A variational approach towards supercritical elliptic systems; Existence and symmetry breaking phenomena *

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Abstract

In this paper, we delve into the investigation of positive solutions for two distinct elliptic systems. The first system is a Hamiltonian system defined as

$$\begin{cases} -\Delta u \;=\; a(x)|v|^{p-2}v, & x \in \Omega \\ -\Delta v \;=\; b(x)u(e^{u^2}-1), & x \in \Omega \\ u, v > 0, & x \in \Omega \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is an annulus in $\mathbb{R}^N (N \geq 3)$, $p > 2$ and the functions $a(x),\ b(x)$ are positive continuous but not necessarily radial. Due to the presence of the exponential term and the fact that we are not imposing any upper bound on p this problem is supercritical, and standard variational methods may not be used. In this work, we show the existence of a positive solution when the problem enjoys certain mild symmetry and monotonicity conditions. We shall also address the symmetry breaking phenomena where the functions a and b are radial. The second problem we are studying in this paper is the supercritical gradient system described by

$$\begin{cases} -\Delta u \;= a(x)\; u(e^{u^2}-1) + \; pu^{p-1}v^q, & x \in \Omega \\ -\Delta v \;= b(x)\; v(e^{v^2}-1) + qv^{q-1}u^p, & x \in \Omega \\ u, v > 0, & x \in \Omega \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

within the same annular region $\Omega \subset \mathbb{R}^N$. Our contribution for this problem is to prove the existence of a positive solution for all $p, q > 2$.

1 Introduction

The study of Hamiltonian systems has received significant attention over the past decades, with numerous researchers contributing to its development and exploration. Researchers have investigated various aspects of Hamiltonian systems, including the existence of solutions, the multiplicity of solutions, concentration phenomena, positivity properties, symmetry-related behavior, and Liouville theorems. For a comprehensive overview of Hamiltonian systems, encompassing key findings and prevailing trends, valuable references include [3,13,30]. For some latest breakthroughs and recent discoveries in this field, one can explore the works in [4,7,8,12,21,26–28,35].

*A.M. is pleased to acknowledge the support of the Natural Sciences and Engineering Research Council of Canada. A.K. and S.M. are grateful to CNPq (401772/2022-5) and Coordenação de Aperfeiçoamento de Pessoal de Nível Superior of Brazil for their support.

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It is noteworthy that strongly indefinite variational problems in elliptic systems of Hamiltonian type exhibit a complex geometric structure, making the investigation of such problems particularly challenging. Dealing with the challenges in these problems entails the fact that the energy functional is strongly indefinite, which differs from the single equation case. To overcome this challenge, various methods come into play, including the dual variational method (see [1,31]), the Orlicz space approach (see [15]), the generalized linking theory (see [20]), and the reduction method (see [2]).

The study of systems of Hardy-Hénon type equations holds particular importance in the context of Hamiltonian systems. Consider the following system of superlinear elliptic equations

$$\begin{cases} -\Delta u = |x|^{\beta} v^{p-1}, & x \in \Omega \\ -\Delta v = |x|^{\alpha} u^{p-1}, & x \in \Omega \\ u, v > 0, & x \in \Omega \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $0 \in \Omega$, $N \geq 3$, $p, q > 2$, and $\alpha, \beta > -N$. For the specific case where $\alpha = \beta = 0$, the existence or non-existence of solutions is determined by the critical hyperbola,

$$\frac{N}{p} + \frac{N}{q} = N - 2.$$

This hyperbola was first introduced by Mitidieri [24], who proved the non-existence of solutions for (p, q) lying on or above it using a Pohozaev-type identity. The existence of solutions for (p, q) below the critical hyperbola was proven by de Figueiredo and Felmer (see [14]) and by Hulshoff and van der Vorst (see [17,18]) through a variational approach employing fractional Sobolev spaces. An alternative approach, working with Sobolev-Orlicz spaces (allowing for generalization to non-polynomial nonlinearities), is discussed in [10, 15]. The general case where $\alpha \neq 0$ and/or $\beta \neq 0$ were carried out in [6, 16, 22]. In particular, Calanchi and Ruf, in [6] investigate existence, multiplicity, and qualitative properties (such as radial symmetry in the case of a ball) of solutions . They explicitly showed that when

$$\frac{N+\alpha}{p} + \frac{N+\beta}{q} \leq N - 2,$$

with $p, q > 2$, there are no positive solutions u and v for problem (4) within the open unit ball B in $\mathbb{R}^N$, where $N \geq 3$. Later, the authors in [3] furthered the investigation of ground state solutions for the system of equations

$$\begin{cases} -\Delta u = |x|^{\beta} |v|^{q-2} v, & x \in B \\ -\Delta v = |x|^{\alpha} |u|^{p-2} u, & x \in B \\ u = v = 0, & x \in \partial B, \end{cases} \tag{4}$$

where B is the open unit ball of $\mathbb{R}^N$, $N \geq 1$, $\alpha, \beta \geq 0$, $p, q > 2$. The authors established that the hyperbola

$$\frac{N+\alpha}{p} + \frac{N+\beta}{q} = N - 2$$

serves as the exact threshold for the existence of positive solutions to (4). Additionally, they provided detailed conditions on the parameters α, β, p, and q that lead to ground state solutions deviating from radial symmetry. It is worth noting that in the Hardy-Hénon system, the terms $|x|^{\alpha}$ and $|x|^{\beta}$ lead to enhanced compactness properties. However, in this specific paper, the functions a and b in the system (1) are considered to be strictly positive and kept away from zero. Consequently, the presence of these terms doesn't contribute to improved compactness in this particular case.

In [23], Lou, Weth, and Zhang considered the Schrödinger-Hénon system

$$\begin{cases} -\Delta u + \mu_1 u = |x|^{\alpha} \partial_u F(u, v), & x \in B \\ -\Delta v + \mu_2 v = |x|^{\alpha} \partial_v F(u, v), & x \in B \\ u = v = 0, & x \in \partial B, \end{cases} \tag{5}$$

where $B \subset \mathbb{R}^N, N \geq 2$ is the unit ball, $\mu_1, \mu_2 \geq 0$, $\alpha > 0$, and $F : \mathbb{R}^2 \to \mathbb{R}$ is a p-homogeneous C^2-function for some $p > 2$ with $F(u, v) > 0$ for $(u, v) \neq (0, 0)$. They showed that, as $\alpha \to \infty$, the Morse index of nontrivial radial solutions of this problem (positive or sign- changing) tends to infinity. Moreover, in [9], Clapp and Soares studied a related problem

$$-\Delta u_i + u_i = \sum_{j=1}^{l} \beta_{ij} |u_j|^p |u_i|^{p-2} u_i, \quad u_i \in H^1(\mathbb{R}^N), \quad i = 1, \ldots, l,$$

where $N \geq 4$, $1 < p < \frac{N}{N-2}$, and (β_{ij}) represents a symmetric matrix admitting a block decomposition with entries either positive or zero within each block and negative for all remaining entries. The authors established the existence of fully nontrivial solutions, meaning nontrivial solutions component-wise, provided certain conditions are satisfied for the symmetric matrix (β_{ij}). Furthermore, the authors derived the existence of solutions with positive and non-radial sign-changing components for the system of singularly perturbed elliptic equations

$$-\epsilon^2 \Delta u_i + u_i = \sum_{j=1}^{l} \beta_{ij} |u_j|^p |u_i|^{p-2} u_i, \quad u_i \in H_0^1(B_1(0)), \quad i = 1, \ldots, l,$$

where $B_1(0)$ is the unit ball exhibiting two different kinds of asymptotic behavior. The first being solutions whose components decouple as $\epsilon \to 0$, while the second behavior is for solutions whose components remain coupled up to their limit.
In [19] we examined the Neumann problem given by

$$\begin{cases} -\Delta u + u = f(u), & x \in \Omega \\ u > 0, & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{6}$$

where Ω is a bounded annulus in $\mathbb{R}^N (N \geq 3)$ and the function f has either the exponential growth by the means of Trudinger-Moser inequality $f(u) = u(e^{u^2} - 1)$, or is of the power form $f(u) = u|u|^{p-2}$ where p is supercritical. We have demonstrated the existence of a positive non-radial solution in the case of $f(u) = u(e^{u^2} - 1)$ and established the multiplicity of non-radial positive solutions for the case $f(u) = u|u|^{p-2}$ when the annulus is thin.

In our study, we focus on the annular domain Ω with center at the origin, inner radius R_1, and outer radius R_2. The domain Ω is defined as

$$\Omega = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}.$$

Inspired by the work [5], we introduce the variables s and t as

$$s := \sqrt{x_1^2 + \cdots + x_m^2} \quad , \quad t := \sqrt{x_{m+1}^2 + \cdots + x_N^2}. \tag{7}$$

where m and n are positive integers such that $m + n = N$. Using these definitions, we can express Ω as $\Omega = \{x \in \mathbb{R}^N : R_1^2 < s^2 + t^2 < R_2^2\}$. We refer to the subset $\widehat{\Omega}$ of $\mathbb{R}^2$ as

$$\widehat{\Omega} = \{(s, t) \in \mathbb{R}^2 : s > 0,\ t > 0,\ R_1^2 < s^2 + t^2 < R_2^2\}.$$

Throughout the paper, we make the assumption that the function a (respectively, b) is continuous and strictly positive as a function of (s, t) with respect to $m = N - 1$ and $n = 1$, i.e., $a(x) = a(s, t)$. We also introduce the notation that a (resp. b) satisfies condition $(\mathcal{A})$ if it is continuously differentiable with respect to (s, t) and satisfies the inequality $sa_t - ta_s \leq 0$ (resp. $sb_t - tb_s \leq 0$) in $\widehat{\Omega}$.
Our main results related to the system (1) are stated in the following two theorems.

Theorem 1.1 *Suppose Ω is an annular domain in $\mathbb{R}^N$ for $N \geq 3$. Assume that a and b satisfy $(\mathcal{A})$ with respect to $m = N - 1$ and $n = 1$. Then equation (1) has a positive weak solution (u, v).*

Directing our attention to a specific variant of equation (1), we consider

$$\begin{cases} -\Delta u \;=\; |v|^{p-2}v, & x \in \Omega \\ -\Delta v \;=\; u(e^{u^2}-1), & x \in \Omega \\ u, v > 0, & x \in \Omega \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{8}$$

where the conditions in problem (1) are carried over to problem (8), except for the specific values of a and b, both of which are set to be 1. In the following theorem, we demonstrate that the solution of (8) derived from Theorem 1.1 is non-radial.

Theorem 1.2 *Assume $p > 2$. Suppose (u, v) is the solution of (8) obtained in Theorem 1.1. If*

$$(1 + \frac{2N}{\beta})^2 < \frac{p(p-1)}{2},$$

where

$$\beta = \inf_{\psi \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla \psi|^2 \, dx}{\int_\Omega \frac{\psi^2}{|x|^2} \, dx}.$$

Then (u, v) is non-radial.

Observe that β denotes the optimal constant in the classical Hardy inequality on the domain Ω, which is attained since Ω does not contain the origin and is not an exterior domain.
In the context of system (2), our primary contribution lies in the demonstration of the following theorem

Theorem 1.3 *Suppose Ω is an annular domain in $\mathbb{R}^N$ for $N \geq 3$. Assume that a and b satisfy $(\mathcal{A})$ with respect to $m = N - 1$ and $n = 1$, and $p,\ q > 2$. Then equation (2) has a positive weak solution (u, v).*

The paper is structured as follows. In Section 2, we focus on Hamilton system (1) and non-radial solutions, exploring our primary results. In Section 3, our attention shifts to a gradient system (2), where we conduct a thorough examination. Our approach utilizes a variational formulation on convex closed subsets of an appropriate Sobolev space, which plays a crucial role in establishing the key results presented in the paper.

2 Hamilton systems

2.1 A Variational Approach to the Existence of Solutions

This section is devoted to demonstrating the existence of solutions for

$$\begin{cases} -\Delta u \;=\; a(x)|v|^{p-2}v, & x \in \Omega \\ -\Delta v \;=\; b(x)u(e^{u^2}-1), & x \in \Omega \\ u, v > 0, & x \in \Omega \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{9}$$

where Ω is an annulus in $\mathbb{R}^N (N \geq 3)$ and $a(x), b(x)$ are sufficiently smooth positive functions that satisfy condition $(\mathcal{A})$.
To begin, we will briefly review some standard notation used in the theory of Orlicz spaces.

Definition 2.1 *Let $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex increasing function satisfying*

$$\xi(0) = 0 = \lim_{s \to 0^+} \xi(s), \qquad \lim_{s \to \infty} \xi(s) = \infty.$$

We say that a measurable function $u : \mathbb{R}^d \to \mathbb{C}$ belongs to L^ξ if there exists $\lambda > 0$ such that

$$\int_{\mathbb{R}^d} \xi\left(\frac{|u(x)|}{\lambda}\right) dx < \infty.$$

We denote then

$$\|u\|_{L^\xi} = \inf\left\{\lambda > 0; \int_{\mathbb{R}^d} \xi\left(\frac{|u(x)|}{\lambda}\right) dx \le 1\right\}. \tag{10}$$

It is standard that $\|\cdot\|_{L^\xi}$ is a norm. In what follows, we shall fix $\xi(s) = e^{s^2} - 1$ and denote the Orlicz space L^ξ by $\mathcal{L}$ endowed with the norm $\|\cdot\|_{\mathcal{L}}$.
While our results specifically pertain to domains in $\mathbb{R}^N$ with $N \ge 3$, it is important to mention that the Sobolev embedding for bounded domains in the two-dimensional case ($N = 2$) and the Orlicz space $\mathcal{L}$ can be stated as follows.

Lemma 2.2 *Suppose $\mathcal{O}$ is a bounded domain in $\mathbb{R}^2$. Then*

$$\|u\|_{\mathcal{L}(\mathcal{O})} \le \frac{1}{\sqrt{4\pi}}\|u\|_{H^1(\mathcal{O})}. \tag{11}$$

It is worth noting that the embedding (11) can be directly derived from the Trudinger-Moser inequality, which was proven in [29].

Proposition 2.3 *There exists a constant κ such that for any domain $\mathcal{O} \subset \mathbb{R}^2$*

$$\sup_{\|u\|_{H^1(\mathcal{O})}\le 1} \int_{\mathcal{O}} (e^{4\pi u^2} - 1)\, dx \le \kappa. \tag{12}$$

The inequality is sharp: for any growth $e^{\alpha u^2}$ with $\alpha > 4\pi$ the supremum is $+\infty$.

Let us define $p' = \frac{p}{p-1}$ and consider the Banach space $V = W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega) \cap \mathcal{L}$. The norm on V is given by

$$\|u\| = \|u\|_{W^{2,p'}(\Omega)} + \|u\|_{W_0^{1,p'}(\Omega)} + \|u\|_{\mathcal{L}}.$$

For any u in V and any u^* in V^*, the duality pairing is denoted by $\langle u, u^*\rangle$ and defined as

$$\langle u, u^*\rangle = \int_\Omega u(x)u^*(x)\, dx, \qquad \forall\, u \in V,\ \forall\, u \in V^*.$$

Taking inspiration from the ideas presented in Wang [34], we can deduce from equation (9) that

$$v = (-\Delta u)| - \Delta u|^{p'-2} a(x)^{p'-1}.$$

By substituting the expression for v into the second equation of (9), we obtain a scalar equation that corresponds to

$$-\Delta\Big((-\Delta u)| - \Delta u|^{p'-2} a(x)^{p'-1}\Big) = b(x)u(e^{u^2} - 1). \tag{13}$$

Therefore, the functional associated with problem (9) can be expressed using the Euler-Lagrange formulation as

$$I(u) = \frac{1}{p'}\int_\Omega \frac{|-\Delta u|^{p'}}{a(x)^{p'-1}}\, dx - \frac{1}{2}\int_\Omega b(x)(e^{u^2} - u^2 - 1)\, dx.$$

By introducing the functionals $\Psi : V \to \mathbb{R}$ and $\Phi : V \to \mathbb{R}$ defined by

$$\Psi = \frac{1}{p'}\int_\Omega \frac{|-\Delta u|^{p'}}{a(x)^{p'-1}}\, dx$$

and

$$\Phi = \frac{1}{2}\int_\Omega b(x)(e^{u^2} - u^2 - 1)\, dx,$$

respectively, we can express the functional I as $I = \Psi - \Phi$. To enhance compactness, we introduce a convex set K by

$$K = K(m,n) = \{0 < u = u(s,t) \in W_G^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega) : su_t - tu_s \leq 0 \ \ a.e. \ in \ \widehat{\Omega}\}, \tag{14}$$

Here, $W_G^{2,p'}(\Omega)$ denotes the space of functions in $W^{2,p'}(\Omega)$ that are invariant under the action of $G = O(m) \times O(n)$, where $O(k)$ is the orthogonal group in $\mathbb{R}^k$ and $gu(x) := u(g^{-1}x)$. Note that we can express (s,t) in terms of polar coordinates as $s = r\cos(\theta)$, $t = r\sin(\theta)$, where $r = |x| = |(s,t)|$ and θ is the usual polar angle in the (s,t) plane. Using this representation, we can rewrite the set K as a set of functions u that satisfy the inequality $u_\theta \leq 0$ in $\widetilde{\Omega} = \{(\theta, r) : R_1 < r < R_2, \theta \in (0, \frac{\pi}{2})\}$. Let us now introduce a functional $I_K(u) : V \to (-\infty, +\infty]$ by

$$I_K = \Psi_K - \Phi, \tag{15}$$

where Ψ_K is the restriction of Ψ to K defined by

$$\Psi_K(u) = \begin{cases} \Psi(u), & u \in K \\ +\infty, & u \notin K. \end{cases}$$

We will now review the definition of a critical point for lower semi-continuous functions, which was introduced by Szulkin [32].

Definition 2.4 *Let V be a real Banach space and $\Psi : V \to (-\infty, +\infty]$ be proper, convex and lower semi-continuous. Let E be a function on V defined by*

$$E := \Psi - \Phi, \tag{16}$$

where $\Phi \in C^1(V, \mathbb{R})$. A point $u_0 \in V$ is said to be a critical point of E if $u \in Dom(\Psi)$ and if it satisfies the inequality

$$\langle D\Phi(u), u - v\rangle + \Psi(v) - \Psi(u) \geq 0 \ , \quad \forall v \in V.$$

Definition 2.5 *We say that E defined in* (16) *satisfies the Palais–Smale compactness condition (PS) if every sequence u_j such that*

- $E[u_j] \to c \in \mathbb{R}$,
- $\langle D\Phi(u_j), u_j - v\rangle + \Psi(v) - \Psi(u_j) \geq -\epsilon_j \|v - u_j\|, \quad \forall v \in V$,

where $\epsilon_j \to 0$, then $\{u_j\}$ possesses a convergent subsequence.

The following theorem by A. Szulkin [32] is a very useful result called the Mountain Pass Theorem.

Theorem 2.6 *Suppose that $E : V \to (-\infty, +\infty]$ is of the form* (16) *and satisfies the Palais-Smale condition and the Mountain Pass Geometry (MPG):*

1. *$E(0) = 0$.*
2. *There exists $e \in V$ such that $E(e) \leq 0$.*
3. *There exists some ρ such that $0 < \rho <\|e\|$ and for every $u \in V$ with $\|u\| = \rho$ one has $E(u) > 0$.*

Then E has a critical value $c > 0$ which is characterized by

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} E(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], V) : \gamma(0) = 0, \gamma(1) = e\}$.

In the proof of the upcoming theorem, we will rely on the utilization of the following lemma.

Lemma 2.7 *Let H be a reflexive Banach space, and let $f : H \to \mathbb{R}$ be a convex and differentiable functional. If*

$$f(u) - f(\bar{u}) \geq \langle Df(u), u - \bar{u} \rangle, \tag{17}$$

then $Df(u) = Df(\bar{u})$, where $\langle \cdot, \cdot \rangle$ is the duality pairing between H and H^*. In particular, if f is strictly convex, then $u = \bar{u}$.

Proof: Considering the convexity of f, we have

$$f(\bar{u}) - f(u) \geq \langle Df(u), \bar{u} - u \rangle,$$

which can be rearranged as

$$f(u) - f(\bar{u}) \leq \langle Df(u), u - \bar{u} \rangle.$$

This, combined with the inequality (17), leads to

$$f(u) - f(\bar{u}) = \langle Df(u), u - \bar{u} \rangle. \tag{18}$$

Furthermore, for any $v \in H$, we have

$$f(v) - f(u) \geq \langle Df(u), v - u \rangle,$$

or equivalently,

$$f(v) - \langle Df(u), v \rangle \geq f(u) - \langle Df(u), u \rangle.$$

Now, let us define the function $G(v) = f(v) - \langle Df(u), v \rangle$. It follows that $G(v) \geq G(u)$ for all $v \in H$. Additionally, when $v = \bar{u}$, we have from (18) that

$$G(\bar{u}) = f(\bar{u}) - \langle Df(u), \bar{u} \rangle = f(u) - \langle Df(u), u \rangle = G(u),$$

which means G attains its minimum at $v = \bar{u}$, i.e., $DG(\bar{u}) = 0$. Thus,

$$Df(\bar{u}) - Df(u) = 0.$$

In the case where f is strictly convex, the equation

$$\langle Df(u) - Df(\bar{u}), u - \bar{u} \rangle = 0,$$

implies that $u = \bar{u}$. This result directly leads us to the desired conclusion. □

Motivated by the variational principle introduced in [25], the following Theorem establishes a connection between the critical points of I_K and the solutions of the system (9).

Theorem 2.8 *Let $\bar{u}$ be a critical point of the functional I_K. If there exists $\tilde{u} \in K$ and $\tilde{v} \in V$, such that*

$$\begin{cases} -\Delta \tilde{u} = & a(x)|\tilde{v}|^{p-2}\tilde{v} \\ -\Delta \tilde{v} = & b(x)\bar{u}(e^{\bar{u}^2} - 1), \end{cases} \tag{19}$$

then $(\tilde{u}, \tilde{v})$ is a solution of

$$\begin{cases} -\Delta u = & a(x)|v|^{p-2}v \\ -\Delta v = & b(x)u(e^{u^2} - 1). \end{cases}$$

Proof: Since $\bar{u}$ is a critical point of I_K, it follows from Definition 2.4 that

$$\langle D\Phi(\bar{u}), \bar{u} - v \rangle + \Psi(v) - \Psi(\bar{u}) \geq 0\ , \quad \forall v \in V,$$

which implies

$$\frac{1}{p'}\int_\Omega \frac{|-\Delta w|^{p'}}{a(x)^{p'-1}}\,dx - \frac{1}{p'}\int_\Omega \frac{|-\Delta \bar{u}|^{p'}}{a(x)^{p'-1}}\,dx \geq \langle b(x)\bar{u}(e^{\bar{u}^2}-1), w-\bar{u}\rangle, \quad \forall w \in K. \tag{20}$$

Let $F : V \to \mathbb{R}$ be defined by

$$F(w) = \frac{1}{p'}\int_\Omega \frac{|-\Delta w|^{p'}}{a(x)^{p'-1}}\,dx - \frac{1}{2}\int_\Omega b(x)\bar{u}(e^{\bar{u}^2}-1)w\,dx.$$

From the first equation in (19), we have

$$\tilde{v} = \frac{1}{a(x)^{p'-1}}|\Delta\tilde{u}|^{p'-2}(-\Delta\tilde{u}).$$

Consider $\eta \in V$. We can then evaluate

$$\begin{aligned}
\langle F'(\tilde{u}), \eta\rangle &= \int_\Omega \frac{1}{a(x)^{p'-1}}|\Delta\tilde{u}|^{p'-2}(-\Delta\tilde{u})(-\Delta\eta)\,dx - \int_\Omega b(x)\bar{u}(e^{\bar{u}^2}-1)\eta\,dx \\
&= \int_\Omega \tilde{v}(-\Delta\eta)\,dx - \int_\Omega b(x)\bar{u}(e^{\bar{u}^2}-1)\eta\,dx \\
&= \int_\Omega (-\Delta\tilde{v})\eta\,dx - \int_\Omega b(x)\bar{u}(e^{\bar{u}^2}-1)\eta\,dx \\
&= \int_\Omega b(x)\bar{u}(e^{\bar{u}^2}-1)\eta\,dx - \int_\Omega b(x)\bar{u}(e^{\bar{u}^2}-1)\eta\,dx \\
&= 0,
\end{aligned}$$

which implies that $\tilde{u}$ is a critical point of F. It then follows that

$$0 = \langle F'(\tilde{u}), \tilde{u}-\bar{u}\rangle = \int_\Omega \frac{1}{a(x)^{p'-1}}|\Delta\tilde{u}|^{p'-2}(-\Delta\tilde{u})(-\Delta(\tilde{u}-\bar{u}))\,dx - \int_\Omega b(x)\bar{u}(e^{\bar{u}^2}-1)\eta\,dx,$$

from which we obtain

$$\int_\Omega \frac{1}{a(x)^{p'-1}}|\Delta\tilde{u}|^{p'-2}(-\Delta\tilde{u})(-\Delta(\tilde{u}-\bar{u}))\,dx = \int_\Omega b(x)\bar{u}(e^{\bar{u}^2}-1)(\tilde{u}-\bar{u})\,dx.$$

Combining this with (20) for $w = \tilde{u}$, implies

$$\frac{1}{p'}\int_\Omega \frac{|\Delta\tilde{u}|}{a(x)^{p'-1}}\,dx - \frac{1}{p'}\int_\Omega \frac{|\Delta\bar{u}|}{a(x)^{p'-1}}\,dx \geq \int_\Omega \frac{1}{a(x)^{p'-1}}|\Delta\tilde{u}|^{p'-2}(-\Delta\tilde{u})(-\Delta(\tilde{u}-\bar{u}))\,dx.$$

By applying Lemma 2.7, we conclude that $\bar{u} = \bar{v}$. Considering that $\bar{u} = \bar{v}$ in (19), we obtain the desired result. □

To prove the main result in this section, we will employ Theorem 2.8 as a crucial element of our approach. To verify condition (i) in this theorem and prove the existence of a critical point for the nonsmooth functional I_K, we will initiate our efforts by establishing the following theorems. It is important to note that C will represent a positive constant throughout, which may vary and need not remain constant across different contexts.

Theorem 2.9 *For $N \geq 3$, let Ω be an annular domain in $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$, where $n = 1$ and $m = N-1$. Then for every $\alpha \leq 4\pi$, we have*

$$\sup_{u\in K,\ \|u\|_{H^1(\Omega)}\leq 1} \int_\Omega (e^{\alpha u^2}-1)\,dx < \infty, \tag{21}$$

where $K = K(N-1, 1)$ is a convex and closed subset of $H^1(\Omega)$ defined in (14).

Proof: Firstly, we may rewrite the given integral in terms of polar coordinates $(s,t) = (r\cos\theta, r\sin\theta)$. Thus, for any $u = u(s,t) \in K$, we obtain

$$\int_\Omega (e^{\alpha u^2} - 1)\, dx = C \int_{\widehat{\Omega}} (e^{\alpha u(s,t)^2} - 1) s^{N-2}\, ds\, dt = C \int_0^{\frac{\pi}{2}} \int_{R_1}^{R_2} r^{N-2} \cos^{N-2}(\theta)(e^{\alpha u(r,\theta)^2} - 1)\, r\, dr\, d\theta.$$

If we choose $\theta \in [\frac{\pi}{3}, \frac{\pi}{2}]$, since $\theta \to e^{\alpha u(r,\theta)^2}$ is monotone, we obtain that

$$\begin{aligned}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_{R_1}^{R_2} r^{N-2} \cos^{N-2}(\theta)(e^{\alpha u(r,\theta)^2} - 1)\, r\, dr\, d\theta &\leq \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_{R_1}^{R_2} r^{N-2} \cos^{N-2}(\theta - \frac{\pi}{4})(e^{\alpha u(r,\theta-\frac{\pi}{4})^2} - 1)\, r\, dr\, d\theta \\ &\leq \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \int_{R_1}^{R_2} r^{N-2} \cos^{N-2}(\theta)(e^{\alpha u(r,\theta)^2} - 1)\, r\, dr\, d\theta,\end{aligned}$$

and therefore

$$\int_0^{\frac{\pi}{2}} \int_{R_1}^{R_2} r^{N-2} \cos^{N-2}(\theta)(e^{\alpha u(r,\theta)^2} - 1)\, r\, dr\, d\theta \leq 2 \int_0^{\frac{\pi}{3}} \int_{R_1}^{R_2} r^{N-2} \cos^{N-2}(\theta)(e^{\alpha u(r,\theta)^2} - 1)\, r\, dr\, d\theta.$$

On the other hand,

$$\int_0^{\frac{\pi}{3}} \int_{R_1}^{R_2} r^{N-2} \cos^{N-2}(\theta)(e^{\alpha u(r,\theta)^2} - 1)\, r\, dr\, d\theta = \int_{\{\widehat{\Omega}, s \geq \delta\}} (e^{\alpha u(s,t)^2} - 1) s^{N-2}\, ds\, dt, \tag{22}$$

for some positive constant $\delta > 0$. By setting $\mathcal{O} = \{\widehat{\Omega}, s \geq \delta\}$ we have that

$$\begin{aligned} 1 \geq \|u\|^2_{H^1(\Omega)} &= C(m,n) \int_{\widehat{\Omega}} (u_t^2 + u_s^2 + u^2) s^{N-2}\, ds\, dt \\ &\geq C(m,n) \int_{\mathcal{O}} (u_t^2 + u_s^2 + u^2) s^{N-2}\, ds\, dt \\ &\geq C(m,n)\delta^{N-2} \int_{\mathcal{O}} (u_t^2 + u_s^2 + u^2)\, ds\, dt. \end{aligned}$$

Therefore, we have

$$\int_{\mathcal{O}} (u_t^2 + u_s^2 + u^2)\, ds\, dt \leq \frac{1}{C(m,n)\delta^{N-2}}. \tag{23}$$

Looking at the term on the right hand side of (22), and applying Proposition 2.3, we have

$$\int_{\{\widehat{\Omega}, s \geq \delta\}} (e^{\alpha u(s,t)^2} - 1) s^{N-2}\, ds\, dt \leq C \int_{\mathcal{O}} (e^{\alpha u(s,t)^2} - 1)\, ds\, dt < \infty,$$

due to the inequality (23). This completes the proof. □

Remark 2.10 *We would like to highlight that the theorem mentioned above leads to*

$$\|u\|_{\mathcal{L}} \leq \frac{1}{\sqrt{4\pi}} \|u\|_{H^1\Omega)}, \quad \forall u \in K.$$

Also for every $u \in K$ we have

$$\begin{aligned}
\|u\|^2_{H^1(\Omega)} &= C\int_{\widehat{\Omega}} (u_t^2 + u_s^2 + u^2) s^{N-2}\, ds\, dt \\
&\le C\int_{\mathcal{O}} (u_t^2 + u_s^2 + u^2)\, ds\, dt \\
&\le C\Big(\int_{\mathcal{O}} \Big(|D^2u(s,t)|^{p'} + |\nabla u(s,t)|^{p'} + |u(s,t)|^{p'}\Big)\ ds\ dt\Big)^{\frac{2}{p'}} \\
&\le C\Big(\int_{\widehat{\Omega}} \Big(|D^2u(s,t)|^{p'} + |\nabla u(s,t)|^{p'} + |u(s,t)|^{p'}\Big)\ s^{m-1} ds\ dt\Big)^{\frac{2}{p'}} \\
&\le C\Big(\int_{\Omega} \Big(|D^2u|^{p'} + |\nabla u|^{p'} + |u|^{p'}\Big)\ dx\Big)^{\frac{2}{p'}} \\
&= C\|u\|^2_{W^{2,p'}(\Omega)}.
\end{aligned}$$

Consequently, we can conclude that

$$\|u\|_{W^{2,p'}} \le \|u\|_V \le C\|u\|_{W^{2,p'}}, \quad \forall u \in K, \tag{24}$$

for some constant $C > 0$.

Within the following theorem, we establish the embedding for annular domains.

Theorem 2.11 *Assume that Ω is an annulus domain in $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$, where $n \le m$. Suppose K is a convex and closed subset of V defined in* (14) *and*

$$1 \le d < \frac{(n+1)p'}{(n+1) - 2p'}.$$

Then the imbedding $K \subset L^d(\Omega)$ is compact with the obvious interpretation if $(n+1) - 2p' \le 0$.

Proof: We will establish the desired result by demonstrating that the inequality $\|u\|_{L^d} \le C\|u\|_{W^{2,p'}}$ holds for all $u \in K$, which is a sufficient condition. We write (s,t) in terms of polar coordinates $s = r\cos\theta$ and $t = r\sin\theta$. Then for $u = u(s,t)$, we have

$$\int_{\widehat{\Omega}} |u(s,t)|^d s^{m-1} t^{n-1}\ ds\ dt = \int_0^{\frac{\pi}{2}} \int_{R_1}^{R_2} r^{m-1}\cos^{m-1}(\theta) r^{n-1} \sin^{n-1}(\theta) u(r,\theta)^d\ r\ dr\ d\theta.$$

If we choose $\theta \in [\frac{\pi}{3}, \frac{\pi}{2}]$ we see that there exist some constant C such that $\sin\theta \le C\sin(\theta - \frac{\pi}{4})$. Since $\theta \to u(r,\theta)$ is monotone, we obtain that

$$\begin{aligned}
&\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_{R_1}^{R_2} r^{m-1}\cos^{m-1}(\theta) r^{n-1} \sin^{n-1}(\theta) u(r,\theta)^d\ r\ dr\ d\theta \\
&\le C\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_{R_1}^{R_2} r^{m-1}\cos^{m-1}(\theta - \frac{\pi}{4}) r^{n-1} \sin^{n-1}(\theta - \frac{\pi}{4}) u(r,\theta - \frac{\pi}{4})^d\ r\ dr\ d\theta \\
&\le C\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \int_{R_1}^{R_2} r^{m-1}\cos^{m-1}(\theta) r^{n-1} \sin^{n-1}(\theta) u(r,\theta)^d\ r\ dr\ d\theta,
\end{aligned}$$

and therefore

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \int_{R_1}^{R_2} r^{m-1}\cos^{m-1}(\theta) r^{n-1} \sin^{n-1}(\theta) u(r,\theta)^d\ r\ dr\ d\theta \\
&\le C\int_0^{\frac{\pi}{3}} \int_{R_1}^{R_2} r^{m-1}\cos^{m-1}(\theta) r^{n-1} \sin^{n-1}(\theta) u(r,\theta)^d\ r\ dr\ d\theta.
\end{aligned}$$

On the other hand,

$$\int_0^{\frac{\pi}{3}} \int_{R_1}^{R_2} r^{m-1} \cos^{m-1}(\theta) r^{n-1} \sin^{n-1}(\theta) u(r,\theta)^d \, r \, dr \, d\theta = \int_{\{\widehat{\Omega}, s\geq\delta\}} |u(s,t)|^d s^{m-1} t^{n-1} \, ds \, dt,$$

for some positive constant $\delta > 0$. We can bound the right hand side above by

$$\int_{\{\widehat{\Omega}, s\geq\delta\}} |u(s,t)|^d s^{m-1} t^{n-1} \, ds \, dt \leq C \int_{\{\widehat{\Omega}, s\geq\delta\}} |u(s,t)|^d t^{n-1} \, ds \, dt.$$

Then by a change of variable $t = |y|$ we obtain

$$\int_{\{\widehat{\Omega}, s\geq\delta\}} |u(s,t)|^d t^{n-1} \, ds \, dt = \int_{\{\Omega_1, s\geq\delta\}} |u(s,y)|^d \, ds \, dy,$$

where $\Omega_1 = \{(s,y) : (s,|y|) \in \widehat{\Omega}\} \subset \mathbb{R}^{n+1}$. If $d < \dfrac{(n+1)p'}{(n+1)-2p'}$ then

$$\begin{aligned}
\Big(\int_{\{\Omega_1, s\geq\delta\}} |u(s,y)|^d \, ds \, dy\Big)^{\frac{p'}{d}} &\leq C \int_{\{\Omega_1, s\geq\delta\}} \left(|D^2 u(s,t)|^{p'} + |\nabla u(s,t)|^{p'} + |u(s,t)|^{p'}\right) \, ds \, dy \\
&\leq C \int_{\{\widehat{\Omega}, s\geq\delta\}} \left(|D^2 u(s,t)|^{p'} + |\nabla u(s,t)|^{p'} + |u(s,t)|^{p'}\right) t^{n-1} ds \, dt \\
&\leq C \int_{\widehat{\Omega}} \left(|D^2 u(s,t)|^{p'} + |\nabla u(s,t)|^{p'} + |u(s,t)|^{p'}\right) t^{n-1} s^{m-1} ds \, dt \\
&\leq C \int_{\Omega} \left(|D^2 u|^{p'} + |\nabla u|^{p'} + |u|^{p'}\right) t^{n-1} \, dx \\
&= C \|u\|^{p'}_{W^{2,p'}(\Omega)}.
\end{aligned}$$

This completes the proof. □

The following proposition will demonstrate the existence of a critical point for the functional I_K.

Proposition 2.12 *Let Ω be a domain in $\mathbb{R}^N$ and $p > 2$. Let $a, b \in C(\overline{\Omega})$ with $a(x) \geq a_0 > 0$ and $b(x) \geq b_0 > 0$, where a_0 and b_0 are constants. Consider the Euler-Lagrange functional $I : V \to \mathbb{R}$ associated to problem* (13),

$$I(u) = \frac{1}{p'} \int_\Omega \frac{|-\Delta u|^{p'}}{a(x)^{p'-1}} \, dx - \frac{1}{2} \int_\Omega b(x)(e^{u^2} - u^2 - 1) \, dx.$$

Let $K = K(N-1, 1)$ be a weakly closed convex subset defined in (14)*. Then the functional I has a critical point $\overline{u}$ on K by means of Definition 2.4.*

Proof: We shall show the functional I_K defined in (15) fulfills both the mountain pass geometry and (PS) compactness condition. Suppose that u_j is a sequence in K such that $I_K(u_j) \to c \in \mathbb{R}, \epsilon_j \to 0$ and

$$\langle D\Phi(u_j), u_j - v\rangle - \Psi(v) - \Psi(u_j) \geq -\epsilon_j \|v - u_j\|_V \quad \forall v \in V. \tag{25}$$

Replacing v by ru_j ($r \in \mathbb{R}$) in (25), it becomes

$$\frac{1 - r^{p'}}{p'} \|u_j\|^{p'}_{W^{2,p'}(\Omega)} + (r-1) \int_\Omega b(x) u_j^2 (e^{{u_j}^2} - 1) \, dx \leq \epsilon_j (r-1) \|u_j\|_V. \tag{26}$$

On the other hand, since $I_K(u_j) \to c$, we have

$$\frac{1}{p'} \|u_j\|^{p'}_{W^{2,p'}(\Omega)} - \frac{1}{2} \int_\Omega b(x)(e^{{u_j}^2} - u_j^2 - 1) \, dx \leq c + 1, \tag{27}$$

for large values of j. Now set $1<r$ and $r^2-1<4(r-1)$. We can take $\alpha>0$ such that

$$\frac{1}{4(r-1)}<\alpha<\frac{1}{r^2-1}.$$

Multiply (26) by α and adding up with (27)) yields that

$$\begin{aligned}&\frac{1+\alpha(1-r^{p'})}{p'}\|u\|^{p'}_{W^{2,p'}(\Omega)}+\alpha(r-1)\int_\Omega b(x)u_j^2(e^{{u_j}^2}-1)\,dx-\frac{1}{2}\int_\Omega b(x)(e^{{u_j}^2}-u_j^2-1)\,dx\\&\le c+1+\alpha\epsilon_j(r-1)\|u_j\|_V.\end{aligned}\tag{28}$$

The choice of α implies that

$$\begin{aligned}&\frac{1+\alpha(1-r^{p'})}{p'}\|u\|^{p'}_{W^{2,p'}(\Omega)}+\frac{1}{4}\int_\Omega b(x)u_j^2(e^{{u_j}^2}-1)\,dx-\frac{1}{2}\int_\Omega b(x)(e^{{u_j}^2}-u_j^2-1)\,dx\\&\le\frac{1+\alpha(1-r^{p'})}{p'}\|u\|^{p'}_{W^{2,p'}(\Omega)}+\alpha(r-1)\int_\Omega b(x)u_j^2(e^{{u_j}^2}-1)\,dx-\frac{1}{2}\int_\Omega b(x)(e^{{u_j}^2}-u_j^2-1)\,dx.\end{aligned}\tag{29}$$

But we also have
$$\frac{1}{4}\int_\Omega a(x)u_j^2(e^{{u_j}^2}-1)\,dx-\frac{1}{2}\int_\Omega b(x)(e^{{u_j}^2}-u_j^2-1)\,dx\ge 0,$$

which, together with (28) and (29), gives

$$\frac{1+\alpha(1-r^{p'})}{p'}\|u\|^{p'}_{W^{2,p'}(\Omega)}\le c+1+\alpha\epsilon_j(r-1)\|u_j\|_{W^{2,p'}(\Omega)}.\tag{30}$$

Since all the coefficients on the left-hand side of the inequality are positive due to the choice of α, we can conclude that

$$\|u\|^{p'}_{W^{2,p'}(\Omega)}\le C(1+\|u_j\|_{W^{2,p'}(\Omega)}),\tag{31}$$

for some constant $C>0$. Standard results in Sobolev spaces allow us to conclude, after possibly passing to a subsequence, that there exists a function $\bar u\in W^{2,p'}(\Omega)$ such that $u_j\rightharpoonup\bar u$ weakly in $W^{2,p'}(\Omega)$. This, in turn, implies that $u_j\to\bar u$ strongly in $L^2(\Omega)$. By setting $v=\bar u$ in (25), and using Hölder's inequality we obtain

$$\begin{aligned}\frac{1}{p'}(\|u_j\|^{p'}_{W^{2,p'}(\Omega)}-\|\bar u\|^{p'}_{W^{2,p'}(\Omega)})&\le\int_\Omega a(x)u_j(e^{{u_j}^2}-1)(\bar u-u_j)\,dx+\epsilon_j\|u_j-\bar u\|_{W^{2,p'}(\Omega)}\\&\le\|a(x)\|_{L^\infty}\|u_je^{u_j^2}-u_j\|_{L^2}\|u_j-\bar u\|_{L^2}+\epsilon_j\|u_j-\bar u\|_{W^{2,p'}(\Omega)}.\end{aligned}\tag{32}$$

Furthermore, since $u_j\in K$, we can apply the concentration compactness principle for the Trudinger-Moser inequality in $H^1(\Omega)$, as presented in Theorem 2.9, along with the continuous embedding provided in Theorem 2.11, to establish that

$$\sup_{j\ge1}\int_\Omega u_j^2(e^{{u_j}^2}-1)^2\,dx<\infty.$$

Hence, from (32), we can conclude that

$$\limsup_{j\rightharpoonup\infty}(\|u_j\|^{p'}_{W^{2,p'}(\Omega)}-\|\bar u\|^{p'}_{W^{2,p'}(\Omega)})\le 0.\tag{33}$$

Using the properties of weak convergence, we also have

$$0\le\liminf(\|u_j\|^{p'}_{W^{2,p'}(\Omega)}-\|\bar u\|^{p'}_{W^{2,p'}(\Omega)}),$$

which together with (33) implies that $u_j \to \bar{u}$ strongly in $W^{2,p'}(\Omega)$. This completes the proof of the (PS) compactness condition for the function I_K. We now verify the mountain pass geometry of the functional I_K. It is clear that $I_K(0) = 0$. Take $w \in K$. Then, for any $\lambda > 0$, we have

$$I_K(\lambda w) = \frac{\lambda^{p'}}{p'} \int_\Omega \frac{|-\Delta u|^{p'}}{a(x)^{p'-1}} \, dx - \frac{1}{2} \int_\Omega b(x)(e^{\lambda^2 w^2} - \lambda^2 w^2 - 1) \, dx.$$

It is now obvious that $I_K(\lambda w) < 0$ for λ sufficiently large. Take $u \in K$ with $\|u\|_{H_0^1} = \rho > 0$. We have

$$\begin{aligned} I_K(u) &= \frac{1}{p'} \int_\Omega \frac{|-\Delta u|^{p'}}{a(x)^{p'-1}} \, dx - \frac{1}{2} \int_\Omega b(x)(e^{u^2} - u^2 - 1) \, dx \\ &\geq \frac{1}{p'} \|u\|^{p'}_{W^{2,p'}(\Omega)} - \frac{1}{2}\nu \int_\Omega \left[e^{u^2} - u^2 - 1\right] dx, \end{aligned}$$

where $\nu = \|a(x)\|_{L^\infty}$. Note that

$$\lim_{x \to 0} \frac{1}{x^2} \left[e^{x^2} - x^2 - 1\right] = 0,$$

and also

$$e^{x^2} - x^2 - 1 \leq C e^{x^2},$$

for some constant $C > 0$. Therefore we obtain

$$e^{x^2} - x^2 - 1 \leq C_\epsilon x^2 + C x^2 e^{x^2}.$$

As a consequence, it follows that

$$I_K(u) \geq \frac{1}{p'} \|u\|^{p'}_{W^{2,p'}(\Omega)} - \frac{1}{2} \nu C \|u\|^2_{L^4(\Omega)} \Big(\int_\Omega (e^{2u^2} + 1) \, dx \Big)^{\frac{1}{2}}.$$

We may now apply Theorem 2.9 and Sobolev imbedding, to conclude that

$$I_K(u) \geq \frac{1}{p'} \|u\|^{p'}_{W^{2,p'}(\Omega)} - C \|u\|^2_{W^{2,p'}(\Omega)} = \frac{1}{4} \rho^{p'} - C \rho^2 \ > 0,$$

provided ρ is small enough. If $u \notin K$, then clearly $I_K(u) > 0$. Therefore the mountain pass geometry holds for the functional I_k. □

In the subsequent propositions, we shall observe the invariance property of the equations in (9) with respect to the convex set $K = K(N-1, 1)$. The proof for these propositions can be found in [11, 19], where detailed explanations and demonstrations are provided.

Proposition 2.13 *Suppose Ω is an annular domain and $\tilde{v} \in K = K(N-1, 1)$. Then there exists $u \in K$ satisfying*

$$\begin{cases} -\Delta u \; = \; a(x)|\tilde{v}|^{p-2}\tilde{v}, & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{34}$$

in the weak sense.

Proposition 2.14 *Suppose Ω is an annular domain and $\tilde{u} \in K = K(N-1, 1)$. Then there exists $v \in K$ satisfying*

$$\begin{cases} -\Delta v \; = \; b(x)\tilde{u}(e^{\tilde{u}^2} - 1), & x \in \Omega \\ v = 0, & x \in \partial\Omega, \end{cases} \tag{35}$$

in the weak sense.

Having established the necessary background, we proceed to demonstrate the first main result of the paper.

Proof of Theorem 1.1. From Proposition 2.12, we deduce the existence of a critical point $\bar{u}$ for the functional I_K within the set K, satisfying $I_K(\bar{u}) = c$, where $c > 0$. Furthermore, it follows from Proposition 2.14 that there exists $\tilde{v} \in K$ such that

$$-\Delta\tilde{v} = b(x)\bar{u}(e^{\bar{u}^2} - 1)\bar{u}.$$

Moreover, applying Proposition 2.13, there exists $\tilde{u} \in K$ satisfying

$$-\Delta\tilde{u} = a(x)|\tilde{v}|^{p-2}\tilde{v}.$$

Consequently, $(\tilde{u}, \tilde{v})$ satisfies the equation

$$\begin{cases} -\Delta\tilde{u} = & a(x)|\tilde{v}|^{p-2}\tilde{v} \\ -\Delta\tilde{v} = & b(x)\bar{u}(e^{\bar{u}^2} - 1). \end{cases}$$

Now, we can apply Theorem 2.8 and conclude that $(\tilde{u}, \tilde{v})$ is a solution of

$$\begin{cases} -\Delta u = & a(x)|v|^{p-2}v \\ -\Delta v = & b(x)u(e^{u^2} - 1). \end{cases}$$

This completes the proof. □

2.2 Non-radial solutions

In this section, we shall prove that the solution obtained in Theorem 1.1 is nonradial, provided $a(x) = b(x) = 1$. That is,

$$\begin{cases} -\Delta u \; = \; |v|^{p-2}v, & x \in \Omega \\ -\Delta v \; = \; u(e^{u^2} - 1), & x \in \Omega \\ u, v > 0, & x \in \Omega \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{36}$$

Before proceeding with the proof, we need to cover some preliminaries. Let $w(x) = w(s,t)$ be a function of (s,t). When expressing w in polar coordinates ($s = r\cos(\theta)$, $t = r\sin(\theta)$), it becomes clear that $w(x)$ can be written as $w(r,\theta)$. Writing the Laplace operator in polar coordinates gives

$$-\Delta w(x) = -w_{rr} - (N-1)\frac{w_r}{r} - \frac{w_{\theta\theta}}{r^2} + \frac{w_\theta}{r^2}h(\theta),$$

where

$$h(\theta) = (m-1)\tan(\theta) - \frac{n-1}{\tan(\theta)}.$$

Let (μ_1, ψ_1) be the second eigenpair of the following eigenvalue problem

$$\begin{cases} -\psi_1''(\theta) + \psi_1'(\theta)h(\theta) = \mu_1\psi_1(\theta) & \text{in } (0, \frac{\pi}{2}), \\ \psi_1'(\theta) > 0 & \text{in } (0, \frac{\pi}{2}), \\ \psi_1'(0) = \psi_1'(\frac{\pi}{2}) = 0. \end{cases} \tag{37}$$

It's worth noting that the first eigenpair is given by $(\mu_0, \psi_0) = (0, 1)$. A straightforward calculation shows that

$$\mu_1 = 2N, \qquad \psi_1(\theta) = \frac{m-n}{N} - \cos(2\theta).$$

Let us recall the definition of the best constant in Hardy inequality for the domain Ω,

$$\beta = \inf_{\psi \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla \psi|^2 \, dx}{\int_\Omega \frac{\psi^2}{|x|^2} \, dx}.$$

In order to prove the existence of a non-radial solution, we start with following lemma.

Lemma 2.15 *Let $p \geq 2$. Assume that (u, v) is a positive solution of (36). Then the following inequality holds true for u and v:*

$$\frac{v^p}{p} \geq \frac{e^{u^2} - u^2}{2}. \tag{38}$$

Proof: It suffices to show that

$$v \geq (\frac{p}{2})^{\frac{1}{p}} (e^{u^2} - u^2)^{\frac{1}{p}}.$$

Define $z(x) = v - (\frac{p}{2})^{\frac{1}{p}} (e^{u^2} - u^2)^{\frac{1}{p}}$. It follows that

$$\begin{aligned}
\nabla z &= \nabla v - \left(\frac{p}{2}\right)^{\frac{1}{p}} \frac{1}{p} (e^{u^2} - u^2)^{\frac{1}{p}-1} (2ue^{u^2} - 2u) \nabla u, \\
\Delta z &= \Delta v - \left(\frac{p}{2}\right)^{\frac{1}{p}} \frac{1}{p} (e^{u^2} - u^2)^{\frac{1}{p}-1} (2ue^{u^2} - 2u) \Delta u \\
&\quad - (\frac{p}{2})^{\frac{1}{p}} \frac{1}{p} \big[(\frac{1}{p} - 1)(e^{u^2} - u^2)^{\frac{1}{p}-2} (2ue^{u^2} - 2u)^2 + (e^{u^2} - u^2)^{\frac{1}{p}-1} (4u^2 e^{u^2} + 2e^{u^2} - 2) \big] |\nabla u|^2.
\end{aligned} \tag{39}$$

We have

$$\begin{aligned}
&(1 - \frac{1}{p})(e^{u^2} - u^2)^{\frac{1}{p}-2} (2ue^{u^2} - 2u)^2 - (e^{u^2} - u^2)^{\frac{1}{p}-1} (4u^2 e^{u^2} + 2e^{u^2} - 2) \\
&= (e^{u^2} - u^2)^{\frac{1}{p}-2} \left((1 - \frac{1}{p})(2ue^{u^2} - 2u)^2 - (e^{u^2} - u^2)(4u^2 e^{u^2} + 2e^{u^2} - 2) \right) \\
&\leq (e^{u^2} - u^2)^{\frac{1}{p}-2} \left((2ue^{u^2} - 2u)^2 - (e^{u^2} - u^2)(4u^2 e^{u^2} + 2e^{u^2} - 2) \right),
\end{aligned} \tag{40}$$

since $p \geq 2$. Consider the function $g(u)$, defined as

$$\begin{aligned}
g(u) &= (2ue^{u^2} - 2u)^2 - (e^{u^2} - u^2)(4u^2 e^{u^2} + 2e^{u^2} - 2) \\
&= e^{u^2} \left(-2e^{u^2} + 4u^4 - 6u^2 + 2 \right) + 2u^2.
\end{aligned}$$

To analyze the behavior of $g(u)$, consider its first derivative

$$\begin{aligned}
g'(u) &= 2ue^{u^2} \left(-2e^{u^2} + 4u^4 - 6u^2 \right) + e^{u^2} \left(-4ue^{u^2} + 16u^3 - 12u \right) + 4u \\
&= -8ue^{2u^2} + 8u^5 e^{u^2} + 4u^3 e^{u^2} - 12ue^{u^2} + 4u.
\end{aligned}$$

Observe that $g'(0) = 0$, and now let's examine the second derivative

$$g''(u) = e^{2u^2}(-8 - 32u^2) + e^{u^2}(16u^6 + 48u^4 - 12u^2 - 12) + 4.$$

We note that $g''(u) < 0$ for all u, which implies that $g(u) \leq 0$. Returning to (40), we have shown that the entire expression is non-positive, and we conclude that

$$(1 - \frac{1}{p})(e^{u^2} - u^2)^{\frac{1}{p}-2} (2ue^{u^2} - 2u)^2 - (e^{u^2} - u^2)^{\frac{1}{p}-1} (4u^2 e^{u^2} + 2e^{u^2} - 2) \leq 0. \tag{41}$$

Now, taking into account (39) and (41), we can conclude that

$$\Delta z \leq \Delta v - (\frac{p}{2})^{\frac{1}{p}}\ \frac{1}{p}(e^{u^2} - u^2)^{\frac{1}{p}-1}(2ue^{u^2} - 2u)\Delta u. \tag{42}$$

Next, we will show that $\Delta z \leq 0$ on the set

$$\{x \in \Omega : z(x) \leq 0\}.$$

It follows from (42) that

$$\begin{aligned}
\Delta z &\leq -u(e^{u^2} - 1) + (\frac{p}{2})^{\frac{1}{p}}\ \frac{1}{p}(e^{u^2} - u^2)^{\frac{1}{p}-1}(2ue^{u^2} - 2u)v^{p-1} \\
&= u(e^{u^2} - 1)\big[(\frac{p}{2})^{\frac{1}{p}-1}(e^{u^2} - u^2)^{\frac{1}{p}-1}v^{p-1} - 1\big] \\
&= (\frac{p}{2})^{\frac{1}{p}-1}v^{p-1}u(e^{u^2} - 1)\big[(e^{u^2} - u^2)^{\frac{1}{p}-1} - (\frac{p}{2})^{1-\frac{1}{p}}v^{1-p}\big].
\end{aligned} \tag{43}$$

Note that when $z < 0$, we have

$$v(\frac{p}{2})^{-\frac{1}{p}} \leq (e^{u^2} - u^2)^{\frac{1}{p}},$$

or equivalently

$$(e^{u^2} - u^2)^{\frac{1}{p}-1} \leq (\frac{p}{2})^{1-\frac{1}{p}}v^{1-p},$$

which, together with (43) follows that $\Delta z \leq 0$, as desired. Consequently, we have

$$\int_{\{x\in\Omega: z(x)\leq 0\}} -\Delta z\ z \leq 0,$$

and thus

$$\int_{\{x\in\Omega: z(x)\leq 0\}} |\nabla z| \leq 0.$$

Therefore

$$\int_{\Omega} |\nabla z^-| \leq 0,$$

which implies that $z > 0$. Therefore, the proof is complete. $\square$

Proof of Theorem 1.2. Assume, for the sake of contradiction, that the solution (u, v) of (36) obtained in Theorem 1.1 is a radial function. Note that $I_K(u) = c > 0$ where the critical value c is characterized by

$$c = \inf_{\gamma\in\Gamma} \sup_{\tau\in[0,1]} I(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0, 1], V) : \gamma(0) = 0 \neq \gamma(1),\ I_K(\gamma(1)) \leq 0\}$. We claim that there exists some element $\gamma \in \Gamma$ such that

$$I_K(\gamma(\tau)) < I_K(u), \quad \forall\, \tau \in [0, 1].$$

This implies that

$$c \leq \max_{\tau\in[0,1]} I_K(\gamma(\tau)) < I_K(u),$$

which contradicts $I_K(u) = c$. Now, in order to prove our claim, let (λ_1, ψ) be the first eigenpair of the following eigenvalue problem

$$\begin{cases} -\varphi''(r) - (N-1)\frac{\varphi'(r)}{r} + \frac{2N\varphi(r)}{r^2} = \lambda_1 v(r)^{\frac{p-2}{2}}(e^{u^2} - u^2 - 1)^{\frac{1}{2}}\varphi(r), & r \in (R_1, R_2), \\ \varphi(r) = 0, & r \in \{R_1, R_2\}. \end{cases}$$

Set $w(r,\theta)=\varphi(r)\psi(\theta)$ where $\psi(\theta)=\dfrac{m-n}{N}-\cos(2\theta)$ being the solution of (37) with $\mu_1=2N$. Note that

$$
\begin{aligned}
-\Delta w(x) &= -w_{rr}-(N-1)\frac{w_r}{r}-\frac{w_{\theta\theta}}{r^2}+\frac{w_\theta}{r^2}h(\theta)\\
&= -\varphi_{rr}(r)\psi(\theta)-(N-1)\varphi_r(r)\psi(\theta)\frac{1}{r}-\frac{\varphi(r)\psi''(\theta)}{r^2}+\frac{\varphi(r)\psi'(\theta)}{r^2}h(\theta)\\
&= -\varphi_{rr}(r)\psi(\theta)-(N-1)\varphi_r(r)\psi(\theta)\frac{1}{r}-\frac{\varphi(r)\psi''(\theta)}{r^2}+\frac{\varphi(r)\psi'(\theta)}{r^2}h(\theta)\\
&= -\varphi_{rr}(r)\psi(\theta)-(N-1)\varphi_r(r)\psi(\theta)\frac{1}{r}+\frac{2N\varphi(r)\psi(\theta)}{r^2}\\
&= \lambda_1 v(|x|)^{\frac{p-2}{2}}\left(e^{u^2(|x|)}-1+2u^2(|x|)e^{u^2(|x|)}\right)^{\frac{1}{2}}w(x) \qquad (44)
\end{aligned}
$$

Let $l>0$ be such that $I_K((u+\sigma w)l)\leq 0$ for all $|\sigma|<1$. Consider

$$\gamma_\sigma(\tau)=\tau(u+\sigma w)l.$$

We have $\gamma_\sigma\in\Gamma$ for all $|\sigma|<1$. Moreover, there exists a unique twice differentiable real function g on a small neighbourhood of zero with $g'(0)=0$ and $g(0)=1/l$ such that

$$\max_{\tau\in[0,1]} I_K(\gamma_\sigma(\tau))=I_K(g(\sigma)(u+\sigma w)l).$$

Now we define $h:\mathbb{R}\to\mathbb{R}$ by

$$h(\sigma)=I_K(g(\sigma)(u+\sigma w)l)-I_K(u).$$

We already know that $h(0)=0$ and $h'(0)=\langle I_K'(u),w\rangle=0$. If we prove that $h''(0)<0$, then there exists σ sufficiently small such that $h(\sigma)<0$, or equivalently,

$$\max_{\tau\in[0,1]} I_K(\gamma_\sigma(\tau))=I_K(g(\sigma)(u+\sigma w)l)<I_K(u).$$

In this way the desired conclusion follows by taking $\gamma=\gamma_\sigma$. The only remaining condition that needs to be checked is $h''(0)<0$. We have

$$
\begin{aligned}
h''(0)=\langle I_K''(u);w,w\rangle &= (p'-1)\int_\Omega |\Delta u|^{p'-2}(-\Delta w)^2dx-\int_\Omega (e^{u^2}-1+2u^2e^{u^2})w^2\,dx\\
&= (p'-1)\lambda_1^2\int_\Omega |\Delta u|^{p'-2}v^{p-2}(e^{u^2}-1+2u^2e^{u^2})w^2(x)dx-\int_\Omega (e^{u^2}-1+2u^2e^{u^2})w^2\,dx\\
&= (p'-1)\lambda_1^2\int_\Omega \left(v^{p-1}\right)^{p'-2}v^{p-2}(e^{u^2}-1+2u^2e^{u^2})w^2(x)dx-\int_\Omega (e^{u^2}-1+2u^2e^{u^2})w^2\,dx\\
&= (p'-1)\lambda_1^2\int_\Omega (e^{u^2}-1+2u^2e^{u^2})w^2(x)dx-\int_\Omega (e^{u^2}-1+2u^2e^{u^2})w^2\,dx\\
&= \left((p'-1)\lambda_1^2-1\right)\int_\Omega (e^{u^2}-1+2u^2e^{u^2})w^2\,dx.
\end{aligned}
$$

Note that $(p'-1)\lambda_1^2-1<0$ if and only if $\lambda_1^2<p-1$. It follows

$$\begin{aligned}
\lambda_1 &= \inf_{0\neq\eta\in H_0^1(\Omega)} \frac{\int_\Omega |\nabla\eta|^2 dx + 2N\int_\Omega \frac{|\eta|^2}{|x|^2}\, dx}{\int_\Omega \eta^2 v^{\frac{p-2}{2}}(e^{u^2}-1+2u^2e^{u^2})^{\frac{1}{2}}\, dx} \\
&\leq \inf_{0\neq\eta\in H_0^1(\Omega)} \frac{(1+\frac{2N}{\beta})\int_\Omega |\nabla\eta|^2\, dx}{\int_\Omega \eta^2 v^{\frac{p-2}{2}}(e^{u^2}-1+2u^2e^{u^2})^{\frac{1}{2}}\, dx} \\
&\leq \frac{(1+\frac{2N}{\beta})\int_\Omega |\nabla v|^2\, dx}{\int_\Omega v v^{\frac{p}{2}}(e^{u^2}-1+2u^2e^{u^2})^{\frac{1}{2}}\, dx} \\
&\leq \frac{(1+\frac{2N}{\beta})\int_\Omega vu(e^{u^2}-1)\, dx}{\int_\Omega v\big(\frac{p}{2}(e^{u^2}-u^2)\big)^{\frac{1}{2}}(e^{u^2}-1+2u^2e^{u^2})^{\frac{1}{2}}\, dx} \\
&\leq \sqrt{\frac{2}{p}}(1+\frac{2N}{\beta}),
\end{aligned}$$

where in the last inequality, we utilize Lemma 2.15 along with the inequality

$$(e^{u^2}-u^2)(e^{u^2}-1+2u^2e^{u^2}) \geq u^2(e^{u^2}-1)^2.$$

If $(1+\frac{2N}{\beta})^2 < \frac{p(p-1)}{2}$, then $(p'-1)\lambda_1^2 - 1 < 0$ and consequently $h''(0) < 0$, as desired. ☐

3 Gradient Systems

In this section we examine the equation

$$\begin{cases} -\Delta u = a(x)\, u(e^{u^2}-1) + \; pu^{p-1}v^q, & x\in\Omega \\ -\Delta v = b(x)\, v(e^{v^2}-1) + qv^{q-1}u^p, & x\in\Omega \\ u,v>0, & x\in\Omega \\ u=v=0, & x\in\partial\Omega, \end{cases} \tag{45}$$

where Ω is a bounded annulus in $\mathbb{R}^N = \mathbb{R}^{N-1}\times\mathbb{R}$ and $p,q>2$. We shall consider the Banach space $V = H_0^1(\Omega)\cap\mathcal{L}$ equipped with the following norm

$$\|u\| = \|u\|_{H_0^1(\Omega)} + \|u\|_{\mathcal{L}}.$$

Let $I : V\times V\to\mathbb{R}$ be the Euler-Lagrange functional corresponding to problem (45), i.e.,

$$\begin{aligned} I(u,v) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \frac{1}{2}\int_\Omega |\nabla v|^2\, dx - \frac{1}{2}\int_\Omega a(x)(e^{u^2}-u^2-1)\, dx \\ -\frac{1}{2}\int_\Omega b(x)(e^{v^2}-v^2-1)\, dx - \int_\Omega |u|^p|v|^q\, dx. \end{aligned} \tag{46}$$

Let

$$\begin{aligned} K = K(N-1,1) = \{&(u,v)\in H_{0,G}^1(\Omega)\times H_{0,G}^1(\Omega) : 0<u=u(s,t),\ 0<v=v(s,t), \\ &su_t - tu_s \leq 0\ \ a.e.\ \ in\ \widehat{\Omega},\ \ sv_t - tv_s \leq 0\ \ a.e.\ \ in\ \widehat{\Omega}\}. \end{aligned} \tag{47}$$

Thus, we define

$$I_K = \Psi_K - \Phi, \tag{48}$$

where

$$\Psi(u,v) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \frac{1}{2}\int_\Omega |\nabla v|^2\, dx,$$

$$\Phi(u,v)=\frac{1}{2}\int_\Omega a(x)(e^{u^2}-u^2-1)\,dx+\frac{1}{2}\int_\Omega b(x)(e^{v^2}-v^2-1)\,dx+\int_\Omega |u|^p|v|^q\,dx,$$

and Ψ_K is the restriction of Ψ to K defined by

$$\Psi_K(u,v)=\begin{cases}\Psi(u,v), & (u,v)\in K\\ +\infty, & (u,v)\notin K.\end{cases}$$

Theorem 3.1 *Let $V=H_0^1(\Omega)\cap\mathcal{L}$ and K be a convex and closed subset defined in (47). Let a and b be non-negative continuously differentiable functions that are not identically zero. Assume that the following two assertions hold:*

(i) The functional $I_K:V\times V\to\mathbb{R}$ defined in (48) has a critical point $(\bar u,\bar v)\in V\times V$ as in Definition 2.4, and;

(ii) There exists $(\tilde u,\tilde v)\in K$ such that

$$\begin{cases}-\Delta\tilde u &= a(x)\ \bar u(e^{\bar u^2}-1)+\ p\bar u^{p-1}\bar v^q, \quad x\in\Omega\\ -\Delta\tilde v &= b(x)\ \bar v(e^{\bar v^2}-1)+q\bar v^{q-1}\bar u^p, \quad x\in\Omega.\end{cases}\tag{49}$$

Then $(\tilde u,\tilde v)\in K$ is a weak solution of the equation

$$\begin{cases}-\Delta u &= a(x)\ u(e^{u^2}-1)+\ pu^{p-1}v^q, \quad x\in\Omega\\ -\Delta v &= b(x)\ v(e^{v^2}-1)+qv^{q-1}u^p, \quad x\in\Omega.\end{cases}\tag{50}$$

Proof: As $(\bar u,\bar v)$ is a critical point of I, we can deduce from Definition 2.4 that

$$\langle D\Phi(\bar u,\bar v),(\bar u,\bar v)-(u,v)\rangle+\Psi(u,v)-\Psi(\bar u,\bar v)\geq 0\ ,\quad \forall (u,v)\in V\times V,$$

or equivalently,

$$\langle D\Phi(\bar u,\bar v),(\bar u-u,\bar v-v)\rangle+\Psi(u,v)-\Psi(\bar u,\bar v)\geq 0\ ,\quad \forall (u,v)\in V\times V,$$

which implies

$$\begin{aligned}&\frac{1}{2}\int_\Omega|\nabla\bar u|^2\,dx+\frac{1}{2}\int_\Omega|\nabla\bar v|^2\,dx-\frac{1}{2}\int_\Omega|\nabla u|^2\,dx-\frac{1}{2}\int_\Omega|\nabla v|^2dx\\ &\leq\int_\Omega\big[a(x)\bar u(e^{\bar u^2}-1)+p\bar u^{p-1}\bar v^q\big](\bar u-u)\,dx\\ &\quad+\int_\Omega\big[b(x)\bar v(e^{\bar v^2}-1)+q\bar v^{q-1}\bar u^p\big](\bar v-v)\,dx,\quad \forall (u,v)\in K.\end{aligned}\tag{51}$$

On the other hand, by (49) we have

$$\int_\Omega\nabla\tilde u\nabla\eta\,dx=\int_\Omega\big[a(x)\bar u(e^{\bar u^2}-1)+p\bar u^{p-1}\bar v^q\big]\eta\,dx,\quad \forall\eta\in K,\tag{52}$$

and

$$\int_\Omega\nabla\tilde v\nabla\zeta\,dx=\int_\Omega\big[a(x)\bar v(e^{\bar v^2}-1)+q\bar v^{q-1}\bar u^p\big]\zeta\,dx,\quad \forall\zeta\in K.\tag{53}$$

By substituting $\eta=\bar u-\tilde u$ in (52) and $\zeta=\bar v-\tilde v$ in (53), we obtain

$$\int_\Omega\nabla\tilde u\nabla(\bar u-\tilde u)\,dx=\int_\Omega\big[a(x)\bar u(e^{\bar u^2}-1)+p\bar u^{p-1}\bar v^q\big](\bar u-\tilde u)\,dx,\tag{54}$$

and

$$\int_\Omega \nabla\tilde{v}\nabla(\bar{v}-\tilde{v})\,dx = \int_\Omega \big[a(x)\bar{v}(e^{\bar{v}^2}-1)+q\bar{v}^{q-1}\bar{u}^p\big](\bar{v}-\tilde{v})\,dx. \tag{55}$$

Now, setting $(u,v)=(\tilde{u},\tilde{v})$ in (51) and considering (54) and (55), we can deduce that

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\nabla\bar{u}|^2\,dx+\frac{1}{2}\int_\Omega |\nabla\bar{v}|^2\,dx-\frac{1}{2}\int_\Omega |\nabla\tilde{u}|^2\,dx-\frac{1}{2}\int_\Omega |\nabla\tilde{v}|^2dx\\ &\le \int_\Omega \nabla\tilde{u}\nabla(\bar{u}-\tilde{u})\,dx+\int_\Omega \nabla\tilde{v}\nabla(\bar{v}-\tilde{v})\,dx,\end{aligned}$$

from which we conclude

$$\frac{1}{2}\int_\Omega |\nabla(\tilde{u}-\bar{u})|^2\,dx+\frac{1}{2}\int_\Omega |\nabla(\tilde{v}-\bar{v})|^2\,dx\le 0.$$

This implies that $(\tilde{u},\tilde{v})=(\bar{u},\bar{v})$ for a.e. $x\in\Omega$. Taking into account that $(\tilde{u},\tilde{v})=(\bar{u},\bar{v})$ in (49), we have completed the proof. □

Lemma 3.2 *The functional I_K satisfies the mountain pass geometry and (PS) compactness condition. Moreover, I_K has a non-trivial critical point in K.*

Proof: Let (u_j,v_j) be a sequence in K such that $I_K(u_j,v_j)\to c\in\mathbb{R}, \epsilon_j\to 0$ and

$$\langle D\Phi(u_j,v_j),(u_j,v_j)-(u,v)\rangle+\Psi(u,v)-\Psi(u_j,v_j)\ge -\epsilon_j\|(u,v)-(u_j,v_j)\|_V,\quad \forall (u,v)\in V. \tag{56}$$

Replacing (u,v) by (ru_j,rv_j) $(r\in\mathbb{R})$ in (56), it becomes

$$\begin{aligned}&\frac{1-r^2}{2}(\|u_j\|^2_{H_0^1(\Omega)}+\|v_j\|^2_{H_0^1(\Omega)})+(r-1)\int_\Omega \big[a(x)u_j(e^{{u_j}^2}-1)+p{u_j}^{p-1}{v_j}^q\big]u_j\,dx\\ &+(r-1)\int_\Omega \big[b(x)v_j(e^{{v_j}^2}-1)+q{v_j}^{q-1}{u_j}^p\big]v_j\,dx\\ &\le \epsilon_j(r-1)\big(\|u_j\|_V+\|v_j\|_V\big).\end{aligned} \tag{57}$$

On the other hand, since $I_K(u_j,v_j)\to c$, we have

$$\begin{aligned}&\frac{1}{2}\|u_j\|^2_{H_0^1(\Omega)}+\frac{1}{2}\|v_j\|^2_{H_0^1(\Omega)}-\frac{1}{2}\int_\Omega a(x)(e^{{u_j}^2}-u_j^2-1)\,dx-\frac{1}{2}\int_\Omega b(x)(e^{{v_j}^2}-v_j^2-1)\,dx-\int_\Omega |u_j|^p|v_j|^q\,dx\\ &\le c+1,\end{aligned} \tag{58}$$

for large values of j. Now set $1<r$ and $r^2-1<4(r-1)$. We can take $\zeta>0$ such that

$$\frac{1}{4(r-1)}<\zeta<\frac{1}{r^2-1}.$$

Multiply (57) by ζ and adding up with (58) yields

$$\begin{aligned}&\frac{1+\zeta(1-r^2)}{2}(\|u_j\|^2_{H_0^1(\Omega)}+\|v_j\|^2_{H_0^1(\Omega)})+\frac{1}{4}\int_\Omega [a(x)u_j^2(e^{{u_j}^2}-1)+pu_j^pv_j^q]\,dx-\frac{1}{2}\int_\Omega a(x)(e^{{u_j}^2}-u_j^2-1)\,dx\\ &+\frac{1}{4}\int_\Omega [b(x)v_j^2(e^{{v_j}^2}-1)+qv_j^qu_j^p]\,dx-\frac{1}{2}\int_\Omega b(x)(e^{{v_j}^2}-v_j^2-1)\,dx-\int_\Omega u_j^pv_j^q\,dx\\ &\le \frac{1+\zeta(1-r^2)}{2}(\|u_j\|^2_{H_0^1(\Omega)}+\|v_j\|^2_{H_0^1(\Omega)})+\zeta(r-1)\int_\Omega [a(x)u_j^2(e^{{u_j}^2}-1)+pu_j^pv_j^q]\,dx-\frac{1}{2}\int_\Omega a(x)(e^{{u_j}^2}-u_j^2-1)\,dx\\ &+\zeta(r-1)\int_\Omega [b(x)v_j^2(e^{{v_j}^2}-1)+qv_j^qu_j^q]\,dx-\frac{1}{2}\int_\Omega b(x)(e^{{v_j}^2}-v_j^2-1)\,dx-\int_\Omega u_j^pv_j^q\,dx\\ &\le c+1+\zeta\epsilon_j(r-1)\|u_j\|_V.\end{aligned} \tag{59}$$

Also as a consequence of the inequality $x^2(e^{x^2}-1)-2(e^{x^2}-x^2-1)\geq 0$, we can deduce that

$$\frac{1}{4}\int_\Omega a(x)u_j^2(e^{{u_j}^2}-1)\,dx-\frac{1}{2}\int_\Omega a(x)(e^{{u_j}^2}-u_j^2-1)\,dx\geq 0,$$
$$\frac{1}{4}\int_\Omega b(x)v_j^2(e^{{v_j}^2}-1)\,dx-\frac{1}{2}\int_\Omega a(x)(e^{{v_j}^2}-v_j^2-1)\,dx\geq 0,$$

which, together with (59) and $p+q>4$, yields

$$\frac{1+\zeta(1-r^2)}{2}(\|u_j\|^2_{H_0^1(\Omega)}+\|v_j\|^2_{H_0^1(\Omega)})\leq c+1+\zeta\epsilon_j(r-1)(\|u_j\|_V+\|v_j\|_V). \tag{60}$$

Since all the coefficients on the left-hand side of the inequality are positive due to the choice of ζ, we can conclude that

$$\|u_j\|^2_{H_0^1(\Omega)}+\|v_j\|^2_{H_0^1(\Omega)}\leq C(1+\|u_j\|_V+\|v_j\|_V), \tag{61}$$

for some constant $C>0$. On the other hand, by Theorem 2.9, we conclude that there exists a constant $C>0$ such that

$$\|u\|_{H_0^1(\Omega)}\leq\|u\|_V\leq C\|u\|_{H_0^1(\Omega)},\qquad \|v\|_{H_0^1(\Omega)}\leq\|v\|_V\leq C\|v\|_{H_0^1(\Omega)}, \tag{62}$$

for every $(u,v)\in K$. Therefore we may define a usual equivalent norm for $(u,v)\in K$ by

$$\|(u,v)\|=\int_\Omega|\nabla u|^2\,dx+\int_\Omega|\nabla v|^2\,dx.$$

This, together with (61), implies that

$$\|(u_j,v_j)\|^2\leq C\big(1+\|(u_j,v_j)\|\big). \tag{63}$$

Standard results in Sobolev spaces allow us to conclude, after possibly passing to a subsequence, that there exist functions $\bar{u},\bar{v}\in H_0^1(\Omega)$ such that $u_j\rightharpoonup\bar{u}$ and $v_j\rightharpoonup\bar{v}$ weakly in $H_0^1(\Omega)$. This, in turn, implies that $u_j\to\bar{u}$ and $v_j\to\bar{v}$ strongly in $L^2(\Omega)$. By setting $u=\bar{u}$ and $v=\bar{v}$ in (56), and using Hölder's inequality we obtain

$$\begin{aligned}
&\frac{1}{2}(\|u_j\|^2_{H_0^1(\Omega)}-\|\bar{u}\|^2_{H_0^1(\Omega)})+\frac{1}{2}(\|v_j\|^2_{H_0^1(\Omega)}-\|\bar{v}\|^2_{H_0^1(\Omega)})\\
&\leq\int_\Omega[a(x)u_j(e^{{u_j}^2}-1)+pu_j^{p-1}v_j^q](\bar{u}-u_j)\,dx+\epsilon_j\|u_j-\bar{u}\|_{H_0^1(\Omega)}\\
&\quad+\int_\Omega[b(x)v_j(e^{{v_j}^2}-1)+qv_j^{q-1}u_j^p](\bar{v}-v_j)\,dx+\epsilon_j\|v_j-\bar{v}\|_{H_0^1(\Omega)}\\
&\leq\Big[\|a(x)\|_{L^\infty}\|e^{u_j^2}-1\|_{L^2}+\|pu_j^{p-2}v_j^q\|_{L^2}\Big]\|u_j(u_j-\bar{u})\|_{L^2}+\epsilon_j\|u_j-\bar{u}\|_{H_0^1(\Omega)}\\
&\quad+\Big[\|b(x)\|_{L^\infty}\|e^{v_j^2}-1\|_{L^2}+\|qv_j^{q-2}u_j^p\|_{L^2}\Big]\|v_j(v_j-\bar{v})\|_{L^2}+\epsilon_j\|v_j-\bar{v}\|_{H_0^1(\Omega)}\\
&\leq\Big[\|a(x)\|_{L^\infty}\|e^{u_j^2}-1\|_{L^2}+\|pu_j^{p-2}\|v_j^q\|_{L^4}\Big]\|u_j\|_{L^4}\|u_j-\bar{u}\|_{L^4}+\epsilon_j\|u_j-\bar{u}\|_{H_0^1(\Omega)}\\
&\quad+\Big[\|b(x)\|_{L^\infty}\|e^{v_j^2}-1\|_{L^2}+\|qv_j^{q-2}\|_{L^4}\|u_j^p\|_{L^4}\Big]\|v_j\|_{L^4}\|v_j-\bar{v}\|_{L^4}+\epsilon_j\|v_j-\bar{v}\|_{H_0^1(\Omega)}.
\end{aligned} \tag{64}$$

Furthermore, since $(u_j,v_j)\in K$, we can apply the concentration compactness principle for the Trudinger-Moser inequality in $H^1(\Omega)$, as presented in Theorem 2.9, along with the continuous embedding provided in Theorem 2.11, and conclude that

$$\limsup_{j\to\infty}(\|u_j\|^2_{H_0^1(\Omega)}-\|\bar{u}\|^2_{H_0^1(\Omega)})+(\|v_j\|^2_{H_0^1(\Omega)}-\|\bar{v}\|^2_{H_0^1(\Omega)})\leq 0. \tag{65}$$

Using the properties of weak convergence, we also obtain

$$0 \leq \liminf(\|u_j\|^2_{H^1_0(\Omega)} - \|\bar{u}\|^2_{H^1_0(\Omega)}) + (\|v_j\|^2_{H^1_0(\Omega)} - \|\bar{v}\|^2_{H^1_0(\Omega)}),$$

which, together with (65), implies that $u_j \to \bar{u}$ and $v_j \to \bar{v}$ strongly in $H^1_0(\Omega)$. This completes the proof of the (PS) compactness condition for the function I_K. We now verify the mountain pass geometry of the functional I_K. It is clear that $I_K(0) = 0$. Take $(u, v) \in K$. Then, for any $\lambda > 0$, we have

$$\begin{aligned} I_K(\lambda u, \lambda v) = \frac{\lambda^2}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\lambda^2}{2}\int_\Omega |\nabla v|^2\,dx - \frac{1}{2}\int_\Omega a(x)(e^{\lambda^2 u^2} - \lambda^2 u^2 - 1)\,dx \\ - \frac{1}{2}\int_\Omega b(x)(e^{\lambda^2 v^2} - \lambda^2 v^2 - 1)\,dx - \lambda^{p+q}\int_\Omega |u|^p|v|^q\,dx. \end{aligned}$$

It is now obvious that $I_K(\lambda u, \lambda v) < 0$ for λ sufficiently large. Take $(u, v) \in K$ with $\|(u, v)\| = \rho > 0$. We have

$$\begin{aligned} I_K(u, v) &= \frac{1}{2}\|(u, v)\| - \frac{1}{2}\int_\Omega a(x)(e^{u^2} - u^2 - 1)\,dx - \frac{1}{2}\int_\Omega b(x)(e^{v^2} - v^2 - 1)\,dx - \int_\Omega |u|^p|v|^q\,dx \\ &\geq \frac{1}{2}\|(u, v)\| - \frac{1}{2}\|a(x)\|_{L^\infty}\int_\Omega (e^{u^2} - u^2 - 1)\,dx - \frac{1}{2}\|b(x)\|_{L^\infty}\int_\Omega (e^{v^2} - v^2 - 1)\,dx - \int_\Omega |u|^p|v|^q\,dx. \end{aligned}$$

With a method similar to the one used in the proof of Proposition 2.12, we can show that

$$e^{x^2} - x^2 - 1 \leq Cx^3 e^{x^2} + \frac{1}{4}x^2\ ,$$

Additionally, we can employ Hölder's inequality and the Sobolev embedding theorem to obtain

$$\int_\Omega u^p v^q\,dx \leq \int_\Omega (u^{2p})^{\frac{1}{2}}\,dx \int_\Omega (v^{2q})^{\frac{1}{2}}\,dx = \|u\|^p_{L^{2p}}\|v\|^q_{L^{2q}} \leq \|(u, v)\|^{p+q}.$$

As a consequence, it follows that

$$I_K(u, v) \geq \frac{1}{4}\|(u, v)\|^2 - C\|(u, v)\|^3 - \|(u, v)\|^{p+q} = \frac{1}{4}\rho^2 - C\rho^3 - \rho^{p+q} > 0,$$

provided ρ is small enough. If $(u, v) \notin K$, then clearly $I_K(u, v) > 0$. Therefore the mountain pass geometry holds for the functional I_K. □

Proposition 3.3 *Suppose Ω is an annular domain and $(\tilde{u}, \tilde{v}) \in K = K(N-1, 1)$. Then there exists $(u, v) \in K$ satisfying*

$$\begin{cases} -\Delta u \ = \ a(x)\tilde{u}(e^{\tilde{u}^2} - 1) + \ p\tilde{u}^{p-1}\tilde{v}^q, & x \in \Omega \\ -\Delta v \ = \ \ b(x)\tilde{v}(e^{\tilde{v}^2} - 1) + q\tilde{v}^{q-1}\tilde{u}^p, & x \in \Omega \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{66}$$

Proof: Let $(\tilde{u}, \tilde{v}) \in K$ be fixed. By setting $\tilde{u}_k(x) = min\{\tilde{u}(x), k\}$ and $\tilde{v}_k(x) = min\{\tilde{v}(x), k\}$ for $k \geq 1$, we have $\tilde{u}_k,\ \tilde{v}_k \in H^1(\Omega)$. Observe that the cut off does not affect the symmetry and also preserves the monotonicity of u and v. Therefore since $\tilde{u},\ \tilde{v} \in H^1_G(\Omega)$ have symmetry, we obtain that $\tilde{u}_k,\ \tilde{v}_k \in H^1_G(\Omega)$ and the monotonicity of $\tilde{u}$, $\tilde{v}$, $\tilde{u}_k$ and $\tilde{v}_k$ should be the same. Now we shall consider the following problem

$$\begin{cases} -\Delta u \ = \ a(x)\tilde{u}_k(e^{\tilde{u}_k^2} - 1) + \ p\tilde{u}_k^{p-1}\tilde{v}_k^q, & x \in \Omega \\ -\Delta v \ = \ \ b(x)\tilde{v}_k(e^{\tilde{v}_k^2} - 1) + q\tilde{v}_k^{q-1}\tilde{u}_k^p, & x \in \Omega \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{67}$$

Taking into account the associated energy on $H^1_G(\Omega)$ and applying standard regularity theory, we can deduce the existence of a unique solution (u_k, v_k) of (67), where $0 \leq u_k, v_k \in H^3_G(\Omega) \cap C^{1,\alpha}(\Omega)$. We want to show that $(u_k, v_k) \in K$. Note that $u_k = u_k(s, t)$ satisfies the equation

$$-u^k_{ss} - u^k_{tt} - \frac{m-1}{s}u^k_s - \frac{n-1}{t}u^k_t = a(s, t)\tilde{u}_k(e^{\tilde{u}_k^2} - 1) + \ p\tilde{u}_k^{p-1}\tilde{v}_k^q, \quad in\ \widehat{\Omega}, \tag{68}$$

with $u_k = 0$ on $(s,t) \in \partial\widehat{\Omega} \setminus \big(\{s=0\} \cup \{t=0\}\big)$. We set $w^k = su_t^k - tu_s^k$. Then differentiating w^k with respect to s and t, we obtain

$$w_s^k = u_t^k + su_{ts}^k - tu_{ss}^k \quad , \quad w_t^k = -u_s^k - tu_{st}^k + su_{tt}^k$$

and

$$w_{ss}^k = 2u_{st}^k + su_{tss}^k - tu_{sss}^k \quad , \quad w_{tt}^k = -2u_{st}^k - tu_{stt}^k + su_{ttt}^k.$$

These together with a computation from the equation (68) imply that for all $(s,t) \in \widehat{\Omega}$

$$-w_{ss}^k - w_{tt}^k - \frac{m-1}{s} w_s^k - \frac{n-1}{t} w_t^k + \frac{m-1}{s^2} w^k + \frac{n-1}{t^2} w^k = H, \tag{69}$$

where

$$\begin{aligned} H &= \tilde{u}_k(e^{\tilde{u}_k^2} - 1)(sa_t - ta_s) + a(s,t)(2\tilde{u}_k^2 e^{\tilde{u}_k^2} + e^{\tilde{u}_k^2} - 1)(s(\tilde{u}_k)_t - t(\tilde{u}_k)_s) \\ &\quad + p(p-1)\tilde{v}_k \tilde{u}_k^{p-2}(s(\tilde{u}_k)_t - t(\tilde{u}_k)_s) + p\tilde{u}_k^{p-1}(s(\tilde{v}_k)_t - t(\tilde{v}_k)_s). \end{aligned}$$

This problem behaves like a two dimensional problem away from the sets $\{s=0\}$ and $\{t=0\}$. Since $\tilde{u}_k \geq 0$ and it has the same monotonicity as $\tilde{u}$, it follows that $H \leq 0$ in $\widehat{\Omega}$. On the portions of $\partial\widehat{\Omega}$ corresponding to $\{s=0\}$ and $\{t=0\}$, we have $u_s^k = 0$ and $u_t^k = 0$, respectively. This is enough to ensure $w^k = 0$ on these portions of the boundary. We will show that $w^k \leq 0$ on $\widehat{\Omega} \setminus (\{s=0\} \cup \{t=0\})$. For a small $\epsilon > 0$, consider $\varphi(s,t) := (w^k(s,t) - \epsilon)^+$. Note that $(w^k(s,t) - \epsilon)^+ = 0$ near $\partial\widehat{\Omega}$. Due to the sufficient regularity of φ, we can multiply inequality (69) by φ and integrate it over $\widehat{\Omega}$, which results in

$$\begin{aligned} &\int_{\widehat{\Omega}} |\nabla_{s,t}(w^k - \epsilon)^+|^2 \, d\mu(s,t) + \int_{\widehat{\Omega}} (w^k - \epsilon)^+ \big(\frac{m-1}{s^2} w^k + \frac{n-1}{t^2} w^k + w^k\big) \, d\mu(s,t) \\ &= \int_{\widehat{\Omega}} (w^k - \epsilon)^+ H \, d\mu(s,t) \leq 0, \end{aligned}$$

and thus

$$\int_{\widehat{\Omega}} |\nabla_{s,t}(w^k - \epsilon)^+|^2 \, d\mu(s,t) + \int_{\widehat{\Omega}} |(w^k - \epsilon)^+|^2 \big(\frac{m-1}{s^2} + \frac{n-1}{t^2} + 1\big) \, d\mu(s,t) \leq 0.$$

We can deduce that $(w^k - \epsilon)^+ = 0$, which implies $w^k \leq \epsilon$ for all $\epsilon > 0$ on $\widehat{\Omega}$. Consequently, As a result, we have $w^k \leq 0$ in $\widehat{\Omega}$, leading to the inequality $su_t^k - tu_s^k \leq 0$. Similarly, we can infer $sv_t^k - tv_s^k \leq 0$. Hence, we conclude that $(u_k, v_k) \in K$. We now proceed to show that u_k and v_k are bounded in $H^1(\Omega)$. Using u_k as a test function, it follows from (67) that

$$\int_\Omega |\nabla u_k|^2 \, dx = \int_\Omega \big[a(x)\tilde{u}_k(e^{\tilde{u}_k^2} - 1) + \; p\tilde{u}_k^{p-1}\tilde{v}_k^q, \big] u_k \, dx. \tag{70}$$

On the other hand,

$$\begin{aligned} &\int_\Omega \big[a(x)\tilde{u}_k(e^{\tilde{u}_k^2} - 1) + \; p\tilde{u}_k^{p-1}\tilde{v}_k^q, \big] u_k \, dx \\ &\leq \|a(x)\|_{L^\infty} \big(\int_\Omega \tilde{u}_k^2 (e^{\tilde{u}_k^2} - 1)^2 \, dx\big)^{\frac{1}{2}} \|u_k\|_{L^2} + p\|\tilde{u}_k^{p-1}\tilde{v}_k^q\|_{L^2}\|u_k\|_{L^2} \\ &\leq \|u_k\|_{L^2} \big(\|a(x)\|_{L^\infty}\|\tilde{u}_k\|_{L^4}\|e^{\tilde{u}_k^2} - 1\|_{L^4} + \|p\tilde{u}_k^{p-1}\|_{L^4}\|\tilde{v}_k^q\|_{L^4}\big) \\ &\leq C\|u_k\|_{H^1}, \end{aligned} \tag{71}$$

where we have used Theorem 2.9 in the last inequality. From (70) and (71) we deduce that

$$\|u_k\|_{H^1}^2 \leq C\|u_k\|_{H^1}.$$

Similarly, for v_k, we obtain

$$\|v_k\|_{H^1}^2 \leq C\|v_k\|_{H^1}.$$

These inequalities imply that both u_k and v_k are bounded. After passing to a subsequences (still denoted by u_k and v_k), we may assume that $u_k \rightharpoonup u$ and $v_k \rightharpoonup v$ weakly in $H^1(\Omega)$ for some u and v. More precisely, we have $u, v \in H^1_G(\Omega)$. On the other hand, since $\tilde{u}_k^2(e^{\tilde{u}_k^2}-1)^2 \to \tilde{u}^2(e^{\tilde{u}^2}-1)^2$ pointwise and the function $y \to y^2(e^{y^2}-1)^2$ is increasing, we can apply the monotone convergence theorem to conclude that $\tilde{u}_k(e^{\tilde{u}_k^2}-1) \to \tilde{u}(e^{\tilde{u}^2}-1)$ in $L^2(\Omega)$. Similarly we have $p\tilde{u}_k^{p-1}\tilde{v}_k \to p\tilde{u}^{p-1}\tilde{v}_k$. Passing to a subsequence deduce that $u_k \to u$ in $W^{2,2}(\Omega)$. It follows that $\nabla u_k \to \nabla u$ in $L^2(\Omega)$, and therefore $u_s^k \to u_s$ a.e. in Ω and $u_t^k \to u_t$ a.e. in Ω. Hence, $u_s^k \to u_s$ a.e. $(s,t) \in \widehat{\Omega}$ and $u_t^k \to u_t$ a.e. $(s,t) \in \widehat{\Omega}$. Indeed, setting $w := su_t - tu_s$, we have $w^k \to w$ a.e. $(s,t) \in \widehat{\Omega}$, and consequently $w \leq 0$ in $\widehat{\Omega}$. This implies that $su_t - tu_s \leq 0$. Likewise, we can demonstrate that $sv_t - tv_s \leq 0$, leading to the desired conclusion that $(u,v) \in K$. □

Proof of Theorem 1.3. By Lemma 3.2 and Proposition 3.3, We can deduce conditions (i) and (ii) in Theorem 3.1 respectively. This demonstrates the existence of a weak solution for (45). □

Funding

A.M. is pleased to acknowledge the support of the Natural Sciences and Engineering Research Council of Canada.
A.K. and S.M. are grateful to CNPq (401772/2022-5) and Coordenação de Aperfeiçoamento de Pessoal de Nível Superior of Brazil for their support.

Author Contribution

This is the collaborative work of Alireza Khatib, Abbas Moameni, and Somayeh Mousavinasr.

Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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