14 letters of the order respectively; this includes the y. For theorem 6 the letter ‘ is used for several of the points.

Thus Book XIV displays in some, but not all, theorems a modified abjad/Syriac order – and Book XV an apparent complete adherence to an abjad/Syriac order.

When I first noticed XIV, ** I naïvely considered the case for a Syriac intermediary almost certain. But the case of Abū Kāmil alone, not to mention

[106] Note that whereas the Greek text [Heiberg, vol. 5, 21] only talks about the area of the rectangle formed by the two lines, without stating precisely what their lengths are, the Arabic text comes right out and tells us: area of a pentagon = (5/6 diagonal) · (3/4 diameter). This result was used for computational purposes by Fibonacci and Piero della Francesca (see *Mathematical History of D.E.M.R.*, Sections 31 B,C).

the anomalies (missing ḥ, y, k) in XIV and other examples that have been brought to my attention, invites greater prudence [107]. It seems to me, though, that what we have seen in XIV and XV together with the 'Syriac connection' of both Qusṭā ibn Lūqā and Ṯābit ibn Qurra makes the assumption of a Syriac predecessor an attractive one, although far from certain. I will therefore proceed with an exploration of possibilities related to this assumption.

The mixture of systems, in XIV in particular where we have a result (theorem 2) in abjad/Syriac order and its lemma in normal Greek-Arabic order, suggests that we are dealing with a revised manuscript. Furthermore we do not have to look very far for a possible revisor, namely Ṯābit ibn Qurra.

The tradition attributes Ṯābit ibn Qurra with the revision [108] of the edition of the *Elements* by Isḥāq ibn Ḥunayn, but this is usually interpreted as only referring to Books I through XIII, presumably because the last two are supposed to be due to Qusṭā ibn Lūqā. But there is no reason why this should

[107] After I had written this Appendix, I came across the discussion of the question of the correspondence between the Greek and Arabic letters in [Diocles-Toomer, 32] which included a reference to [Gandz b]. The latter in turn surveys articles and commentaries by Hultsch, Cantor, Karpinski, Hochheim, Simon, Ruske and the one by Klamroth, as well as giving Gandz's own views. Of particular interest to us are Gandz's observations concerning three texts of the *Elements*. In the al-Ḥajjāj text (only Books I-IV are considered) the waw and ya occur only occasionally. In the al-Ṭūsī version the same is true except, as in the Isḥāq-Ṯābit manuscript, in Books XIV and XV (see Section 3E). In Gerard of Cremona's translation of al-Nayrīzī 's commentary on the *Elements*, the Latin equivalents of the waw and ya are generally absent, but Gandz gives some examples including a complete 22 letter abjad based sequence. A non-Euclidean example is given by the work "On the Measurement of Solids of Revolution" by al-Karābīsī who lived before 950. The abjad order with waw and ya is used throughout. Gandz argues that the Arabic writers followed the Arabic alphabet; in particular that the fact that the w and y are missing should not be attributed to a transcription of the Greek, but rather to the order of the Arabic alphabet. One of his reasons is the appearance of the w and y in Books XIV and XV both of which he attributes, following Klamroth, to Qusṭā ibn Lūqā. Yet says Gandz, Qusṭā ibn Lūqā was under Greek influence with the same being true of al Karābīsī. The appearance of w and y is attributed [p. 97] to these authors " ... evidently being free of grammatical considerations". However as I stated, the statement about Qusṭā ibn Lūqā 's complete editorship of both XIV and XV is far from certain. Furthermore Gandz [p. 91] follows Furlani apropos the Syriac fragment and does not even consider the possibility of a Syriac original. Toomer, citing the Arabic translation of Apollonius' *Conics*, states that in Arabic versions of Greek texts, "Each Greek letter is represented by the Arabic letter which has the same numerical value. There are no general exceptions to this rule, and very few individual exceptions, and there can be no doubt that the translators were fully aware of the correspondence ... ". Toomer's list [p. 33], differs at the end from the list, based on the Arabic *Elements*, given by Klamroth [p. 288].

[108] Kapp [p. 85] speaks about two revisions. This may be an echo of the al-Ḥajjāj story or it may, as De Young [a, 156] has suggested, explain some anomalies in the history of the Arabic transmission of the *Elements*.

automatically be assumed to be true, for the manuscript explicitly describes the last two books as "The XIV (XVth) Book of Euclid by Hypsicles"; nor as indicated earlier does there seem to be any reason why we should attribute the editorship of Book XV to Qusṭā ibn Lūqā [109]. Thus a revision of Books XIV and XV by Ṯābit ibn Qurra is certainly not out of the question especially when we note that there is historical information that links Ṯābit ibn Qurra both with Books XIV and XV and with revisions of the work of Qusṭā ibn Lūqā. Indeed we are informed by Ibn al-Qifṭī that Ṯābit ibn Qurra wrote a commentary on Books XIV and XV and by al Musta'īn that he completed the version of *Spherics* by Theodosius of Bithynia that had been started by Qusṭā ibn Lūqā [110].

If we wished to take into account the evidence given above that Ṯābit ibn Qurra's Arabic manuscripts, including at least one that appears to be a translation from the Syriac, are characterized by the use of a modified abjad/Syriac order then we might wish to identify those theorems in XIV that apparently use a modified abjad/Syriac order and all of XV with the revision of Ṯābit ibn Qurra. There are of course problems with this assumption for on the one hand those theorems in XIV that have a w avoid using h, something which is not true in the Ṯābit ibn Qurra manuscripts discussed above, and on the other hand Book XV follows that abjad/Syriac order completely, including the y which is avoided in the Ṯābit ibn Qurra manuscripts. Furthermore a skeptical reader can legitimately ask why Ṯābit ibn Qurra used abjad/Syriac order in his revisions of Books XIV and XV, but not in revisions of Books I-XIII.

Finally there is the question of the source of Books XIV and XV. As stated in Section 3A we have it on the authority of al-Tūsī that the al-Ḥajjāj version had fifteen books, and Leiden 399.1 says that same is true of the Isḥāq ibn Ḥunayn version [111]. But we also know from the incipit and explicit of the Isḥāq ibn Ḥunayn-Ṯābit ibn Qurra version of Section 3B that Book XIV of that version was translated by Qusṭā ibn Lūqā. Presumably this latter statement implies that Qusṭā ibn Lūqā did not simply revise a former version. So perhaps he had found a manuscript which was so superior to the previous one that he was not simply credited with a revision as was the case with Isḥāq ibn Ḥunayn and Ṯābit ibn Qurra. Since it is Book XIV alone, and not both Books XIV and XV, which are attributed to Qusṭā ibn Lūqā it is possible that there is some relation with the fact that the main portion of the Greek-Latin manuscript discussed in Section 5A only contains Books I-XIII and XV and that another copy does not even

[109] See fn. 88; as I noted there both Plooj and Sezgin state, without giving any reasons, that Ṯābit ibn Qurra also revised Books XIV and XV.

[110] [Kapp, 1935, 65, number 86]; [Gabrieli, 354, number xv].

[111] For the statement from Leiden manuscript see [Euclid-Leiden, 3]. There is no mention of Ṯābit ibn Qurra in the list.

contain a combined XIV-XV. There were evidently several traditions involving Books XIV and XV, separately or combined, but as with XIV, ** itself the matter is far from clear [112].

Acknowledgements. – First of all I wish to thank two fine young scholars who read the Latin and Arabic manuscripts for me: Dr Len Curchin of the University of Waterloo and Dr Ismail Idris of Carleton University and Ain Shams University, Cairo. In the same vein I express my appreciation to all those in the past and present who have done the work of finding and editing the various texts that I have used; it is when doing research of this nature that one really appreciates one's predecessors. I would also like to thank the following scholars for sending me material and answering my inquiries: Drs L. Berggren (Simon Fraser), H. Busard (Venlo), G. De Young (Stonehill), J. Hogendijk (Utrecht), Y. Langermann (Jerusalem), G. Saliba (Columbia), and the referee.

My research was supported by The Social Science and Humanities Research Council of Canada (leave fellowship) and the Faculty of Graduate Studies and Research at Carleton University.

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In several cases I have indicated that a microfilm (mf.) or photostat of a manuscript is at Carleton University or in my possession. In the first case the microfilm may be obtained on inter-library loan; in the second I would be pleased to send photocopies upon request. The practice of placing microfilms in libraries, and stating so, in articles, so that others may easily have access to them would greatly benefit and facilitate research especially in view of the waiting periods (three months minimum at the British Library to + ∞ at the Kara Mustafa Library, Istanbul) one experiences.

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