Solutions of supercritical semilinear non-homogeneous Elliptic problems

Maryam Basiri† Abbas Moameni‡

Abstract

Considering a semilinear elliptic equation
\[
\begin{cases}
-\Delta u + \lambda u = \mu g(x,u) + b(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
in a bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary, we apply a new variational principle introduced in [12, 13] to show the existence of a strong solution, where $g$ can have critical growth. To be more accurate, assuming $G(x,\cdot)$ is the primitive of $g(x,\cdot)$ and $G^*(x,\cdot)$ is the Fenchel dual of $G(x,\cdot)$, we shall find a minimum of the functional $I[\cdot]$ defined by
\[
I[u] = \int_\Omega \mu G^*(x, -\Delta u + \lambda u - b(x)) \, dx - \int_\Omega \mu G(x,u) + b(x) u \, dx,
\]
over a convex set $K$, consisting of bounded functions in an appropriate Sobolev space. The symmetric nature of the functional $I[\cdot]$, provided by existence of a function $G$ and its Fenchel dual $G^*$, alleviate the difficulty and shorten the process of showing the existence of solution for problems with supercritical nonlinearity. It also makes it an ideal choice among the other energy functionals including Euler-Lagrange functional.

1 Introduction

In this paper, we are interested in showing the existence of a solution for the boundary value problem of the form
\[
\begin{cases}
-\Delta u + \lambda u = f(x,u,\mu) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $\mu$ and $\lambda$ are two parameters, and $f : \Omega \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is a Carathéodory function. There have been numerous studies about elliptic equation such as (1), and to prove the existence of solutions for this problem either topological methods or variational tools have been used in various publications. A particular case of (1), which has been investigated extensively, is the following problem
\[
\begin{cases}
-\Delta u + \lambda u = \mu |u|^{p-2}u + b(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where $\mu > 0$ and $b \in L^2(\Omega)$. Working on (2) with $\lambda = 0$ and $\mu = 1$, in [2] the author has proved that for $2 < p < 2^*$ where $2^* = 2n/(n-2)$ for $n \geq 3$ and $2^* = \infty$ for $n = 2$, problem (2) admits infinitely many weak solutions for an open dense set of $b$ in $W^{-1,2}(\Omega)$. In [6] the author used a new version of Krasnoselskii's fixed point theorem for the sum of two operators given in [5] and proved the existence of

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a strong solution for (2) with \( \lambda \) close to zero for any \( p > 2 \) if \( n = 3 \), for \( 1 < p < (2n - 4)/(n - 4) \) if \( n > 4 \), and finally for \( 2 < p \leq 3 \) if \( n = 4 \). In [3] and [16], with different methods, they also proved if \( b \in L^2(\Omega) \) then problem (2) possesses infinite number of distinct solutions for \( 2 < p < p_N \) where \( p_N \) is a constant less than the critical exponent \( 2^* \). In addition, in [7] and [15], they consider the problem (2) in \( \mathbb{R}^n \).

Here, as a consequence of our result, we prove that if \( b \in L^2(\Omega) \), \( \lambda \geq 0 \), and \( \mu > 0 \) is small then problem (2) has a strong solution for any \( p > 2 \) if \( n \leq 4 \), and for \( 2 < p \leq (2n - 4)/(n - 4) \) if \( n > 4 \). The importance of our result is that for this range of \( p \), our problem includes supercritical nonlinearity as well. Furthermore, when \( \partial \Omega \) is smooth enough, the new principle enables us to generalize this result by just increasing the regularity of \( b(x) \). In other words, by setting \( b \in W^{k,2}(\Omega) \), we can show the existence of a strong solution \( u \in W^{k+2,2}(\Omega) \) for \( p > 2 \), if \( n \leq 2(k + 2) \), and for \( 2 < p \leq 2(n - 2(k + 1))/(n - 2(k + 2)) \) if \( 2(k + 2) < n < 2(k + 2) + 4/(k - 1) \). It is worth noting that the smoothness of the solution is also determined in this result.

In this paper, we also consider a more general case of (2). To unveil another aspect of the new variational principle established in [12, 13] and its application, we use it as a main tool to prove there is a solution for the following equation

\[
\begin{cases}
-\Delta u + \lambda u = \mu g(x, u) + b(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(3)

Note that when \( g \) has critical growth, the usual variational methods, such as mountain pass approach, do not apply here, as we do not have the compactness of Sobolev embeddings. However, as we see in this paper the new variational approach will rectify this matter. Indeed, we shall prove the following result.

**Theorem 1.1.** Suppose \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) which \( \partial \Omega \) is of class of \( C^{1,1} \). Consider the problem (3), with \( \lambda \geq 0 \) and \( \mu \) positive. Furthermore, assume \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function which is increasing in \( u \) and satisfies the following growth condition:

\[ |g(x, u)| \leq a|u|^{p-1} + c, \]

(4)

where \( a > 0 \), \( b \in L^d(\Omega) \) for \( d \geq 2 \), and

\[
\begin{cases}
2 < p \leq \frac{(2n - 2d)}{(n - 2d)} & \text{if } n > 2d, \\
2 < p & \text{if } n \leq 2d.
\end{cases}
\]

(5)

Then, problem (3) admits at least one solution provided \( \mu \) is small.

In order to utilize the new principle, we first need to define the energy functional \( I[\cdot] \) for our problem. Let \( V = H^1_0(\Omega) \cap L^p(\Omega) \), and consider a closed convex subset \( K = \{ u \in V; \|u\|_{W^{2,2}(\Omega)} \leq R \} \) of \( V \), where \( R \) is a constant to be determined. Let \( V^* \) be the topological dual of \( V \) and define the linear operator \( A : V \to V^* \) by \( A := -\Delta + \lambda I \).

Set \( f(x, y, \mu) = \mu g(x, y) + b(x) \) and define

\[
F(x, y, \mu) = \int_0^y f(x, s, \mu)ds = \mu G(x, y) + b(x)y,
\]

where \( G : \Omega \times \mathbb{R} \to \mathbb{R} \), and \( G(x, y) = \int_0^y g(x, s)ds \). Now define \( \varphi : V \to \mathbb{R} \) by

\[
\varphi(u) := \int_{\Omega} F(x, u, \mu)dx = \int_{\Omega} \mu G(x, u) + ub(x)dx.
\]

Note that we can rewrite the problem (3) as

\[
Au = D\varphi(u),
\]

2
We endow $X$ (where $D\varphi$ in the next sections. devote this section to a few of these important definitions and results which are going to be used frequently fundamental definitions and results from convex analysis for lower semi-continuous functions. Therefore, we I In order to define the functional $I$ in section 3. Furthermore, in section 4 by adapting the principle in our case, we prove the main result for section 2, next for the convenience of the reader we give a brief exposition of the new variational principle used throughout the paper.

2 Preliminaries and notations

In order to define the functional $I[\cdot]$ and apply the variational method, it is essential to review some fundamental definitions and results from convex analysis for lower semi-continuous functions. Therefore, we devote this section to a few of these important definitions and results which are going to be used frequently in the next sections.

Consider the duality pairing $(X, X^*, \langle \cdot, \cdot \rangle)$, that is two real Banach spaces $X$ and $X^*$, and a bilinear map $(x, x^*) \rightarrow \langle x, x^* \rangle$ into $\mathbb{R}$ which separates points:

$$\forall x \in X, \exists x^* \in X^*: \langle x, x^* \rangle \neq 0,$$

$$\forall x^* \in X^*, \exists x \in X: \langle x, x^* \rangle \neq 0.$$ We endow $X$ and $X^*$ with the corresponding weak topologies, $\sigma(X, X^*)$.

Let $F : X \rightarrow \mathbb{R} \cup \{+\infty\}$. We define its domain to be the set of points $x \in X$ where $F(x)$ is finite:

$$\text{Dom}(F) := \{ x \in X | F(x) < +\infty \}.$$ We shall say that $F$ is proper if it is not identically $+\infty$ (see [8, p.79] for more details).

Definition 2.1 (Weak lower semi-continuity). A function $F : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is called weakly lower semi-continuous if

$$F(x) \leq \liminf_{n \rightarrow \infty} F(x_n),$$

for each $x \in X$ and any sequence $\{x_n\}$ converging to $x$ in the weak topology $\sigma(X, X^*)$. The manuscripts is organized as follows. We first review some preliminary definitions and results in 2. In our case, looking for a minimizer in a small closed ball in $H^1(\Omega)$ instead of the whole space $H^1_0(\Omega)$, and using the boundedness of elements of $K$, provide us with the needed compactness as we can use the compact embeddings for Sobolev spaces. Showing that the critical point of the functional in the small set corresponds to its critical point in $K$. For our problem, the existence of solution for the linear equation can be easily proved by applying $L^p$ estimate for Dirichlet problems. Finally to prove that this solution is indeed in $K$, we make use of the Elliptic regularity result.

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Definition 2.2 (The sub-differential of a convex function). Consider the proper convex function $F : X \to (-\infty, +\infty]$, the sub-differential of $F$ is the mapping $\partial F : X \to X^*$ defined by

$$
\partial F(x) = \left\{ x^* \in X^*; F(x) - F(u) \leq \langle x - u, x^* \rangle, \forall u \in X \right\}.
$$

The elements $x^* \in \partial F(x)$ are called sub-gradients of $F$ at $x$. The set of those $x$ for which $\partial F(x) \neq \emptyset$ is called the domain of $\partial F$ and is denoted by $\text{Dom}(\partial F)$. Clearly, if $F$ is not the constant $+\infty$, $\text{Dom}(\partial F)$ is a subset of $\text{Dom}(F)$. The function $F$ is said to be sub-differentiable at $x$, if $x \in \text{Dom}(\partial F)$ (see [4, p.82] for more details).

The following definitions and results are from [8, 9] where they can be referred for the proofs.

Definition 2.3 (Gâteaux-differentiability). A function $F : A \to \mathbb{R} \cup \{+\infty\}$, finite in a neighborhood of $x$, is Gâteaux-differentiable at $x$ if there exists some $x^* \in X^*$ such that:

$$
\forall y \in X, \lim_{h \to 0} \frac{F(x + hy) - F(x)}{h} = \langle x^*, y \rangle.
$$

We then write $DF(x) = x^*$.

When $F$ is lower semi-continuous, convex and finite on some neighborhood of $x \in X$, it must be continuous at $x$, and hence, sub-differentiable at $x$, but it does not need to be differentiable in any sense. So, sub-differentiability is a strict extension of differentiability, as the following result shows:

Proposition 2.1. If $F : X \to \mathbb{R} \cup \{+\infty\}$ is a convex and lower semi-continuous function and if its sub-differential at $x$ is a singleton, $\partial F(x) = \{x^*\}$, then $F$ is Gâteaux-differentiable at $x$ and $x^* = DF(x)$. If $F : X \to \mathbb{R} \cup \{+\infty\}$ is Gâteaux-differentiable and continuous at $x$, then $\partial F(x) = \{DF(x)\}$ (see Proposition 9, [8, p.92] for more details).

We define Fenchel dual of a function as below:

Definition 2.4. Let $F : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function. The function $F^* : X \to \mathbb{R} \cup \{+\infty\}$ defined by

$$
F^*(x^*) = \sup_{x \in X} \{ \langle x, x^* \rangle - F(x) \}
$$

is called the Fenchel dual or Legendre transform of $F$.

As an easy consequence of Definition (2.4), we have Fenchel’s inequality,

$$
F(x) + F^*(x^*) \geq \langle x, x^* \rangle, \quad \forall x \in X, \forall x^* \in X^*.
$$

Proposition 2.2. If $F$ is a convex and lower semi-continuous function, then $(F^*)^* = F$ and the three properties are equivalent:

1. $x^* \in \partial F(x)$,
2. $x \in \partial F^*(x^*)$,
3. $F(x) + F^*(x^*) = \langle x, x^* \rangle$.

Theorem 2.5. Let $\Omega$ be a bounded open domain in $\mathbb{R}^n$ and $f$ a nonnegative normal integrand of $\Omega \times \mathbb{R}$. Put the spaces $L^\alpha(\Omega)$ with $1 \leq \alpha \leq \infty$, in duality with $L^{\alpha'}(\Omega)$, i.e. $1/\alpha + 1/\alpha' = 1$, and define a functional on $L^\alpha(\Omega)$, by

$$
F(u) = \int_{\Omega} f(x, u(x))dx,
$$

whence its polar $F^*$ on $L^{\alpha'}(\Omega)$ is

$$
F^*(u^*) = \sup_{u \in L_{\alpha}^{\infty}} \left\{ \langle u, u^* \rangle - \int_{\Omega} f(x, u(x))dx \right\}
$$

4
\[ \begin{align*}
&= \sup_{u \in L^m_0} \left\{ \int_{\Omega} \left[ u(x)u^*(x) - f(x, u(x)) \right] dx \right\}.
\end{align*} \]

Now assume that there exists \( u_0 \in L^\infty \) such that \( F(u_0) < \infty \). Then for all \( u^* \in L^\alpha(\Omega) \), we have
\[ F^*(u^*) = \int_{\Omega} f^*(x; u^*(x)) dx, \]
where
\[ f^*(x, s) = \sup_{t \in \mathbb{R}} \{ ts - f(x, t) \}. \]

See proposition 2.1 [9, p.271] for a proof.

### 2.1 Notation

In this short section we give a quick review for the notations used in this paper (see [1] for more details).

- **\( C^m(\Omega) \):** The vector space consisting of all functions \( u \) which together with all their partial derivatives \( D^\alpha u \) of orders \( |\alpha| \leq m \) are continuous on \( \Omega \).
- **\( C^0(\Omega) \equiv C(\Omega) \)**
- **\( C^\infty(\Omega) \equiv \bigcap_{m=0}^\infty C^m(\Omega) \)**
- **\( C^\infty_0(\Omega) \):** The vector space consists of all those functions in \( C^\infty(\Omega) \) that have compact support in \( \Omega \).
- **\( C^m_B(\Omega) \):** The space of functions \( u \in C^m(\Omega) \) for which \( D^\alpha u \) is bounded on \( \Omega \) for \( 0 \leq |\alpha| \leq m \). \( C^m_B(\Omega) \) is a Banach space with norm given by
  \[ \| u \|_{C^m_B(\Omega)} = \max_{0 \leq |\alpha| \leq m} \sup_{x \in \Omega} |D^\alpha u(x)|. \]
- **\( C^m(\bar{\Omega}) \):** The vector space consists of all functions \( u \in C^m(\Omega) \) for which \( D^\alpha u \) is bounded and uniformly continuous on \( \Omega \) for \( 0 \leq |\alpha| \leq m \). \( C^m(\bar{\Omega}) \) is a closed subspace of \( C^m_B(\Omega) \), and therefore also a Banach space with the same norm.
- **\( C^{m,\lambda}(\bar{\Omega}) \):** The vector space consists of functions \( u \) for which, for \( 0 \leq |\alpha| \leq m \), \( D^\alpha u \) satisfies in \( \Omega \) a Hölder condition of exponent \( \lambda \), for \( 0 < \lambda \leq 1 \), that is, there exists a constant \( K \) such that
  \[ |D^\alpha u(x) - D^\alpha u(y)| \leq K|x - y|^\lambda, \quad x, y \in \Omega. \]
  \( C^{m,\lambda}(\bar{\Omega}) \) is a Banach space with norm given by
  \[ \| u \|_{C^{m,\lambda}(\bar{\Omega})} = \| u \|_{C^{m}(\bar{\Omega})} + \max_{0 \leq |\alpha| \leq m} \sum_{x \neq y \in \Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^\lambda}. \]
- **\( L^p(\Omega) \):** The class of all measurable functions \( u \) defined on \( \Omega \) for which
  \[ \int_{\Omega} |u(x)|^p dx < \infty, \]
  and the norm is
  \[ \| u \|_{L^p(\Omega)} = \left( \int_{\Omega} |u(x)|^p dx \right)^{1/p}. \]
- **\( L^\infty(\Omega) \):** The vector space of all functions \( u \) that are essentially bounded on \( \Omega \), and
  \[ \| u \|_{L^\infty(\Omega)} = \text{ess sup}_{x \in \Omega} |u(x)|. \]
- **\( W^{m,p}(\Omega) \):** The space of \( k \) times weakly differentiable functions with their derivatives up to order \( k \) in \( L^p(\Omega) \), that is
  \[ W^{m,p}(\Omega) = \{ u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m \}. \]
- **\( W^{m,p}_0(\Omega) \):** the closure of \( C^\infty_0(\Omega) \) in the space \( W^{m,p}(\Omega) \).
3 A variational principle

Now using the definitions and result from the previous section, we give a short introduction to the variational principle which is adapted from [14].

Let $V$ be a reflexive Banach space, $V^*$ be its topological dual and $K$ be a closed convex subset of $V$. Assume that $\varphi : V \to \mathbb{R}$ is convex, Gâteaux differentiable (with Gâteaux derivative $D\varphi(u)$) and lower semi-continuous and that $A : V \to V^*$ is a positive linear symmetric operator. Let $\varphi^*$ be the Fenchel dual of $\varphi$, i.e.

$$\varphi^*(u^*) = \sup\{\langle u^*, u \rangle - \varphi(u) ; u \in V\}, \quad u^* \in V^*;$$

where the pairing between $V$ and $V^*$ is denoted by $\langle \cdot, \cdot \rangle$. Define the function $\psi_K : V \to (-\infty, +\infty]$ by

$$\psi_K(u) = \begin{cases} \varphi^*(Au), & u \in K \\ +\infty, & u \not\in K, \end{cases}$$

(6)

and set $\text{Dom}(\psi_K) = \{ u \in V ; \psi_K(u) < \infty \}$. Consider the functional $I_K : V \to (-\infty, +\infty]$ defined by

$$I_K[w] := \psi_K(w) - \varphi(w).$$

A point $u \in \text{Dom}(\psi_K)$ is said to be a critical point of $I_K$ if

$$\psi_K(v) - \psi_K(u) \geq \langle D\varphi(u), v - u \rangle, \quad \forall v \in V.$$

We shall now recall the following variational principle established in [13].

**Theorem 3.1.** Let $V$ be a reflexive Banach space and $K$ be a closed convex subset of $V$. Let $\varphi : V \to \mathbb{R}$ be a Gâteaux differentiable convex and lower semi-continuous function, and let the linear operator $A : \text{Dom}(A) \subset V \to V^*$ be symmetric and positive. Assume that $u$ is a critical point of $I_K(w) = \psi_K(w) - \varphi(w)$, and that there exists $v \in K$ satisfying the linear equation,

$$Av = D\varphi(u).$$

Then $u \in K$ is a solution of the equation,

$$Au = D\varphi(u).$$

(7)

In the remainder of this paper, we develop the proof of main result as a consequence of this theorem.

4 Existence results via a new variational principle

This section is devoted to the proof of the main results of this paper. We also discuss the general case of (2) at the end of this section.

Before showing the main result, we state and prove the adapted version of Theorem 3.1 to our case. In order to do that, first we set up $V$, $K$, $\varphi$, and $\psi$ accordingly. Set $V$ to be the Banach space $H^1_0(\Omega) \cap L^p(\Omega)$ with the norm

$$\|u\|_V := \|u\|_{H^1_0(\Omega)} + \|u\|_{L^p(\Omega)}.$$

Consider a closed convex subset of $V$, $K = \{ u \in V ; \|u\|_{W^{1,2}(\Omega)} \leq R \}$, where $R$ is a positive real number to be chosen later. Let $V^*$ be the topological dual of $V$, and denote by $\langle \cdot, \cdot \rangle$ the pairing between $V$ and $V^*$ which is defined by

$$\langle v, v^* \rangle = \int_\Omega v(x)v^*(x)dx, \quad \forall v \in V, \forall v^* \in V^*.$$

For $v \in V$ define the linear operator $A : \text{Dom}(A) \subset V \to V^*$ by $Av := -\Delta v + \lambda v$, with $\lambda \geq 0$. Set $f(x, y, \mu) = \mu g(x, y) + b(x)$, with $\mu > 0$, and let $F(x, y, \mu)$, to be its primitive, i.e.

$$F(x, y, \mu) = \int_0^y f(x, s, \mu)ds = \mu G(x, y) + b(x)y,$$
where \( G : \Omega \times \mathbb{R} \to \mathbb{R} \) is defined by \( G(x, y) = \int_0^y g(x, s)ds \), and \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) satisfies the growth condition 
\[ |g(x, y)| \leq a|y|^{p-1} + c. \]
Now define \( \varphi : V \to \mathbb{R} \) by
\[ \varphi(u) := \int_{\Omega} F(x, u, \mu)dx = \int_{\Omega} \mu G(x, u) + ub(x)dx. \]  
(8)
Note that we can rewrite the problem (3) as
\[ Au = D\varphi(u). \]
Now let \( G^* : \Omega \times \mathbb{R} \to (-\infty, +\infty] \) be the Fenchel dual of \( G \), i.e,
\[ G^*(x, s) = \sup_{t \in \mathbb{R}} \{ ts - G(x, t) \}. \]
As shown in Lemma 4.2, for \( u \in V \), we find the Fenchel dual \( \varphi^* : V^* \to \mathbb{R} \) of \( \varphi \) by
\[ \varphi^*(Au) = \int_{\Omega} \mu G^*(x, \frac{Au - b(x)}{\mu})dx. \]
Again, consider the set \( K = \{ u \in V ; \|u\|_{W^{2,4}(\Omega)} \leq R \} \), and define \( \psi \) as follows
\[ \psi(u) := \varphi^*(\Delta u + \lambda u) = \begin{cases} \int_{\Omega} \mu G^*(x, \frac{-\Delta u + \lambda u - b(x)}{\mu})dx, & u \in K, \\ +\infty & u \notin K, \end{cases} \]
with \( \text{Dom}(\psi) = \{ u \in V ; \psi(u) < \infty \} \). As \( G^* \) is a convex function and \( K \) is a convex set, It can be easily seen that \( \psi \) is also a convex function.
Finally, set \( I[u] = \psi(u) - \varphi(u) \), i.e.
\[ I[u] = \int_{\Omega} \mu G^*(x, \frac{-\Delta u + \lambda u - b(x)}{\mu})dx - \int_{\Omega} \mu G(x, u) + ub(x)dx. \]
(10)
For reader’s convenience, we shall now rephrase and prove Theorem 3.1 for our case as follows.

**Corollary 4.1.** Assume that \( u \) is a local minimum of
\[ I[w] := \psi(w) - \varphi(w), \]  
(11)
where \( \psi \) and \( \varphi \) are given in (9) and (8), respectively. If there exists \( v \in \text{Dom}(\psi) \) satisfying the linear equation,
\[ \begin{cases} -\Delta v + \lambda v = \mu g(x, u) + b(x), & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases} \]
(12)
then \( u \) is a solution of the equation,
\[ \begin{cases} -\Delta u + \lambda u = \mu g(x, u) + b(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \]

**Proof.** First note that \( u \) being a local minimum of \( I \), implies \( u \) is a critical point of \( I \). Indeed, as \( \psi \) is convex, we can conclude that for all \( t > 0 \) and all \( w \in V \), we have that
\[ 0 \leq I[(1-t)u + tw] - I[u] = \psi((1-t)u + tw) - \psi(1-t)u + tw) - \psi(u) + \varphi(u) \]
\[ \leq (1-t)\psi(w) + t\psi(w) - \varphi((1-t)u + tw) - \psi(u) + \varphi(u) \]
\[ = t(\psi(w) - \psi(u)) - \varphi((1-t)u + tw) + \varphi(u). \]
(13)
Now if we divide (13) by \( t \) and take the limit when \( t \) approaches \( 0^+ \), by definition of Gâteaux-derivative we obtain
\[ \psi(w) - \psi(u) \geq \lim_{t \to 0^+} \frac{\varphi(u + t(w - u)) - \varphi(u)}{t} = \langle D\varphi(u), w - u \rangle. \]
we have a>

for Lemma 4.2. Assume that \( b \) is lower semi-continuous. We first prove some property of the functional for \( K \).

In addition, for \( u \in \text{Dom}(\psi) \), obviously we have \( Au = -\Delta u + \lambda u \in L^d(\Omega) \) and therefore \( \psi(u) = \varphi^*(Au) \).

By assumption, \( Av = D\varphi(u) \) for \( v \in \text{Dom}(\psi) \), so we can substitute \( w = v \) in (14) to find that

\[
\varphi^*(Av) - \varphi^*(Au) = \psi(v) - \psi(u) \geq \langle D\varphi(u), v - u \rangle = \langle Av, v - u \rangle.
\]

Furthermore, from Proposition 2.2, \( Av = D\varphi(u) \) yields \( u \in \partial \varphi^*(Av) \), so by definition we get

\[
\varphi^*(Av) - \varphi^*(Au) \geq \langle u, Au - Av \rangle.
\]

Adding (15) to (16) implies that

\[
\langle u, Au - Av \rangle + \langle Av, v - u \rangle \leq 0.
\]

Note that \( A \) is symmetric, so applying the property of a symmetric operator, we get

\[
\langle u - v, Au - Av \rangle \leq 0
\]

which means that

\[
\int_{\Omega} |\nabla u - \nabla v|^2 \, dx + \lambda \int_{\Omega} |u - v|^2 \, dx \leq 0,
\]

which clearly forces \( u = v \). It then follows that \( Au = Av = D\varphi(u) \) as desired.

Based on Corollary 4.1, the proof of Theorem 1.1 can be broken down into two main steps; first to show that the functional \( I[\cdot] \) defined in (11) has a minimizer, and second to prove the linear equation (12) has a solution in \( K \). In the remainder of this section before the proof of main results, we prove some preliminary results needed in either step of the proof.

In the first part of the proof, to show that \( I[\cdot] \) has a critical point, we need to prove that \( I[\cdot] \) is weakly lower semi-continuous. We first prove some property of \( \varphi^*(u) \).

**Lemma 4.2.** Assume that \( \varphi : V \rightarrow \mathbb{R} \) is defined by \( \varphi(u) = \int_{\Omega} F(x, u, \mu) \, dx = \int_{\Omega} \mu G(x, u) + b(x)u \, dx \), where \( b \in W^{2,d}(\Omega) \), \( G(x, u) = \int_0^u g(x, s) \, ds \) and

\[
|g(x, u)| \leq a|u|^{p-1} + c,
\]

for \( a > 0 \), and \( p > 2 \). Let \( \varphi^* : V^* \rightarrow \mathbb{R} \) be the Fenchel dual of \( \varphi \). Then for each \( h \in L^p(\Omega) \) with \( 1/p + 1/p' = 1 \) we have

\[
\varphi^*(h) = \mu \int G^*(x, \frac{h(x) - b(x)}{\mu}) \, dx.
\]

**Proof.** Take \( h \in L^p(\Omega) \). From the definition of Fenchel dual and the density of \( V \) in \( L^p(\Omega) \), we observe that

\[
\varphi^*(h) = \sup_{v \in V} \{ \langle v, h \rangle - \varphi(v) \}
\]

\[
= \sup_{v \in V} \left\{ \int_{\Omega} v(x)h(x) \, dx - \int_{\Omega} \mu G(x, v) \, dx - \int_{\Omega} b(x)v(x) \, dx \right\}
\]

\[
= \sup_{v \in L^p(\Omega)} \left\{ \int_{\Omega} v(x)h(x) \, dx - \int_{\Omega} \mu G(x, v) \, dx - \int_{\Omega} b(x)v(x) \, dx \right\}
\]

\[
= \mu \sup_{v \in L^p(\Omega)} \left\{ \int_{\Omega} v(x) \frac{h(x) - b(x)}{\mu} \, dx - G(x, v) \, dx \right\}.
\]
Since $G(x,0) = 0$ and $\varphi(0) = 0 < \infty$, as a consequence of Theorem 2.5, we obtain

$$\varphi^*(h) = \int_{\Omega} \mu G^*(x, \frac{h(x) - b(x)}{\mu}) dx.$$ 

**Remark 4.3.** In the special case, when $g(x,u) = |u|^{p-2}u$, by similar calculation we find that

$$\varphi^*(h) = \frac{1}{p} \int_{\Omega} \mu \left| \frac{h(x) - b(x)}{\mu} \right|^p dx,$$

for $h \in L^{p'}(\Omega)$.

**Lemma 4.4.** The functional $I : V \to (-\infty, +\infty]$, defined by

$$I[w] = \int_{\Omega} \mu G^*(x, -\Delta w + \lambda w - b(x)) dx - \int_{\Omega} \mu G(x,w) + b(x) w dx,$$  \hspace{1cm} (17)

is weakly lower semi-continuous.

**Proof.** First, we claim that $\psi(\cdot)$ is weakly lower semi-continuous. Let $\{u_n\}$ be a sequence in $V$ that converges weakly to some $u \in V$. If $\liminf_{n \to \infty} \psi(u_n) = \infty$ we have the result, so let us assume that $\liminf_{n \to \infty} \psi(u_n) < \infty$. By definition of $\psi(\cdot)$, we can deduce that $\{u_n\} \subset K$ and hence $\|u_n\|_{W^{2,d}(\Omega)} \leq R$, for all $n$. Then going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ weakly in $W^{2,d}(\Omega)$.

For $v \in L^p(\Omega)$, by definition of $\varphi^*(-\Delta u_n + \lambda u_n)$, we have that

$$\psi(u_n) = \varphi^*(-\Delta u_n + \lambda u_n) \geq \int_{\Omega} v(x)(-\Delta u_n + \lambda u_n) dx - \varphi(v).$$

Let $d'$ be the conjugate of $d$, i.e. $1/d + 1/d' = 1$. So, $d \geq 2$ yields that $d' \leq 2$. Considering the fact that $p > 2 \geq d'$ we can use the embedding $L^{d'}(\Omega) \hookrightarrow L^{d'}(\Omega)$, and obtain that $v \in L^{d'}(\Omega)$. In addition, $-\Delta u_n + \lambda u_n$ is obviously in $L^{d'}(\Omega)$. Hence, from the definition of weak convergence in $W^{2,d}(\Omega)$, we can deduce

$$\liminf_{n \to \infty} \psi(u_n) = \liminf_{n \to \infty} \varphi^*(-\Delta u_n + \lambda u_n) \geq \int_{\Omega} v(x)(-\Delta \bar{u} + \lambda \bar{u}) dx - \varphi(v).$$

Taking sup over all $v \in L^{d'}(\Omega)$ implies that

$$\liminf_{n \to \infty} \psi(u_n) = \liminf_{n \to \infty} \varphi^*(-\Delta u_n + \lambda u_n) \geq \varphi^*(\Delta \bar{u} + \lambda \bar{u}) = \psi(\bar{u}),$$

which means that $\psi(\cdot)$ is weakly lower semi-continuous.

In addition, we prove that $\varphi(u)$ is continuous. Recall that $u_n \rightharpoonup \bar{u}$ in $W^{2,d}(\Omega)$. Moreover, having $\partial \Omega \in C^{1,1}$, which implies the cone condition, we can use the compact embeddings $W^{2,d}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leq q < dn/(n-2d)$ if $2d < n$ and for $1 < q < \infty$ if $n \leq 2d$, to conclude that up to a subsequence, $u_n \rightharpoonup \bar{u}$ strongly in $L^{p}(\Omega)$, for the range of $p$ in (5), and hence $u_n \to u$ a.e.. Therefore, from the continuity of $G(x,\cdot)$, implied by continuity of $g(x,\cdot)$ as a Carathéodory function, we can deduce that $G(x,u_n) \to G(x,\bar{u})$ a.e., and so $F(x,u_n,\mu) = \mu G(x,u_n) + b(x) u_n \to \mu G(x,\bar{u},\mu) + b(x) \bar{u} = F(x,\bar{u},\mu)$, for a.e. Furthermore, considering the growth condition (4) and the fact that $G(x,\cdot)$ is the primitive of $g(x,\cdot)$ we obtain

$$G(x,u_n) \leq \frac{\mu a}{p} |u_n|^p + cu_n,$$

which yields

$$F(x,u_n,\mu) \leq \frac{\mu a}{p} |u_n|^p + (b(x) + \mu c) u_n.$$
It can be easily proved that \( \frac{\mu a}{p} |u_n|^p + (b(x) + \mu c)u_n \in L^1(\Omega) \). Indeed, by applying H"older inequality we find that
\[
\int \frac{\mu a}{p} |u|^p + (b(x) + \mu c)u \, dx \leq \frac{\mu a}{p} \|u\|_{L^p(\Omega)}^p + \|b(x) + \mu c\|_{L^q(\Omega)} \|u\|_{L^q(\Omega)} \\
\leq \frac{\mu a}{p} \|u\|_{L^p(\Omega)}^p + C \|b(x)\|_{L^q(\Omega)} \|u\|_{L^p(\Omega)} < \infty,
\]
for an appropriate constant \( C \). Therefore, by Lebesgue Dominated Converges theorem we find that
\[
\lim_{n \to \infty} \int_\Omega F(x, u_n, \mu) \, dx = \lim_{n \to \infty} \int_\Omega \mu G(x, u_n) + u_n b(x) \, dx \\
= \int_\Omega \lim_{n \to \infty} \mu G(x, u_n) + u_n b(x) \, dx = \int_\Omega \mu G(x, \bar{u}) + \bar{u} b(x) \, dx = \int_\Omega F(x, \bar{u}, \mu) \, dx,
\]
and hence
\[
\varphi(u_n) \to \varphi(\bar{u}),
\]
which means that \( \varphi(\cdot) \) is continuous.

Recall \( u_n \to \bar{u} \). By definition we have
\[
\lim \inf_{n \to \infty} I[u_n] = \lim \inf_{n \to \infty} (\psi(u_n) - \varphi(u_n)),
\]
and since \( \varphi(u) \) is continuous, we obtain
\[
\lim \inf_{n \to \infty} I[u_n] = \lim \inf_{n \to \infty} \psi(u_n) - \lim_{n \to \infty} \varphi(u_n) \geq \psi(\bar{u}) - \varphi(\bar{u}) = I[\bar{u}],
\]
which means that \( I[\cdot] \) is weakly lower semi-continuous.

In the second part of our proof, where we show that the linear equation \( Av = D\varphi(\bar{u}) \) admits a solution \( \bar{v} \in K \), the following result is required.

**Lemma 4.5.** Let \( d \geq 2 \) and consider the linear equation
\[
\begin{cases}
-\Delta v + \lambda v = l(x) & \text{in } \Omega \\
v = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where \( l \in L^d(\Omega) \) and \( \lambda \geq 0 \). Then for the weak solution \( \bar{v} \) of (19) we have \( \|\bar{v}\|_{L^d(\Omega)} \leq C_d \|l\|_{L^d(\Omega)} \) for some constant \( C_d \) depending only on \( d \).

**Proof.** Let \( l_n \) be a sequence of smooth functions approaching \( l \) in \( L^d(\Omega) \) and assume that \( v_n \) is a solution of
\[
\begin{cases}
-\Delta v_n + \lambda v_n = l_n(x) & \text{in } \Omega \\
v_n = 0 & \text{on } \partial \Omega.
\end{cases}
\]
As \( v_n \) is a smooth solution of (20), we can multiply (20) by \( |v_n|^{d-2}v_n \) and integrate to obtain
\[
\int_\Omega (-\Delta v_n + \lambda v_n)|v_n|^{d-2}v_n \, dx = \int_\Omega l_n |v_n|^{d-2}v_n \, dx,
\]
by integration by part and applying Hölder inequality on the right hand side, we get
\[
\int_\Omega (d - 1)|\nabla v_n|^2|v_n|^{d-2} + \lambda |v_n|^d \leq \|l_n\|_{L^d(\Omega)} \|v_n|^{d-1}\|_{L^{d'}(\Omega)} \\
\leq \|l_n\|_{L^d(\Omega)} \|v_n|^{d-1}\|_{L^{d'}(\Omega)}^{d'},
\]
where \( d' = d/(d-1) \) is the conjugate of \( d \). Note that
\[
\int_\Omega |\nabla v_n|^2|v_n|^{d-2} \geq \left( \frac{2}{d} \right)^2 \int_\Omega |\nabla v_n|^2 \geq C \left( \frac{2}{d} \right)^2 \int_\Omega |v_n|^d = C \left( \frac{2}{d} \right)^2 \|v_n|^{d'}.
\]
where $C$ is a constant depending on $\Omega$. Hence it follows that
\[
\|v_n\|_{L^4(\Omega)} \leq C_d \|I_n\|_{L^4(\Omega)},
\] (21)
for a constant $C_d$. The desired result follows by letting $n \to \infty$.

Now that we have all the required results, we can state the proof of the main theorem.

**Proof of Theorem 1.1.** Recall that
\[
K = \{ u \in V; \| u \|_{W^{2,4}(\Omega)} \leq R \},
\]
with $R$ a constant to be determined. To prove that (3) has a solution, using Corollary 4.1, first we must show that $I$ has a local minimum in $K$, i.e.
\[
\exists \tilde{u} \in K \quad \text{such that} \quad I[\tilde{u}] = \inf_{u \in K} I[u].
\]
It follows from the growth condition (4) that for a constant $C_1 > 0$, $G(x,u) \leq \frac{2}{p} |u|^p + cu + C_1$. Then, for $u \in K$, that is $\|u\|_{W^{2,4}(\Omega)} \leq R$, by using Hölder inequality we have
\[
\varphi(u) = \int_{\Omega} \mu G(x,u) + ub(x)dx \leq \int_{\Omega} \frac{\mu a}{p} |u|^p + (b(x) + \mu c)u + \mu C_1 \, dx
\]
\[
\leq \frac{\mu a}{p} \|u\|_{L^p(\Omega)}^p + \|b(x) + \mu c\|_{L^4(\Omega)} \|u\|_{L^4(\Omega)} + \mu C_1 |\Omega|,
\]
where $d'$ is the Sobolev conjugate of $d$, i.e. $1/d + 1/d' = 1$. It follows from Theorem 5.3 that the embedding $W^{2,d}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for $1 < q < \infty$ if $n \leq 2d$, and for $1 < q \leq nd/(n-2d)$ if $n > 2d$. Recalling that by assumption $2 < p \leq (2n-2d)/(n-2d)$ if $n > 2d$, and $p > 2$ if $n < 2d$ we can conclude the continuous embedding $W^{2,d} \hookrightarrow L^p(\Omega)$, for this range of $p$. In addition, $d \geq 2$ implies that $d' \leq 2$ so clearly $d' < (2n-2d)/(n-2d)$ for any $n$ and $d$ and hence the embedding $W^{2,d}(\Omega) \hookrightarrow L^{d'}(\Omega)$ is also continuous. Thus we get
\[
\varphi(u) \leq \frac{\mu a}{p} C_3 \|u\|_{W^{2,4}(\Omega)}^p + \|b(x) + \mu c\|_{L^4(\Omega)} C_4 \|u\|_{W^{2,4}(\Omega)} + C_2,
\]
for appropriate $C_2, C_3, C_4$. Since $\|u\|_{W^{2,4}(\Omega)} \leq R$ and $b \in L^d(\Omega)$ we obtain
\[
\varphi(u) \leq \left( C_5 R^p + C_4 \|b(x) + \mu c\|_{L^4(\Omega)} R + C_2 \right) < \infty,
\]
for $C_5 = \mu a C_3/p$. As a result we see that
\[
I[u] = \varphi^*(Au) - \varphi(u) \geq \varphi^*(Au) - \left( C_5 R^p + C_4 \|b(x) + \mu c\|_{L^4(\Omega)} R + C_2 \right) > -\infty.
\]
So, as $\inf_{u \in K} I[u] > -\infty$, we can consider a minimizing sequence in $K$ i.e.
\[
\exists\{|u_n\} \subset K; \quad \text{such that} \quad I[u_n] \to \inf_{u \in K} I[u] := m.
\]
Now we shall show that $I[\cdot]$ is bounded from below and attains its infimum in $K$. Note that $\{u_n\} \subset K$ implies $\|u_n\|_{W^{2,4}(\Omega)} \leq R$, for all $n$, so we can deduce that $\{u_n\}$ is bounded in $W^{2,d}(\Omega)$. Going if necessary to a subsequence, and recalling that $W^{2,d}(\Omega)$ is reflexive, we can assume there exists a $\tilde{u} \in W^{2,d}(\Omega)$ that $u_n \rightharpoonup \tilde{u}$ weakly in $W^{2,d}(\Omega)$. Moreover, as $K$ is convex and closed, and therefore weakly closed, $\tilde{u}$ is also in $K$. From Lemma 4.4, we know that $I[\cdot]$ is weakly lower semi-continuous, so we can deduce that
\[
m \leq I[\tilde{u}] \leq \liminf_{n \to \infty} I[u_n] = m,
\]
which means that $\bar{u} \in K$ is a minimizer of $I[\cdot]$.

Following Theorem 3.1, to show the existence of solution for (3), in the next step, we need to prove that there exists a $\bar{v} \in V \cap K$ satisfying

$$
\begin{cases}
-\Delta v + \lambda v = \mu g(x, \bar{u}) + b(x), & \text{in } \Omega, \\
v = 0, & \text{on } \partial \Omega.
\end{cases}
$$

First we claim that (22) is satisfied by a $\bar{v} \in V$. Indeed, from the growth condition (4) it can be easily seen that

$$
\|g(x, \bar{u})\|_{L^a(\Omega)} \leq a\|\bar{u}^{p-1}\|_{L^a(\Omega)} + C_6,
$$

for an appropriate constant $C_6$. Moreover, as shown in Lemma 5.12, for the range of $p$ in (5), we have $|\bar{u}|^{p-1} \in L^d(\Omega)$, and also

$$
\|\bar{u}^{p-1}\|_{L^d(\Omega)} \leq \left( \int_{\Omega} |\bar{u}|^{d(p-1)} \, dx \right)^{\frac{1}{d}} \leq \|\bar{u}\|^{p-1}_{L^{d(p-1)}(\Omega)} \leq C_p \|\bar{u}\|^{p-1}_{W^{2,d}(\Omega)},
$$

so, by using triangle inequality, from (23) we obtain

$$
\|f(x, \bar{u}, \mu)\|_{L^a(\Omega)} = \|\mu g(x, \bar{u}) + b(x)\|_{L^a(\Omega)} \leq \|\mu g(x, \bar{u})\|_{L^a(\Omega)} + \|b(x)\|_{L^a(\Omega)} \leq \mu a \|\bar{u}^{p-1}\|_{L^d(\Omega)} + \|b(x)\|_{L^d(\Omega)} + \mu C_6.
$$

Hence, we deduce that $f(x, \bar{u}, \mu) = \mu g(x, \bar{u}) + b(x) \in L^d(\Omega)$ as we have $|\bar{u}|^{p-1}, b \in L^d(\Omega)$. Therefore, since $\partial \Omega$ is of class of $C^{1,1}$, by Theorem 5.9 we can assert that there exists a $\bar{v} \in W^{2,d}(\Omega)$ solution of (22). We can also apply Theorem 5.7 to obtain

$$
\|\bar{v}\|_{W^{2,d}(\Omega)} \leq C_7(\|\bar{v}\|_{L^a(\Omega)} + \|\mu g(x, \bar{u}) + b(x)\|_{L^a(\Omega)}),
$$

for an appropriate constant $C_7$ defined in Theorem 5.7. Now to show $\|\bar{v}\|_{W^{2,d}(\Omega)} \leq R$, recall that from Lemma (4.5) we have

$$
\|\bar{v}\|_{L^d(\Omega)} \leq C_d(\mu a \|\bar{u}^{p-1}\|_{L^a(\Omega)} + \|b(x)\|_{L^d(\Omega)} + \mu C_8),
$$

which yields

$$
\|\bar{v}\|_{L^d(\Omega)} \leq C_d(\mu a \|\bar{u}^{p-1}\|_{L^a(\Omega)} + \|b(x)\|_{L^d(\Omega)} + \mu C_8),
$$

Thus, by substituting (23) and (26) in (25) we have

$$
\|\bar{v}\|_{W^{2,d}(\Omega)} \leq C_9(\mu a \|\bar{u}^{p-1}\|_{L^a(\Omega)} + \mu \|b(x)\|_{L^d(\Omega)} + \mu C_{10})
$$

where $C_9 = C_7(C_d + 1)$ and $C_{10} = (C_d C_8 + C_6)/(C_d + 1)$. Therefore, on substituting (24) in (27), we obtain

$$
\|\bar{v}\|_{W^{2,d}(\Omega)} \leq C_9(\mu a C_p \|\bar{u}\|_{L^{d(p-1)}(\Omega)} + \|b(x)\|_{L^d(\Omega)} + \mu C_{10}).
$$

Now as $\bar{u} \in K$, i.e. $\|\bar{u}\|_{W^{2,d}(\Omega)} \leq R$, we get

$$
\|\bar{v}\|_{W^{2,d}(\Omega)} \leq C_9(\mu a C_p R^{p-1} + \|b(x)\|_{L^d(\Omega)} + \mu C_{10}).
$$

Next, recall that $b \in L^d(\Omega)$, so we can choose $R$ such that $C_9 \|b(x)\|_{L^d(\Omega)} < R$, then by fixing $R$, we can choose $\mu$ such that

$$
C_9(\mu a C_p R^{p-1} + \|b(x)\|_{L^d(\Omega)} + \mu C_{10}) \leq R.
$$

As a result for this specific value of $\mu$ we have

$$
\|\bar{v}\|_{W^{2,d}(\Omega)} \leq R
$$

which means that $\bar{v} \in K$.

Finally, as $I$ has a local minimum $\bar{u}$ and there exists a $\bar{v} \in V \cap K$ satisfying (22), by Corollary 4.1 we can conclude that (3) has a solution.

In the special case, When $d = 2$, our assumption on $b$, coincides with the one assumed in [2, 3, 16, 6], however, as one can see in the following Corollary, regarding the range of $p$, the comparable result in our case is still stronger than those achieved in the cited papers. We can restate our result for this case as follows.
**Corollary 4.6.** Suppose $\Omega$ is a bounded domain in $\mathbb{R}^n$ which $\partial \Omega$ is of class of $C^{1,1}$. Consider the problem
\[
\begin{cases}
-\Delta u + \lambda u = \mu g(x,u) + b(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega, 
\end{cases}
\]  
(28)

with $\lambda \geq 0$ and $\mu$ positive. Furthermore, assume $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which is increasing and satisfies the following growth condition:
\[
|g(x,u)| \leq a|u|^{p-1} + c, 
\]
where $a > 0$, $b \in L^2(\Omega)$, and $p \in \mathbb{R}$ is sufficiently small. Then problem (28) admits at least one solution provided $\mu$ is small.

As mentioned in the introduction, when it come to problem (2), in addition to proving the existence of solution for the problem with supercritical nonlinearity, by choosing $b$ to be smooth enough and applying Theorem 5.6, we can also determine the smoothness of the solution. Therefore we dedicate the rest of this section to the proof of this interesting result which we can state as follows.

**Theorem 4.7.** Suppose $\Omega$ is a bounded domain in $\mathbb{R}^n$ which $\partial \Omega$ is of class of $C^{k+2}$. Consider following problem
\[
\begin{cases}
-\Delta u + \lambda u = \mu |u|^{p-2}u + b(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega, 
\end{cases}
\]  
(30)

where $\lambda \geq 0$ and $\mu > 0$ are two parameters. If $b \in W^{k,2}(\Omega)$, and
\[
\begin{cases}
2 < p \leq \frac{2(n-4)}{(n-4)} & \text{if } n > 4, \\
2 < p & \text{if } n \leq 4. 
\end{cases}
\]
Then problem (30) admits at least one solution for $\mu$ sufficiently small.

Before the proof, first we set up $I[\cdot]$ for this special case for reader’s convenience. In this case we must set $f(x, y, \mu) = \mu|y|^{p-2}y + b(x)$, and define
\[
F(x, y, \mu) = \int_0^y \mu|s|^{p-2}s + b(x)ds = \frac{\mu}{p}|y|^p + yb(x).
\]
Hence, we can let $\varphi : V \to \mathbb{R}$ to be
\[
\varphi(u) := \int_{\Omega} F(x, u, \mu)dx = \int_{\Omega} \frac{\mu}{p}|u|^p + ub(x)dx,
\]
so by definition $\varphi(u)$ is Gâteaux differentiable, convex and lower semi-continuous. Now, we rewrite (30) in the following form
\[
Au = D\varphi(u).
\]
Recalling Remark 4.3, we find the Fenchel dual $\varphi^*(\cdot) : V^* \to \mathbb{R}$ of $\varphi$ by
\[
\varphi^*(Au) = \frac{\mu^{1-p'}}{p'} \int_{\Omega} |Au - b|^{p'}dx,
\]
where as before $p'$ is the conjugate of $p$. Consider the set $K = \{ u \in V; \|u\|_{W^{k+2}(\Omega)} \leq R \}$, and define $\psi$ as bellow
\[
\psi(u) := \varphi^*(\Delta u + \lambda u) = \begin{cases}
\frac{\mu^{1-p'}}{p'} \int_{\Omega} |\Delta u + \lambda u - b(x)|^{p'}dx & u \in K, \\
+ \infty & u \notin K,
\end{cases}
\]
(32)
with \( \text{Dom}(\psi) = \{ u \in V; \psi(u) < \infty \} \). It is easy to check that \( \psi(\cdot) \) is a convex function. Now, we can define the new energy functional \( I[\cdot] \) as follows

\[
I[u] = \psi(u) - \varphi(u) = \frac{1}{p} \int_{\Omega} |\Delta u + \lambda u - b|^{p} dx - \frac{\mu}{p} \int_{\Omega} |u|^{p} + ub(x) dx.
\] (33)

Note that as this \( I[\cdot] \) is a special case of (10), in spite of the different definition of the set \( K \) for this case, it can be proved in much the same way as in Lemma 4.4 that (33) is weakly lower semi-continuous as well.

We shall now prove Theorem 4.7 again by applying Corollary 4.1. It is worth noting that this proof takes the same steps as in proof of Theorem 1.1, however for the convenience of the reader we bring the proof here.

**Proof of Theorem 4.7.** Applying Theorem 3.1, we start the proof by showing that \( I \) has a nontrivial critical point, which is a minimizer, i.e.

\[ \exists \bar{u} \in V \quad \text{such that} \quad I[\bar{u}] = \inf_{u \in V} I[u]. \]

Using Hölder inequality on \( \varphi \), we obtain

\[
\varphi(u) = \int_{\Omega} F(x, \mu, u) = \int_{\Omega} \frac{\mu}{p} |u|^{p} + b(x)u dx
\]

\[
= \frac{\mu}{p} \int_{\Omega} |u|^{p} dx + \int_{\Omega} b(x)u dx
\]

\[
\leq \frac{\mu}{p} \|u\|_{L^{p}(\Omega)} + \|b(x)\|_{L^{2}(\Omega)} \|u\|_{L^{2}(\Omega)}.
\] (34)

As having \( \partial \Omega \in C^{k+2} \) implies the cone condition, we can consider the following compact embeddings, which we have from Theorem 5.3,

\[
\begin{align*}
\text{if} & \quad 2(k + 2) \geq n, \quad W^{k+2,2}(\Omega) \hookrightarrow L^{q}(\Omega), \quad \text{for} \quad 1 < q < \infty, \\
\text{if} & \quad 2(k + 2) < n, \quad W^{k+2,2}(\Omega) \hookrightarrow L^{q}(\Omega), \quad \text{for} \quad 1 < q < \frac{2n}{n - 2(k + 2)},
\end{align*}
\]

so it can be deduced by assumption (31) that \( W^{k+2,2}(k\Omega) \hookrightarrow L^{p}(\Omega) \) for the range of \( p \) that we have in our problem. We also know that \( W^{k,2}(\Omega) \hookrightarrow L^{2}(\Omega) \). As a result, from (34) we get

\[
\varphi(u) \leq \frac{\mu}{p} C_{1}^{'} \|u\|_{W^{k+2,2}}^{p} + C_{2}^{'} \|b(x)\|_{W^{k,2}(\Omega)} \|u\|_{W^{k+2,2}(\Omega)}.
\]

where \( C_{1}^{'} \) and \( C_{2}^{'} \) are constant. Recalling that \( \|u\|_{W^{k+2,2}(\Omega)} \leq R \), and \( b \in W^{k,2}(\Omega) \), we thus have

\[
\varphi(u) \leq \left( C_{1}^{'} \frac{\mu}{p} R^{p} + C_{2}^{'} \|b(x)\|_{W^{k,2}(\Omega)} R \right) < \infty.
\] (35)

Hence,

\[
I[u] = \varphi^{*}(Au) - \varphi(u) \geq \varphi^{*}(Au) - \left( C_{1}^{'} \frac{\mu}{p} R^{p} + C_{2}^{'} \|b(x)\|_{W^{k,2}(\Omega)} R \right) > -\infty
\]

So, as \( \inf_{u \in K} I[u] > -\infty \), we can consider a minimizing sequence in \( K \), i.e.

\[ \exists \{u_{n}\} \subset K; \quad \text{such that} \quad I[u_{n}] \rightarrow \inf_{u \in K} I[u] := m. \]

Now by an argument similar to the one in the proof of Theorem 1.1, as a result of the boundedness of \( \{u_{n}\} \subset K \) in \( W^{k+2,2}(\Omega) \), that is \( \|u_{n}\|_{W^{k+2,2}(\Omega)} \leq R \), we can consider a \( \bar{u} \in W^{k+2,2}(\Omega) \) that up to a subsequence \( u_{n} \rightharpoonup \bar{u} \) weakly in \( W^{k+2,2}(\Omega) \), and so \( \bar{u} \in K \). Then recalling the lower semi continuity of \( I[\cdot] \), deduced from Lemma (4.4), again we obtain

\[
m \leq I[\bar{u}] \leq \liminf_{n \rightarrow \infty} I[u_{n}] = m,
\]
which means that \( \bar{u} \in K \) is a minimizer of \( I[|.|] \).

Continuing as in the proof of Theorem 1.1, to show the existence of solution for (30), in the next step, we need to prove that there exists a \( \bar{v} \in V \cap K \) satisfying

\[
\begin{align*}
-\Delta v + \lambda v &= \mu |\bar{u}|^{p-2} \bar{u} + b(x), & \text{in } \Omega, \\
v &= 0, & \text{on } \partial \Omega,
\end{align*}
\]

(36)

First, we show that a \( \bar{v} \in V \) satisfies this linear equation. Consider the Euler-Lagrange energy functional for (36),

\[
E(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 + \lambda |v|^2 dx - \int_{\Omega} |\bar{u}|^{p-2} \bar{u} v + b(x) v dx,
\]

which is well-defined as \( v \in H^1_0(\Omega), b \in W^{k,2}(\Omega), \) and \( \bar{u} \in W^{k+2,2}(\Omega) \) yields \( \bar{u} \in L^2(p-1)(\Omega) \) for the range of \( p \) that we are considering in (31) for this problem. Set

\[
E(\bar{v}) = \inf_{v \in V} E(v),
\]

then, obviously \( \bar{v} \) is a critical point of \( E(v) \) and satisfies (36).

Furthermore, we must show \( \bar{v} \in K \), i.e. \( \| \bar{v} \|_{W^{k+2,2}(\Omega)} \leq R \).

First note that from the property of norm, triangle inequality, we obtain

\[
\| \mu |\bar{u}|^{p-1} + b(x) \|_{W^{k,2}(\Omega)} \leq \mu \| \bar{u} \|_{W^{k,2}(\Omega)}^{p-1} + \| b(x) \|_{W^{k,2}(\Omega)}.
\]

As shown in Lemma 5.10, \( |\bar{u}|^{p-1} \in W^{k,2}(\Omega) \) and we have

\[
\| \bar{u} \|_{W^{k,2}(\Omega)} \leq C_p \| \bar{u} \|_{W^{k+2,2}(\Omega)}^{p-1}.
\]

Also, by assumption we have \( b \in W^{k,2}(\Omega) \). Hence, from (37) we can deduce that \( f(x, \bar{u}, \mu) = \mu |\bar{u}|^{p-2} \bar{u} + b \in W^{k,2}(\Omega) \). Beside this, we also have that \( \bar{v} \in H^1_0(\Omega) \) is a solution of (36), and \( \partial \Omega \) is of class of \( C^{k+2} \), so we can apply Theorem 5.6 to find

\[
\| \bar{v} \|_{W^{k+2,2}(\Omega)} \leq C_3^* \left( \| \bar{v} \|_{L^2(\Omega)} + \| \mu |\bar{u}|^{p-1} + b(x) \|_{W^{k,2}(\Omega)} \right),
\]

(39)

where \( C_3^* = C(n, \lambda, K', k, \partial \Omega) \) defined in Theorem 5.6. Moreover, in Lemma 4.5 we have proved that

\[
\| \bar{v} \|_{L^2(\Omega)} \leq C_k \left( \mu \| \bar{u} \|_{W^{k,2}(\Omega)}^{p-1} + \| b(x) \|_{W^{k,2}(\Omega)} \right),
\]

(40)

Hence by applying (37) and (40) in (39), we have

\[
\| \bar{v} \|_{W^{k+2,2}(\Omega)} \leq C_4 \left( \mu \| \bar{u} \|_{W^{k,2}(\Omega)}^{p-1} + \| b(x) \|_{W^{k,2}(\Omega)} \right)
\]

(41)

where \( C_4 = C_3^* (C_k + 1) \). Consequently, by substituting (38) in (41) we get

\[
\| \bar{v} \|_{W^{k+2,2}(\Omega)} \leq C_4 \left( C_p \mu \| \bar{u} \|_{W^{k+2,2}(\Omega)}^{p-1} + \| b(x) \|_{W^{k,2}(\Omega)} \right).
\]

Now as \( \bar{u} \in K \), i.e., \( \| \bar{u} \|_{W^{k+2,2}(\Omega)} \leq R \). We obtain

\[
\| \bar{v} \|_{W^{k+2,2}(\Omega)} \leq C_4 \left( C_p \mu R^{p-1} + \| b(x) \|_{W^{k,2}(\Omega)} \right).
\]

Furthermore, recall that \( b \in W^{k,2}(\Omega) \), so we can choose \( R \) such that \( C_4 \| b(x) \|_{W^{k,2}(\Omega)} < R \), then by fixing \( R \), we can choose \( \mu \) such that

\[
C_4 \left( C_p \mu R^{p-1} + \| b(x) \|_{W^{k,2}(\Omega)} \right) \leq R.
\]

Thus, for this specific value of \( \mu \), we have

\[
\| \bar{v} \|_{W^{k+2,2}(\Omega)} \leq R,
\]

which means that \( \bar{v} \in K \).

As a result, since \( I[|.|] \) has a local minimum \( \bar{u} \) and there exists a \( \bar{v} \in V \cap K \) satisfying (36), by Corollary 4.1 we can conclude that (30) has a solution. \( \square \)
5 Appendix

To make the paper self-contained, in this section we bring some results to which we referred frequently throughout the proof of the main theorems. First we state Sobolev embedding and elliptic regularity theorems without proof. Divided into a sequence of Lemmas, we next prove some specific results for our case as an application of these theorems. We start with the standard compactness and embedding results in Sobolev spaces. The following embedding theorems are proved in [1].

**Theorem 5.1. (The Rellich-Kondrachov Theorem)** Let $\Omega$ be a domain in $\mathbb{R}^n$, $\Omega_0$ be a bounded sub-domain of $\Omega$, and $\Omega^k_0$ be the intersection of $\Omega_0$ with a $k$-dimensional plane in $\mathbb{R}^n$. Let $j \geq 0$ and $m \geq 1$ be integers, and let $1 \leq p < \infty$.

**Part I** If $\Omega$ satisfies the cone condition and $mp \leq n$, then the following embeddings are compact:

\[
W^{j+m,p}(\Omega) \hookrightarrow W^{j,q}(\Omega^k_0) \quad \text{if} \quad 0 < n - mp < k \quad \text{and} \quad 1 \leq q < \frac{kp}{n - mp},
\]

\[
W^{j+m,p}(\Omega) \hookrightarrow W^{j,q}(\Omega^k_0) \quad \text{if} \quad n = mp, 1 \leq k \leq n \quad \text{and} \quad 1 \leq q < \infty.
\]

**Part II** If $\Omega$ satisfies the cone condition and $mp > n$, then the following embeddings are compact:

\[
W^{j+m,p}(\Omega) \hookrightarrow C^j_B(\Omega_0) \quad \text{if} \quad 1 \leq q < \infty.
\]

**Part III** If $\Omega$ satisfies the strong local Lipschitz condition, then the following embeddings are compact:

\[
W^{j+m,p}(\Omega) \hookrightarrow C^j(\Omega^k_0) \quad \text{if} \quad mp > m,
\]

\[
W^{j+m,p}(\Omega) \hookrightarrow C^{j,\lambda}(\Omega^k_0) \quad \text{if} \quad mp > n \geq (m-1)p \quad \text{and} \quad 0 < \lambda < m - \left(\frac{n}{p}\right).
\]

**Part IV** If $\Omega$ is an arbitrary domain in $\mathbb{R}^n$, the given embeddings are compact provide $W^{j+m,p}(\Omega)$ is replaced by $W^{j+m,p}_0(\Omega)$.

**Remark 5.2.** It is important to consider the following points about the assumption of the theorem:

1. If $\Omega$ is bounded, we may have $\Omega_0 = \Omega$ in the statement of the theorem.

2. If $k = n$, then $\Omega^k_0 = \Omega$ in the statement of the theorem.

3. We say that $\Omega$ satisfies the cone condition if there exist a finite cone $C$ such that each $x \in \Omega$ is the vertex of a finite cone $C_x$ contained in $\Omega$ and congruent to $C$. Note that $C_x$ need not to obtained from $C$ by parallel translation, but simply by rigid motion.

4. If $\Omega$ is a bounded domain with a locally Lipschitz boundary, it satisfies the cone condition as well. (See para. 4.11 in [1, p. 94])

**Theorem 5.3. (The Sobolev embedding Theorem)** Let $\Omega$ be a domain in $\mathbb{R}^n$ and, for $1 \leq k \leq n$, let $\Omega^k_0$ be the intersection of $\Omega$ with a plane of dimension $k$ in $\mathbb{R}^n$. (If $k = n$, then $\Omega^k_0 = \Omega$.) Let $j \geq 0$ and $m \geq 1$ be integers and $1 \leq p \leq \infty$.

**Part I** Suppose $\Omega$ satisfies the cone condition.

Case A If either $mp > n$ or $m = n$ and $p = 1$, then

\[
W^{j+m,p}(\Omega) \hookrightarrow C^j_B(\Omega),
\]
Moreover, if \(1 \leq k \leq n\).

\[
W^{j+m,p}(\Omega) \hookrightarrow W^{j,q}(\Omega_k) \quad \text{for } p \leq q \leq \infty,
\]

and, in particular,

\[
W^{m,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for } p \leq q \leq \infty.
\]

**Case B.** If \(1 \leq k \leq n\) and \(mp = n\), then

\[
W^{j+m,p}(\Omega) \hookrightarrow W^{j,q}(\Omega_k), \quad \text{for } p \leq q < \infty,
\]

and, in particular,

\[
W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \text{for } p \leq q < \infty.
\]

**Case C.** Let \(mp < n\) and either \(n - mp < k \leq n\) or \(p = 1\) and \(n - m \leq k \leq n\), then

\[
W^{j+m,p}(\Omega) \hookrightarrow W^{j,q}(\Omega), \quad \text{for } p \leq q \leq p^* = \frac{kp}{n - mp},
\]

In particular,

\[
W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \text{for } p \leq q \leq p^* = \frac{kp}{n - mp}.
\]

The embedding constants for the embeddings above depend only on \(n, m, p, q, j, k\), and the dimension of the cone \(C\) in the cone condition.

**Part II** Suppose \(\Omega\) satisfies the strong local Lipschitz condition. Then the target space \(C^j_B(\Omega)\) of the embedding (42) can be replaced with the smaller space \(C^j(\Omega)\), and the embedding can be further refined as follows:

If \(mp > n > (m - 1)p\), then

\[
W^{j+m,p}(\Omega) \hookrightarrow C^{j,\lambda}_{\#}(\Omega) \quad \text{for } 0 < \lambda \leq m - \left(\frac{n}{p}\right),
\]

and if \(n = (m - 1)p\), then

\[
W^{j+m,p}(\Omega) \hookrightarrow C^{j,\lambda}_{\#}(\Omega) \quad \text{for } 0 < \lambda < 1.
\]

Also, if \(n = m - 1\) and \(p = 1\), then the later embedding holds for \(\lambda = 1\) as well.

**Part III** All the embeddings in the Part I and II are valid for arbitrary domains \(\Omega\) if the \(W^{j,\#}\)-space undergoing the embedding is replaced with the corresponding \(W^j_0\)-space. (See Theorem 4.12 in [1, page. 85] for a proof).

**Remark 5.4.** If \(\Omega_k\) (or \(\Omega\)) has finite volume, then embeddings, (43)-(45) also hold for \(1 \leq q < p\) in addition to the values of \(q\) asserted in the statement of the theorem.

Next we review a few Elliptic regularity results from [11].

**Theorem 5.5.** (Elliptic Regularity Theorem) Let \(\Omega\) be a bounded domain in \(\mathbb{R}^n\) and \(\partial \Omega\) be of class of \(C^2\). Also assume that \(u \in W^{1,2}(\Omega)\) is a weak solution of the equation \(Lu = f\) in \(\Omega\), where \(Lu = D_i(a^{ij}(x)D_ju + b^i(x)u) + c^i(x)D_iu + d(x)u\), with \(L\) strictly elliptic in \(\Omega\), the coefficients \(a^{ij}, b^i, c^i, d, i = 1, \ldots, n\) are uniformly Lipschitz continuous in \(\Omega\), the coefficients \(c^i, d, i = 1, \ldots, n\) are essentially bounded in \(\Omega\), the function \(f\) is in the \(L^2(\Omega)\), and there exists a function \(\varphi \in W^{2,2}(\Omega)\) for which \(u - \varphi \in W^{1,2}_0(\Omega)\). Then we have also \(u \in W^{2,2}(\Omega)\) and

\[
\|u\|_{W^{2,2}(\Omega)} \leq C(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + \|\varphi\|_{W^{2,2}(\Omega)}),
\]

where \(C = C(n, \lambda, K, \partial \Omega)\).
Theorem 5.6. \textbf{(Regularity Theorem)} Let $u \in W^{1,2}(\Omega)$ be a weak solution of the equation $Lu = f$ in $\Omega$, where $Lu = D_i(a^{ij}(x)D_ju + b^i(x)u) + c(x)Du + d(x)u$, with $L$ strictly elliptic in $\Omega$, the coefficients $a^{ij}, b^i \in C^{k,1}(\Omega)$, the coefficients $c, d \in C^{k-1,1}(\Omega)$, and the function $f \in W^{k,2}(\Omega)$, $k \geq 1$. In addition, let us assume that $\Omega \in C^{k+2}$ and that there exists a function $\varphi \in W^{k+2,2}(\Omega)$ for which $u - \varphi \in W^{1,2}_0(\Omega)$. Then we have also $u \in W^{k+2,2}(\Omega)$ and

$$
\|u\|_{W^{k+2,2}(\Omega)} \leq C(\|u\|_{L^2(\Omega)} + \|f\|_{W^{k,2}(\Omega)} + \|\varphi\|_{W^{k+2,2}(\Omega)}),
$$

where $C = C(n, \lambda, k', k, \partial \Omega)$, and $k' = \max\{(\lambda, |c|, |d|, f) \in \Omega\}$.

Theorem 5.7. \textbf{(Lp estimates)} Let $\Omega$ be a domain in $\mathbb{R}^n$ with a $C^{1,1}$ boundary portion $T \subset \partial \Omega$. Let $u \in W^{2,p}(\Omega)$, $1 < p < \infty$, be a strong solution of $Lu = f$ in $\Omega$ with $u = 0$ on $T$, in the sense of $W^{1,p}(\Omega)$, where $f \in L^p(\Omega)$ and the coefficients of $L$ satisfy, for positive constants $\lambda, \Lambda$,

- $a^{ij} \in C^0(\Omega \cup T)$, $b^i, c \in L^\infty(\Omega)$;
- $a^{ij} \geq \lambda|\epsilon|^2$, for all $\epsilon \in \mathbb{R}^n$;
- $|a^{ij}|, |b^i|, |c| \leq \Lambda$,

where $i, j = 1, \ldots, n$. Then, for any domain $\Omega' \subset \subset \Omega \cup T$,

$$
\|u\|_{W^{2,p}(\Omega')} \leq C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}),
$$

where $C$ depends on $n, p, \lambda, \Lambda, T, \Omega', \Omega$, and the moduli of continuity of the coefficients $a^{ij}$ on $\Omega'$ (see Theorem 9.13 in [11, p. 239]).

Remark 5.8. When $T = \partial \Omega$ in Theorem 5.7, we may take $\Omega' = \Omega$ to obtain a global $W^{2,p}(\Omega)$ estimate.

Theorem 5.9. \textbf{(Existence of a strong solution for Dirichlet problem)} let $\Omega$ be a $C^{1,1}$ domain in $\mathbb{R}^n$, and let $Lu = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u$, where the operator $L$ is strictly elliptic in $\Omega$, with coefficients $a^{ij} \in C^0(\Omega)$, $b^i, c \in L^\infty(\Omega)$, with $i, j = 1, \ldots, n$ and $c \leq 0$. Then, if $f \in L^p(\Omega)$ and $\varphi \in W^{2,p}(\Omega)$, with $1 < p < \infty$, the Dirichlet problem $Lu = f$ in $\Omega$, $u - \varphi \in W^{1,p}_0(\Omega)$ has a unique solution $u \in W^{2,p}(\Omega)$ (see Theorem 9.15 in [11, p. 241]).

Applying the preceding standard results to our case, we can prove the following interesting results which are essential in the proof of the main theorems.

**Proposition 5.1.** Let $p > 2$ be a real number and $k$ be an integer with $0 \leq k \leq p - 1$. Let $u \in W^{k+2,2}(\Omega)$ where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with $\partial \Omega \in C^{k+2}$. Furthermore, assume that $m \in \{0, \ldots, k\}$ and $s_1, \ldots, s_k \in \{0, \ldots, m\}$ satisfying $s_1 + s_2 + \cdots + s_k = m$ and $1s_1 + 2s_2 + \cdots + ks_k = k$. If either of following assertions holds,

- $k > 1$ and either
  - $i-a)$ $n \leq 2(k + 2)$, or
  - $i-b)$ $2(k + 2) < n < 2(k + 2) + \frac{1}{k-1}$, and $p - 1 \leq \frac{n-2k}{n-2(k+2)}$.

- $k = 0, 1$ and either
  - $ii-a)$ $n \leq 2(k + 2)$, or
  - $ii-b)$ $n > 2(k + 2)$, and $p - 1 \leq \frac{n-2k}{n-2(k+2)}$.

Then there exists a constant $C = C(p, \Omega, m)$ independent of $u$ such that

$$
\left\|u^{p-1-m}|Du|^{s_1} |D^2u|^{s_2} \cdots |D^k u|^{s_k}\right\|_{L^2(\Omega)} \leq C\|u\|_{W^{k+2,2}(\Omega)}^{p-1}.
$$

(47)
Proof. Let \( j \leq m \) be an integer such that there exist \( i_1, i_2, \ldots, i_j \in \{1, \ldots, k\} \) with \( i_1 < i_2 < \cdots < i_j \), and that \( s_{i_1}, s_{i_2}, \ldots, s_{i_j} \neq 0 \) where \( s_{i_1} + s_{i_2} + \cdots + s_{i_j} = m \) and \( i_1 s_{i_1} + i_2 s_{i_2} + \cdots + i_j s_{i_j} = k \). We shall show that

\[
\left\| u^{p-1-m}|D^i u|^{s_{i_1}} |D^j u|^{s_{i_2}} \cdots |D^k u|^{s_{i_j}} \right\|_{L^2(\Omega)} \leq C \|u\|_{W^{k+2,2}(\Omega)}^{p-1},
\]

for an appropriate constant \( C \). By applying the general Hölder inequality to the left hand side of (48), we obtain

\[
\left( \int_{\Omega} \left( u^{p-1-m}|D^i u|^{s_{i_1}} |D^j u|^{s_{i_2}} \cdots |D^k u|^{s_{i_j}} \right)^2 dx \right)^{\frac{1}{2}} = \left( \int_{\Omega} u^{2(p-1-m)\rho_0} dx \right)^{\frac{1}{\rho_0}} \left( \int_{\Omega} |D^i u|^{2s_{i_1} \rho_1} dx \right)^{\frac{1}{\rho_1}},
\]

where \( \frac{1}{\rho_0} + \frac{1}{\rho_1} + \cdots + \frac{1}{\rho_j} = 1 \) and \( 1 \leq \rho_0, \ldots, \rho_j \leq \infty \). Recalling that \( u \in W^{k+2,2}(\Omega) \), we wish to choose \( \rho_0, \ldots, \rho_j \) in a way that every term in (49) is controlled by \( \|u\|_{W^{k+2,2}(\Omega)} \). As \( \partial_0 \subset C^{k+2} \), which implies the cone condition, we can make use of the embeddings in Theorems 5.1 and 5.3 to find various spaces where \( W^{k+2,2}(\Omega) \) is embedded into. Considering the fact that these embeddings depend directly on \( n \), we verify the result in three different ranges of \( n \): for \( n = 2(k+2) > 0 \), \( n = 2(k+2) < n < 2(k+2)+\frac{4}{m-1} \).

\( i) \ k > 1: \)

\( i-a.1) \) Let \( n = 2(k+2) \).

To prove (48), for this value of \( n \), we can apply part I-B of Theorem 5.3. In case of \( \rho_0 \), we consider the embedding \( W^{k+2,2}(\Omega) \hookrightarrow L^q(\Omega) \) for \( 1 < q < \infty \). Applying this in (49), we can see that \( u \in L^q(\Omega) \) for any choice of \( 1 < q < \infty \), so for any value of \( \rho_0 = 2(p-1-m) \) we have

\[
\left( \int_{\Omega} u^{2(p-1-m)\rho_0} dx \right)^{\frac{1}{\rho_0}} \leq \|u\|_{L^{2(p-1-m)\rho_0}(\Omega)}^{p-1-m} \leq C_0 \|u\|_{W^{k+2,2}(\Omega)}^{p-1-m}.
\]

Furthermore, as \( n = 2(k+2) \) implies \( 2(k+2) = n \) for \( 1 \leq l \leq j \), we can apply part I-C of Theorem 5.3 and choose \( \rho_l \) in such a way that the embeddings \( W^{k+2,2}(\Omega) \hookrightarrow W^{s_{i_l} + (k+2-i_l),2}(\Omega) \hookrightarrow W^{s_{i_l} + 2s_{i_l},\rho_l}(\Omega) \) holds true.

By setting \( 2s_{i_l} \rho_l = 2n/(n-2(k+2-i_l)), \) the largest value for which the continuous embedding can be applied, we find \( \rho_l \) for \( l = 1, \ldots, j \) such that

\[
\rho_l = \frac{n}{n-2(k+2-i_l)} s_{i_l}.
\]

In order to use Hölder inequality, we should verify that each \( \rho_l > 1 \) and \( 0 < \sum_{l=1}^j \frac{1}{\rho_l} < 1 \). Note that

\[
\sum_{l=1}^j \frac{1}{\rho_l} = \sum_{l=1}^j \frac{(n-2(k+2-i_l))s_{i_l}}{n} \leq \frac{(n-2(k+2)) \sum_{l=1}^j s_{i_l} + 2 \sum_{l=1}^j i_l s_{i_l}}{n} = \frac{(n-2(k+2))m + 2k}{n} = \frac{2k}{2(k+2)},
\]

which is obviously positive and less than one. Also, note that each \( \rho_l \) is positive as \( n = 2(k+2) \) which means that \( n > 2(k+2-i_l) \) for all \( l \). Therefore, as \( \rho_l > 0 \) for all \( l \) and \( \sum_{l=1}^j \frac{1}{\rho_l} < 1 \), we can conclude that \( \rho_l > 1 \) for all \( l \).

Hence, we do not have any issues in using the Hölder inequality. It also follows that for \( l = 1, \ldots, j \),

\[
\left( \int_{\Omega} |D^i u|^{2s_{i_l} \rho_l} dx \right)^{\frac{1}{\rho_l}} \leq \|u\|_{W^{s_{i_l} + 2s_{i_l},\rho_l}(\Omega)}^{s_{i_l}} \leq C_l \|u\|_{W^{s_{i_l} + 2s_{i_l},\rho_l}(\Omega)}^{s_{i_l}}.
\]
Now, we choose \( r_0 \) such that \( \frac{1}{r_0} = 1 - \sum_{i=1}^{j} \frac{1}{r_i} \). Since \( \sum_{i=1}^{j} \frac{1}{r_i} < 1 \), one has that \( r_0 > 1 \) as well. Hence, combining (50), (51), and (49) we obtain that

\[
\left( \int_{\Omega} (u^{p-1-m}|D^{1}u|^{s_1} |D^{2}u|^{s_2} \cdots |D^{j}u|^{s_j})^{2} \, dx \right)^{\frac{1}{2}} 
\leq \|u\|_{L^{2(p-1-m)r_0}(\Omega)}^{s_1} \|u\|_{W^{1,2s_2r_1}(\Omega)}^{s_2} \cdots \|u\|_{W^{j,2s_jr_j}(\Omega)}^{s_j} 
\leq C_0 \|u\|_{W^{k_1+2,2}(\Omega)}^{s_1} C_1 \|u\|_{W^{k_2+2,2}(\Omega)} \cdots C_j \|u\|_{W^{k_2+2,2}(\Omega)}^{s_j} 
\leq C \|u\|_{W^{k_1+2,2}(\Omega)}^{s_1} + 2 \).
\]

where \( C = C_0 C_1 \cdots C_j \). Therefore, we have the result for this case.

**i-a.2** Let \( n < 2(k + 2) \).

For this range of \( n \), we have the embedding \( W^{k+2,2}(\Omega) \hookrightarrow C^0_B(\Omega) \), which follows from Theorem 5.3. As \( u \in W^{k+2,2}(\Omega) \) implies that \( u \in C^0_B(\Omega) \) and \( \Omega \) is bounded, for any choice of \( r_0 \) the first integral in the right hand side of (49) is controlled by \( \|u\|_{W^{k+2,2}(\Omega)} \).

To deal with the rest of the terms in (49), we need to investigate it for different ranges of \( n \) in order to use the desired embeddings. For \( 2(k + n - i_1) < n < n(k + 2) \), we can use the embedding \( W^{k+2,2}(\Omega) \hookrightarrow W^{2s,\infty}(\Omega) \) for \( l = 1, \ldots, j \).

Hence, as in the previous case, we choose \( r_l \) for \( l = 1, \ldots, j \) such that

\[
r_l = \frac{n}{(n-2(k+2-i_1))s_{i_l}}.
\]

As shown before, for \( l = 1, \ldots, j \) we have \( r_l > 1 \). In particular, we can also show that \( 0 < \sum_{l=1}^{j} \frac{1}{r_l} < 1 \).

Indeed,

\[
\sum_{l=1}^{j} \frac{1}{r_l} = \sum_{l=1}^{j} \frac{n}{(n-2(k+2-i_1))} s_{i_l} = \frac{(n-2(k+2)) \sum_{l=1}^{j} s_{i_l} + 2 \sum_{l=1}^{j} i_1 s_{i_l}}{n} = \frac{(n-2(k+2))m + 2k}{n}. \tag{52}
\]

But (52) is positive as from the assumption we have

\[
n - 2(k + 2 - i_1) > 0.
\]

So, we can deduce that

\[
m(n-2(k+2)) + 2k = m(n-2(k+2 - \frac{k}{m}) = m(n-2(k+2 - \frac{\sum_{l=1}^{j} i_1 s_{i_l}}{\sum_{l=1}^{j} s_{i_l}})) \geq m(n-2(k+2 - \frac{i_1 \sum_{l=1}^{j} s_{i_l}}{\sum_{l=1}^{j} s_{i_l}})) = m(n-2(k+2 - i_1)) \geq 0.
\]

Also, for (52) to be less than one we must have \( (n-2(k+2))m + 2k)/n < 1 \) or equivalently

\[
n < 2(k + 2) + \frac{4}{m-1}, \tag{53}
\]

which is true as we already have \( n < 2(k + 2) \). Now, with these choices of \( r_l \), we can apply the embedding \( W^{k+2,2}(\Omega) \hookrightarrow W^{2s,\infty}(\Omega) \) and obtain that for \( l = 1, \ldots, j \),

\[
\left( \int_{\Omega} |D^{i_1}u|^{2s_1r_1} \, dx \right)^{\frac{1}{2}} \leq \|u\|_{W^{i_1,2s_1r_1}(\Omega)}^{s_1} \leq C_l \|u\|_{W^{k+2,2}(\Omega)}.
\]

(54)
Next, we choose \( r_0 \) such that \( \frac{1}{r_0} = 1 - \sum_{l=1}^{j} \frac{1}{r_l} \). Using the embedding \( W^{k+2,2}(\Omega) \hookrightarrow C_B^0(\Omega) \) for \( r_0 \) we have

\[
\left( \int_{\Omega} u^{2(p-1-m)}^r \, dx \right)^{\frac{1}{s}} \leq \|u\|_{C_B^0(\Omega)}^{p-1-m} \|\Omega\| < C_0 \|\Omega\| \|u\|_{W^{k+2,2}(\Omega)}^{p-1-m} \leq C' \|\Omega\| \|u\|_{W^{k+2,2}(\Omega)}^{p-1-m},
\]

(55)

where \( |\Omega| \) is the measure of the set \( \Omega \). Hence, putting together (54), (55) and (49) we obtain

\[
\left( \int_{\Omega} (u^{p-1-m} |D^{i_1}u|^{s_{i_1}} |D^{i_2}u|^{s_{i_2}} \cdots |D^{i_j}u|^{s_{i_j}})^2 \, dx \right)^{\frac{1}{2}} \\
\leq |\Omega| \|u\|_{C_B^0(\Omega)}^{p-1-m} \|u\|_{W^{k+2,2}(\Omega)}^{s_{i_1}} \|u\|_{W^{k+2,2}(\Omega)}^{s_{i_2}} \cdots \|u\|_{W^{k+2,2}(\Omega)}^{s_{i_j}} \\
\leq C' \|u\|_{W^{k+2,2}(\Omega)}^{p-1-m} C_1 \|u\|_{W^{k+2,2}(\Omega)}^{s_{i_1}} \cdots C_j \|u\|_{W^{k+2,2}(\Omega)}^{s_{i_j}} \\
\leq C \|u\|_{W^{k+2,2}(\Omega)}^{p-1-m} + 2 \|u\|_{W^{k+2,2}(\Omega)}^{2} + \sum_{l=1}^{j} C \|u\|_{W^{k+2,2}(\Omega)}^{p-1-m},
\]

where \( C = C_0 C_1 \cdots C_j \).

Now, we deal with (49) for \( 2(k+2-i_2) < n \leq 2(k+2-i_1) \). Note that for this range, we have \( n \leq 2(k+2-i_1) \). So in case of \( i_1 \), by virtue of part \( I \)-A of Theorem 5.3, since \( 2(k+2-i_2) < n < 2(k+2-i_1) \), we can use either the embedding \( W^{k+2,2}(\Omega) = W^{k+2,2}(\Omega) \hookrightarrow C_B^0(\Omega) \) or \( W^{k+2,2}(\Omega) = W^{k+2,2}(\Omega) \hookrightarrow W^{1,q}(\Omega) \) for any \( 1 < q < \infty \). Furthermore, if \( n = 2(k+2-i_1) \) from part \( I \)-B of Theorem 5.3, we have the embedding \( W^{k+2,2}(\Omega) = W^{k+2,2}(\Omega) \hookrightarrow W^{1,q}(\Omega) \) for any \( 1 < q < \infty \). Therefore, to be consistent we use \( W^{k+2,2}(\Omega) \hookrightarrow W^{1,q}(\Omega) \) for \( i_1 \). Using a similar argument as in the previous cases, since we have \( u \in C_B^0(\Omega) \) and \( D^{i_1}u \in W^{1,q}(\Omega) \) for \( 1 < q < \infty \), for any choices of \( r_0 \) and \( r_1 \) the first two terms in (49) are controlled by \( \|u\|_{W^{k+2,2}(\Omega)} \).

Also, as \( 2(k+2-i_2) < n \), we can use the embeddings \( W^{k+2,2}(\Omega) \hookrightarrow W^{\alpha,\beta}(\Omega) \) for \( l = 2, \ldots, j \). Hence, we can choose \( r_l \) for \( j = 2, \ldots, l \) as before,

\[
r_l = \frac{n}{n - 2(k+2-i_l)}.
\]

Obviously, for each \( r_l \) we have \( r_l > 1 \). To verify that \( 0 < \sum_{l=2}^{j} \frac{1}{r_l} = \sum_{l=2}^{j} \frac{1}{r_l} \) we need to have

\[
\sum_{l=2}^{j} \frac{1}{r_l} = \frac{n - 2(k+2)}{m} + \frac{4i_1}{m - 1 - 1} < \frac{n}{n - 2(k+2 - i_1)} < 1.
\]

In is easily seen that (56) holds if and only if

\[
n < 2(k+2) + \frac{4+2i_1}{m - 1 - s_{i_1}},
\]

which is already satisfied as in this case we have \( m - 1 > s_{i_1} \) and \( n \leq 2(k+2 - i_1) \). Furthermore, we choose \( r_0 \) and \( r_1 \) such that \( \frac{1}{r_0} + \frac{1}{r_1} = 1 - \sum_{l=2}^{j} \frac{1}{r_l} \). Then by using the mentioned embeddings we obtain

\[
\left( \int_{\Omega} (u^{p-1-m} |D^{i_1}u|^{s_{i_1}} |D^{i_2}u|^{s_{i_2}} \cdots |D^{i_j}u|^{s_{i_j}})^2 \, dx \right)^{\frac{1}{2}} \\
\leq |\Omega| \|u\|_{C_B^0(\Omega)}^{p-1-m} \|u\|_{W^{k+2,2}(\Omega)}^{s_{i_1}} \|u\|_{W^{k+2,2}(\Omega)}^{s_{i_2}} \cdots \|u\|_{W^{k+2,2}(\Omega)}^{s_{i_j}} \\
\leq C' \|u\|_{W^{k+2,2}(\Omega)}^{p-1-m} C_1 \|u\|_{W^{k+2,2}(\Omega)}^{s_{i_1}} \cdots C_j \|u\|_{W^{k+2,2}(\Omega)}^{s_{i_j}} \\
\leq C \|u\|_{W^{k+2,2}(\Omega)}^{p-1-m} + 2 \|u\|_{W^{k+2,2}(\Omega)}^{2} + \sum_{l=1}^{j} C \|u\|_{W^{k+2,2}(\Omega)}^{p-1-m},
\]

for an appropriate constant \( C \).

Similarly for \( l = 2, \ldots, j - 1 \) and \( 2(k+2-i_{l+1}) < n \leq 2(k+2-i_l) \), we see that either \( u \in C_B^0(\Omega) \), which means \( D^{i_1}u \in C_B^0(\Omega) \), or \( u \in W^{1,q}(\Omega) \) for \( 1 < q < \infty \). Finally for \( n < 4 = 2(k+2) \) we have \( u \in C_B^0(\Omega) \). Then we obtain

\[
\left( \int_{\Omega} (u^{p-1-m} |D^{i_1}u|^{s_{i_1}} |D^{i_2}u|^{s_{i_2}} \cdots |D^{i_j}u|^{s_{i_j}})^2 \, dx \right)^{\frac{1}{2}} \leq
\]
\[
\leq \| \Omega \| \| u \|_{C^p(\Omega)}^{p-1-m} \| D^i u \|_{C^p(\Omega)}^{s_i} \cdots \| D^i u \|_{C^p(\Omega)}^{s_i} \leq C_0 \| u \|_{W^{k+2,2}(\Omega)}^{p-1} \| u \|_{W^{k+2,2}(\Omega)}^{s_i} \leq C \| u \|_{W^{k+2,2}(\Omega)}^{p-1},
\]

as desired.

By a rather similar argument one can repeat the proof for other cases \textit{i-b, ii-a and ii-b}. 

\[\square\]

As a consequence of Proposition 5.1 we can prove the following Lemma.

**Lemma 5.10.** Let \( f(u(x)) = u(x)^p - 1 \), where \( u \in W^{k+2,2}(\Omega) \) with \( \Omega \) being a bounded domain in \( \mathbb{R}^n \), \( \partial \Omega \in C^{k+2} \) and \( 0 \leq k \leq p - 1 \). Then we have the following inequality

\[
\| u^{p-1} \|_{W^{k,2}(\Omega)} \leq C_p \| u \|_{W^{k+2,2}(\Omega)}^{p-1},
\]

provided that \( 1 < p - 1 \leq (n - 2k)/(n - 2(k + 2)) \) if \( 2(k + 2) < n < 2(k + 2) + 4/(k - 1) \), or \( p > 2 \) if \( n \leq 2(k + 2) \).

**Proof.** To prove the result we estimate \( D^\alpha(u^{p-1}) \), where \( \alpha \) is a multi-index with \( |\alpha| \leq k \). By an easy calculation, we find that for \( |\alpha| = k' \), where \( 0 \leq k' \leq k \), \( D^\alpha(u^{p-1}) \) can be written as a finite sum of terms in the following form

\[
u^{p-1-m'} |D u|^{s_1} |D^2 u|^{s_2} \cdots |D^k u|^{s_k'} \tag{57}
\]

where \( 1 \leq m' \leq k' \) and \( s_1, s_2, \ldots, s_k' \in \{0, 1, \ldots, m'\} \) with \( s_1 + s_2 + \cdots + s_k' = m' \) and \( 1s_1 + 2s_2 + \cdots + k's_k' = k' \). Thus, we just need to prove that each term of the form (57) is being controlled by \( \| u \|_{W^{k+2,2}(\Omega)} \). When \( k' = k \), we have shown in Proposition 5.1 that for \( u \in W^{k+2,2}(\Omega) \), all terms in the form (57) are controlled by \( \| u \|_{W^{k+2,2}(\Omega)} \) as long as \( 2 < p - 1 \leq (n - 2k)/(n - 2(k + 2)) \) if \( 2(k + 2) < n < 2(k + 2) + 4/(k - 1) \), or \( p > 2 \) if \( n \leq 2(k + 2) \).

Following the same steps as in Proposition 5.1, similarly for \( 1 \leq k' \leq k \), it can be proved that for \( |\alpha| = k' \), \( D^\alpha(u^{p-1}) \) is dominated by \( \| u \|_{W^{k+2,2}(\Omega)} \) as long as \( 2 < p - 1 \leq (n - 2k')/(n - 2(k + 2)) \) if \( 2(k + 2) < n < 2(k + 2) + 4/(k - 1) \), or \( p > 2 \) if \( n \leq 2(k + 2) \).

As one can see for \( n \leq 2(k + 2) \), we have the result. However, if \( 2(k + 2) < n < 2(k + 2) + 4/(k - 1) \), we need to choose \( p \) such that for all \( 0 \leq k' \leq k \), the conditions \( 1 < p - 1 \leq (n - 2k')/(n - 2(k + 2)) \) are satisfied. But it is obvious that for \( 0 \leq k' \leq k \), we have

\[
\frac{(n - 2k)}{(n - 2(k + 2))} \leq \frac{(n - 2k')}{(n - 2(k + 2))}
\]

Hence by choosing \( 1 < p - 1 \leq (n - 2k)/(n - 2(k + 2)) \) we have the desired estimate. \[\square\]

**Remark 5.11.** Note that the assumptions on \( p - 1 \) in Proposition 5.1 and Lemma 5.10 are equivalent to the following assumptions on \( p \),

\[\bullet \ 2 < p \leq \frac{2(n-2(k+1))}{n-2(k+2)} \text{ if } 2(k + 2) < n < 2(k + 2) + \frac{4}{k-1}, \text{ or} \]

\[\bullet \ 2 < p \text{ if } n \leq 2(k + 2). \]

Following the same argument as in the proof of Proposition 5.1, we can show an analogous result for \( u \in W^{2,d}(\Omega) \).

**Lemma 5.12.** Let \( f(u(x)) = u(x)^p - 1 \), where \( u \in W^{2,d}(\Omega) \) with \( \Omega \) being a bounded domain in \( \mathbb{R}^n \), \( \partial \Omega \in C^{1,1} \). Then we have

\[
\| u^{p-1} \|_{L^4(\Omega)} \leq C_p \| u \|_{W^{2,d}(\Omega)}^{p-1},
\]

provided that \( 1 < p - 1 \leq n/(n - 2d) \) if \( 2d < n \), or \( p > 2 \) if \( n \leq 2d \).
Proof. We consider the following cases,
i) For $n \leq 2d$: Recalling parts I-A and I-B of Theorem 5.3, we can use the embedding $W^{2,d}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 < q < \infty$. Therefore, we can set $q = d(p - 1)$ to obtain
\[
\left( \int_{\Omega} |u|^{d(p-1)} dx \right)^{\frac{1}{d}} \leq \|u\|_{L^{d(p-1)}(\Omega)}^{p-1} \leq C_p \|u\|_{W^{2,d}(\Omega)}^{p-1},
\]
for an appropriate constant $C_p$.

ii) For $n > 2d$: Using the embedding $W^{2,d}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 < q \leq \frac{dn}{n - 2d}$, which we have from Part I-C of Theorem 5.3, we find that $u \in L^{d(p-1)}(\Omega)$ for $d(p-1) \leq \frac{dn}{n - 2d}$. Hence, for $p - 1 \leq \frac{n}{n - 2d}$ we obtain
\[
\left( \int_{\Omega} |u|^{d(p-1)} dx \right)^{\frac{1}{d}} \leq \|u\|_{L^{d(p-1)}(\Omega)}^{p-1} \leq C_p \|u\|_{W^{2,d}(\Omega)}^{p-1},
\]
as claimed. 

Remark 5.13. The assumptions on $p - 1$ in Lemma 5.12 are equivalent to the following assumptions on $p$,

- $2 < p \leq \frac{(2n-2d)}{n-2d}$ if $2d < n$, or
- $2 < p$ if $n \leq 2d$.

References


[16] M. Struwe, Infinitely many critical points for functionals which are not even and applications to super linear boundary value problems, Manuscripta mathematica, 32 (1980), 335-364.


