A Neumann problem with supercritical and exponential growth in dimension $N \geq 3$ *

Alireza Khatib † Abbas Moameni‡ Somayeh Mousavinasr §

Abstract

We shall consider the problem

$$\begin{cases} -\Delta u + u = f(u), & x \in \Omega \\ u > 0, & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded annulus in $\mathbb{R}^N (N \geq 3)$ and the function f has either the exponential growth by the means of Trudinger-Moser inequality $f(u) = u(e^{u^2} - 1)$, or is of the power form $f(u) = u|u|^{p-2}$ where p is supercritical. It is standard that this problem always possess a positive radial solution. Our main goal in this paper is to prove the existence of a positive non-radial solution for the case $f(u) = u(e^{u^2} - 1)$, and to prove the multiplicity of non-radial positive solutions for the case $f(u) = u|u|^{p-2}$ when the annulus is thin. We shall first state our results for a general annulus when the right hand side of (1) is of the form $a(x)f(u)$ where the function $a(x)$ belongs to a class of sufficiently smooth non-negative functions which enjoys certain symmetry and monotonicity properties. This class includes the case where the function a is radial.

1 Introduction

In this paper we study the existence of positive solutions of the Neumann problem given by

$$\begin{cases} -\Delta u + u = a(x)f(u), & x \in \Omega \\ u > 0, & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

where Ω is an annulus in $\mathbb{R}^N (N \geq 3)$ and the function $a(x)$ is a non-negative sufficiently smooth function which has some symmetry and monotonicity properties. When f has a subcritical nonlinearity, one can utilize a standard variational approach to obtain solutions of (2). In the case of supercritical f, (2) cannot be treated using standard variational techniques due to the absence of appropriate Sobolev embeddings. We shall address both cases $f(u) = u(e^{u^2} - 1)$ and $f(u) = u|u|^{p-2}$. We work on an appropriate convex subset of $H^1(\Omega)$ where Sobolev imbeddings can be improved due to the monotonicity of the underlying functions, allowing suitable supercritical nonlinearities to be handled.
In [2], the authors considered a variant of (2) given by $-\Delta u + u = |x|^\alpha u^p$ in B_1 (the unit ball in $\mathbb{R}^N$ centered at the origin), with $\frac{\partial u}{\partial \nu} = 0$ on ∂B_1. They employ a shooting method to prove the existence of a positive,

*A.M. is pleased to acknowledge the support of the Natural Sciences and Engineering Research Council of Canada. A.K. and S.M. are grateful to CNPq (200197/2023-1) and Coordenação de Aperfeiçoamento de Pessoal de Nível Superior of Brazil for their support.

†Universidade Federal do Amazonas, Manaus-AM, Brazil, alireza@ufam.edu.br

‡School of Mathematics and Statistics, Carleton University, Ottawa, ON, Canada, momeni@math.carleton.ca

§Universidade Federal do Amazonas, Manaus-AM, Brazil, somayeh@ufam.edu.br

radially increasing solution for every $p > 1$ and $\alpha > 0$. Additionally, they conduct numerical computations to observe the existence of oscillating solutions.
In [20] Serra and Tilli considered the variant of (2) where $a(x)$ is replaced with $a(|x|)$, $\Omega = B_1$ and $f(u)$ is still a supercritical nonlinearity. They then considered the associated classical energy

$$E(u) := \frac{1}{2}\int_\Omega (|\nabla u|^2 + u^2)\, dx - \int_\Omega a(|x|)F(u)\, dx,$$

where $F'(u) = f(u)$. To overcome a lack of compactness caused by the supercritical nature of f, they seek critical points of E over the cone $\{u \in H^1_{rad}(B_1) : 0 \le u, u \textit{ is increasing}\}$. Their main contribution lies in demonstrating that critical points of E on the cone are indeed critical points across the entire space. Subsequently, many other studies have successfully utilized this technique. In [12] Grossi and Noris dealt with

$$\begin{cases} -\Delta u + V(|x|)u = |u|^{p-2}u, & x \in B_1 \\ u > 0, & x \in B_1, \end{cases} \tag{3}$$

under both Neumann and Dirichlet homogeneous boundary conditions. The authors assume that $V(|x|) \ge 0$ and $V \not\equiv 0$. Under these assumptions, they prove the existence of a radial solution $u_p = u_p(r)$ of (3) with a Neumann boundary condition, where p is sufficiently large. The solution u_p satisfies $u_p(|x|) \to \frac{G(|x|,1)}{G(1,1)}$ as $p \to \infty$, where $G(r, z)$ is the Green's function associated with the one-dimensional operator

$$\mathcal{L}(u) = -u'' \frac{-N-1}{r} u' + V(r)u, \qquad u'(0) = u'(1) = 0.$$

In [8], positive solutions to equation

$$\begin{cases} -\Delta u + u = a(x)|u|^{p-2}u, & x \in B_1 \\ u > 0, & x \in B_1 \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial B_1, \end{cases} \tag{4}$$

were obtained using a new variational principle, under the same assumptions as earlier works. Specifically, they assumed $a(x) > 0$ and $p > 1$, and established the existence of nontrivial solutions for a range of values of p. Notably, in the case of $a(x) = 1$, their approach allowed them to show that the solution was nonconstant, which was already a known result. However, their method also enabled them to deal directly with a supercritical nonlinearity, without requiring any cut off procedures. For further results regarding these Neumann problems on radial domains see [1, 12, 4, 3, 5, 6, 17].
In [7], the authors examined (2) where Ω is a bounded domain in $\mathbb{R}^N$ with certain symmetry assumptions, namely Ω was a domain of m revolution. They assumed that the coefficient a also satisfied some symmetry conditions. In the case where $f(u) = u^{p-1}$, which corresponds to a supercritical result when $p > 2^* := \frac{2N}{N-2}$, they were able to obtain positive nontrivial monotonic solutions, provided $2 < p < 2^*_m := \frac{2m}{m-2}$ for some $m < N$. For Neumann problems on general domains see [9, 11, 13, 14, 10, 15, 18, 24].
Throughout this paper, we work with the annulus Ω centered at the origin and having inner radius R_1 and outer radius R_2, which is defined as

$$\Omega = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}.$$

We will begin our discussion by considering the following equation with a Trudinger-Moser exponential nonlinearity:

$$\begin{cases} -\Delta u + u = a(x)u(e^{u^2} - 1), & x \in \Omega \\ u > 0, & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{5}$$

Let $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$ where $m, n \ge 1$ and $m + n = N$. The variables s and t are defined as

$$s := \sqrt{x_1^2 + \cdots + x_m^2} \quad , \quad t := \sqrt{x_{m+1}^2 + \cdots + x_N^2}. \tag{6}$$

Using these definitions, Ω can be expressed as $\Omega = \{x \in \mathbb{R}^N : R_1^2 < s^2 + t^2 < R_2^2\}$. We denote $\widehat{\Omega}$ to be the subset of $\mathbb{R}^2$ defined by

$$\widehat{\Omega} = \{(s,t) \in \mathbb{R}^2 : s > 0,\ t > 0,\ R_1^2 < s^2 + t^2 < R_2^2\}.$$

For our applications we consider problem (5) in the case where a is a continuous function of (s,t) and it satisfies the following property:

$\mathcal{A}(m,n)$: $a = a(x)$ is a function of (s,t) where s and t are given in (6), and $a(s,t)$ is continuously differentiable function with respect to (s,t) and $sa_t - ta_s \leq 0$ in $\widehat{\Omega}$.

When there is no confusion we just use $\mathcal{A}$ instead of $\mathcal{A}(m,n)$. Typical examples of functions a satisfying $\mathcal{A}(m,n)$ are $a(x) = 1$, and $a(x) = |x|^\alpha$ for $\alpha \geq 2$.

Our main results related to the Neumann problem (5) are stated in the following two theorems.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^N$ be an annulus domain. Assume that a satisfies condition ($\mathcal{A}$). Then there exists a positive weak solution to the Neumann problem (5).*

In our next result we will show that even when the coefficient $a(x)$ in (5) is radial, the solution obtained in Theorem 1.1 is nonradial. To illustrate this, we consider the problem

$$\begin{cases} -\Delta u + u = a(|x|)u(e^{u^2} - 1), & x \in \Omega \\ u > 0, & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{7}$$

with a radial coefficient $a(|x|)$ and show that the solution obtained in Theorem 1.1 does not possess radial symmetry.

Theorem 1.2 *Let u be the solution of (7) obtained in Theorem 1.1. Assume that $\beta > N$ where*

$$\beta = \inf_{\psi \in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla\psi|^2\, dx + \int_\Omega \psi^2\, dx}{\int_\Omega \frac{\psi^2}{|x|^2}\, dx}. \tag{8}$$

Then u is nonradial. In particular, if $R_1 = R$, $R_2 = R+1$, then β is sufficiently large for large values of R, which leads to the existence of a positive weak nonradial solution of (7).

Observe that β denotes the optimal constant in the classical Hardy inequality on the domain Ω, which is attained since Ω does not contain the origin and is not an exterior domain. In addition to our study of the Trudinger-Moser exponential growth problem, we also investigate the semilinear Neumann problem with power-type nonlinearity given by

$$\begin{cases} -\Delta u + u = a(x)|u|^{p-2}u, & x \in \Omega \\ u > 0, & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{9}$$

We are now list our results regarding the existence and multiplicity of positive solutions for the Neumann problem (9).

Theorem 1.3 *Assume that $\Omega \subset \mathbb{R}^N$ is an annulus, and a satisfies ($\mathcal{A}(m,n)$) with $n \leq m$ and $N = m+n$. Assume that $1 \leq p < \dfrac{2(n+1)}{n-1}$ for $n > 1$, and $2 < p < \infty$ for $n = 1$. Then (9) has a positive weak solution.*

Next we shall prove that the solution obtained in Theorem 1.3 is nonradial when radii R_1, R_2 satisfy certain conditions. To do so, we consider $a(x) = a(|x|)$ is radial, that is

$$\begin{cases} -\Delta u + u = a(|x|)|u|^{p-2}u, & x \in \Omega \\ u > 0, & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{10}$$

It should be noted that for this specific case, problem (10) always have a positive radial solution. The following theorem will prove that it is also possible to obtain non-radial solutions.

Theorem 1.4 *Suppose that $N = m + n$ with $m, n \geq 1$. Let u be the solution of equation (10) obtained in Theorem 1.3. If $p > 2$ and*

$$p - 2 > 2N/\beta,$$

where β is the optimal constant in the Hardy inequality given in (8), then u is nonradial.

The following theorem concerns with the existence of multiple positive solutions for (10).

Theorem 1.5 *Given a natural number $1 \leq k \leq \lfloor \frac{N}{2} \rfloor$, if $2 < p < \infty$ satisfies $2 + \frac{2N}{\beta} < p < \frac{2k+2}{k-1}$ when $k > 1$, and $2 + \frac{2N}{\beta} < p$ when $k = 1$, then the equation (10) has k positive distinct nonradial solutions. Moreover, if $R_1 = R$, $R_2 = R + 1$, and $k = \lfloor \frac{N}{2} \rfloor$, then for large values of R, the value of β is sufficiently large, which results in the existence of $\lfloor \frac{N}{2} \rfloor$ distinct positive nonradial weak solutions of (10) when*

$$2 < p < \frac{2\lfloor \frac{N}{2} \rfloor + 2}{\lfloor \frac{N}{2} \rfloor - 1}.$$

The paper is structured as follows. In Section 2, we present our main results related to the Trudinger-Moser exponential growth equation (5). In Section 3, we focus on a semilinear Neumann problem with a power-type nonlinearity, namely equation (9).

2 Semilinear Neumann elliptic problem with a Trudinger-Moser exponential growth

In this section we deal with

$$\begin{cases} -\Delta u + \; u \; = \; a(x)u(e^{u^2} - 1), & x \in \Omega \\ u > 0, & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{11}$$

where Ω is an annulus in $\mathbb{R}^N (N \geq 3)$ and the function $a(x)$ is a nonnegative sufficiently smooth function which satisfies condition $(\mathcal{A})$.
Let us first briefly recall some standard notations from the theory of Orlicz spaces.

Definition 2.1 *Let $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex increasing function such that*

$$\xi(0) = 0 = \lim_{s \to 0^+} \xi(s), \qquad \lim_{s \to \infty} \xi(s) = \infty.$$

We say that a measurable function $u : \Omega \to \mathbb{R}$ belongs to L^ξ if there exists $\lambda > 0$ such that

$$\int_{\mathbb{R}^d} \xi \left(\frac{|u(x)|}{\lambda} \right) dx < \infty.$$

We denote then

$$\|u\|_{L^\xi} = \inf \left\{ \lambda > 0, \; \int_\Omega \xi \left(\frac{|u(x)|}{\lambda} \right) dx \leq 1 \right\}. \tag{12}$$

It is standard that $\|\cdot\|_{L^\xi}$ is a norm. In what follows, we shall fix $\xi(s) = e^{s^2} - 1$ and denote the Orlicz space L^ξ by $\mathcal{L}$ endowed with the norm $\|\cdot\|_{\mathcal{L}}$.
Even though our results are for domains in $\mathbb{R}^N$ with $N \geq 3$, but it is worth noting that the Sobolev embedding for bounded domains in dimension $N = 2$ for the Orlicz space $\mathcal{L}$ can be expressed as follows:

Lemma 2.2 *Suppose $\mathcal{O}$ is a bounded domain in $\mathbb{R}^2$. Then*

$$\|u\|_{\mathcal{L}(\mathcal{O})} \leq \frac{1}{\sqrt{4\pi}} \|u\|_{H^1(\mathcal{O})}. \tag{13}$$

We would like to emphasize that the embedding (13) can be directly inferred from the Trudinger–Moser inequality, which was proven in [19].

Proposition 2.3 *There exists a constant κ such that for any domain $\mathcal{O} \subset \mathbb{R}^2$*

$$\sup_{\|u\|_{H^1(\mathcal{O})} \leq 1} \int_{\mathcal{O}} (e^{4\pi u^2} - 1)\, dx \leq \kappa, \tag{14}$$

The inequality is sharp: for any growth $e^{\alpha u^2}$ with $\alpha > 4\pi$ the supremum is $+\infty$.

We shall consider the Banach space $V = H^1(\Omega) \cap \mathcal{L}$, and its topological dual as V^*. The norm on V and the duality pairing between V and V^* are given by

$$\|u\|_V = \|u\|_{H^1(\Omega)} + \|u\|_{\mathcal{L}},$$

$$\langle u, u^* \rangle = \int_\Omega u(x) u^*(x)\, dx, \qquad \forall\, u \in V,\ \forall\, u^* \in V^*.$$

Let $I : V \to \mathbb{R}$ be the Euler-Lagrange functional corresponding to (11) i.e.,

$$I(u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + u^2)\, dx - \frac{1}{2} \int_\Omega a(x)(e^{u^2} - u^2 - 1)\, dx.$$

We can decompose I as the difference of two functionals, namely, Ψ and Φ, where

$$\Psi = \frac{1}{2} \int_\Omega (|\nabla u|^2 + u^2)\, dx \quad , \quad \Phi = \frac{1}{2} \int_\Omega a(x)(e^{u^2} - u^2 - 1)\, dx,$$

and we have

$$I = \Psi - \Phi.$$

Note that Φ is a continuously differentiable function on V and Ψ is a proper (i.e. $Dom(\Psi) \neq \emptyset$), convex and lower semi-continuous. Consider

$$H^1_G = \{u \in H^1(\Omega) : gu = u, \forall g \in G\}$$

where $G := O(m) \times O(n)$ where $O(k)$ is the orthogonal group in $\mathbb{R}^k$ and $gu(x) := u(g^{-1}x)$. To solve equations defined on an annulus $\Omega = \{x \in \mathbb{R}^N : R_1^2 < s^2 + t^2 < R_2^2\}$, it is often useful to relate the equation to a new equation defined on $\widehat{\Omega} = \{(s,t) \in \mathbb{R}^2 : s > 0,\ t > 0,\ R_1^2 < s^2 + t^2 < R_2^2\}$. By transforming the problem to a new domain, one can take advantage of its simpler geometry and use techniques that are better suited to the problem at hand. Assume that $u(x)$ is a solution to the equation

$$\begin{cases} -\Delta u(x) + u(x) = f(x), & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{15}$$

with f being a function defined on Ω with the same symmetry as the domain (i.e., for any $g \in G$, we have $gf(x) := f(g^{-1}x)$). Then, u can be written as $u = u(s,t)$ and it satisfies the equation

$$-u_{ss} - u_{tt} - \frac{m-1}{s} u_s - \frac{n-1}{t} u_t + u = f(s,t), \quad in\ \widehat{\Omega}, \tag{16}$$

with $\dfrac{\partial u}{\partial \nu} = s u_s + t u_t = 0$ on $(s,t) \in \partial\widehat{\Omega} \setminus \big(\{s = 0\} \cup \{t = 0\}\big)$. If u is sufficiently smooth then $u_s = 0$ on $\partial\widehat{\Omega} \cap \{s = 0\}$ and $u_t = 0$ on $\partial\widehat{\Omega} \cap \{t = 0\}$ after considering the symmetry properties of u.

In order to improve compactness, we now define the convex set K as the set of functions which are monotonic in an angle. More precisely, $K = K(m,n)$ is given by

$$K = K(m,n) = \{0 \le u \in H^1_G : su_t - tu_s \le 0 \ \ a.e. \ \ in \ \widehat{\Omega}\}. \tag{17}$$

Note that we can express (s,t) in terms of polar coordinates as $s = r\cos(\theta)$, $t = r\sin(\theta)$, where $r = |x| = |(s,t)|$ and θ is the usual polar angle in the (s,t) plane. Using this representation, we can rewrite the set K as a set of functions u that satisfy the inequality $u_\theta \le 0$ in

$$\widetilde{\Omega} = \big\{(\theta, r) : R_1 < r < R_2, \ \theta \in (0, \frac{\pi}{2})\big\}.$$

Let us now introduce the functional $I_K(u) : V \to (-\infty, +\infty]$ by

$$I_K = \Psi_K - \Phi, \tag{18}$$

where Ψ_K is the restriction of Ψ to K defined by

$$\Psi_K(u) = \begin{cases} \Psi(u), & u \in K \\ +\infty, & u \notin K. \end{cases}$$

We will now recall the definition of a critical point for lower semi-continuous functions introduced by Szulkin [22].

Definition 2.4 *Let V be a real Banach space and $\Psi : V \to (-\infty, +\infty]$ be proper, convex and lower semi-continuous. Let E be a function on V defined by*

$$E := \Psi - \Phi, \tag{19}$$

where $\Phi \in C^1(V, \mathbb{R})$. A point $u_0 \in V$ is said to be a critical point of E if $u \in Dom(\Psi)$ and if it satisfies the inequality

$$\langle D\Phi(u), u - v\rangle + \Psi(v) - \Psi(u) \ge 0 \ , \quad \forall v \in V.$$

Definition 2.5 *We say that E defined in (19) satisfies the Palais–Smale compactness condition (PS) if every sequence u_j such that*

- $E[u_j] \to c \in \mathbb{R}$,
- $\langle D\Phi(u_j), u_j - v\rangle + \Psi(v) - \Psi(u_j) \ge -\epsilon_j \|v - u_j\|, \quad \forall v \in V,$

where $\epsilon_j \to 0$, then $\{u_j\}$ possesses a convergent subsequence.

The following theorem by A. Szulkin [22] is a very useful result called the Mountain Pass Theorem.

Theorem 2.6 *Suppose that $E : V \to (-\infty, +\infty]$ is of the form (19) and satisfies the Palais-Smale condition and the Mountain Pass Geometry (MPG):*

1. *$E(0) = 0$.*
2. *There exists $e \in V$ such that $E(e) \le 0$.*
3. *There exists some ρ such that $0 < \rho < \|e\|$ and for every $u \in V$ with $\|u\| = \rho$ one has $E(u) > 0$.*

Then E has a critical value $c > 0$ which is characterized by

$$c = \inf_{\gamma \in \Gamma} \sup_{\tau \in [0,1]} E(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0,1], V) : \gamma(0) = 0, \gamma(1) = e\}$.

Inspired by the variational principle proposed in [16], we prove the following result.

Theorem 2.7 *Let $V = H^1(\Omega) \cap \mathcal{L}$ and K be a convex and closed subset defined in (17). Let a be a non-negative continuously differentiable function that is not identically zero. Assume that the following two assertions hold:*

(i) The functional $I_K : V \to \mathbb{R}$ defined in (18) has a critical point $\bar{u} \in V$ as in Definition 2.4, and;

(ii) There exists $\bar{v} \in K$ with $\dfrac{\partial \bar{v}}{\partial \nu} = 0$ on the boundary of Ω such that

$$-\Delta \bar{v} + \bar{v} \ = \ a(x)\bar{u}(e^{\bar{u}^2} - 1),$$

in the weak sense, i.e.,

$$\int_\Omega \nabla \bar{v} \nabla \eta \, dx + \int_\Omega \bar{v}\eta \, dx = \int_\Omega a(x)\bar{u}(e^{\bar{u}^2} - 1)\eta \, dx, \qquad \forall \eta \in V. \tag{20}$$

Then $\bar{u} \in K$ is a weak solution of the equation

$$\begin{cases} -\Delta u + \ u \ = \ a(x)u(e^{u^2} - 1), & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{21}$$

Proof: Since $\bar{u}$ is a critical point of I, it follows from Definition 2.4 that

$$\langle D\Phi(\bar{u}), \bar{u} - v\rangle + \Psi(v) - \Psi(\bar{u}) \geq 0 \ , \quad \forall v \in V,$$

which means

$$\frac{1}{2}\int_\Omega |\nabla \bar{u}|^2 + \bar{u}^2 \, dx - \frac{1}{2}\int_\Omega |\nabla v|^2 + v^2 \, dx \leq \int_\Omega a(x)\bar{u}(e^{\bar{u}^2} - 1)(\bar{u} - v) \, dx, \quad \forall v \in K. \tag{22}$$

On the other hand, by (ii), there exists $\bar{v} \in K$ satisfying (20). Thus, by substituting $\eta = \bar{u} - \bar{v}$ in (20) one gets

$$\int_\Omega \nabla \bar{v} \nabla(\bar{u} - \bar{v}) \, dx + \int_\Omega \bar{v}(\bar{u} - \bar{v}) \, dx = \int_\Omega a(x)\bar{u}(e^{\bar{u}^2} - 1)(\bar{u} - \bar{v}) \, dx. \tag{23}$$

Now by setting $v = \bar{v}$ in (22), and taking into account the equality (23) we obtain that

$$\frac{1}{2}\int_\Omega |\nabla \bar{u}|^2 + \bar{u}^2 \, dx - \frac{1}{2}\int_\Omega |\nabla \bar{v}|^2 + \bar{v}^2 \, dx \leq \int_\Omega \nabla \bar{v} \nabla(\bar{u} - \bar{v}) \, dx + \int_\Omega \bar{v}(\bar{u} - \bar{v}) \, dx,$$

from which we deduce

$$\frac{1}{2}\int_\Omega |\nabla \bar{u} - \nabla \bar{v}|^2 \, dx + \frac{1}{2}\int_\Omega |\bar{u} - \bar{v}|^2 \, dx \leq 0, \tag{24}$$

which implies that $\bar{u} = \bar{v}$ for a.e. $x \in \Omega$. Taking into account that $\bar{u} = \bar{v}$ in (20) we have that $\bar{u}$ is a weak solution of (21).

□

Our approach to prove the main result in this section involves applying Theorem 2.7. In order to demonstrate the validity of condition (i) in this theorem and proof of the existence of a critical point for the nonsmooth functional I_K, we will begin by establishing the following theorems, which provides important insights into the problem at hand. Hereafter C will denote a positive constant, not necessarily the same one.

Theorem 2.8 *For $N \geq 3$, let Ω be an annular domain in $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$, where $n = 1$ and $m = N - 1$. Then for every $\alpha \leq 4\pi$, we have*

$$\sup_{u \in K, \ \|u\|_{H^1(\Omega)} \leq 1} \int_\Omega (e^{\alpha u^2} - 1) \, dx < \infty, \tag{25}$$

where $K = K(N-1, 1)$ is a convex and closed subset of $H^1(\Omega)$ defined in (17). In particular we have $K \cap H^1(\Omega) \subset V$.

Proof: To prove the statement, we make use of polar coordinates $(s,t) = (r\cos\theta, r\sin\theta)$ and write the integral in terms of r and θ. For a given function $u = u(s,t) \in K$, the integral becomes

$$\int_\Omega (e^{\alpha u^2} - 1)\, dx = C \int_{\widehat{\Omega}} (e^{\alpha u(s,t)^2} - 1) s^{N-2}\, ds\, dt = C \int_0^{\frac{\pi}{2}} \int_{R_1}^{R_2} r^{N-2} \cos^{N-2}(\theta)(e^{\alpha u(r,\theta)^2} - 1)\, r\, dr\, d\theta.$$

If we choose $\theta \in [\frac{\pi}{3}, \frac{\pi}{2}]$, since $\theta \to e^{\alpha u(r,\theta)^2}$ is monotone, we obtain that

$$\begin{aligned}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_{R_1}^{R_2} r^{N-2} \cos^{N-2}(\theta)(e^{\alpha u(r,\theta)^2} - 1)\, r\, dr\, d\theta &\leq \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_{R_1}^{R_2} r^{N-2} \cos^{N-2}(\theta - \frac{\pi}{4})(e^{\alpha u(r,\theta-\frac{\pi}{4})^2} - 1)\, r\, dr\, d\theta \\ &\leq \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \int_{R_1}^{R_2} r^{N-2} \cos^{N-2}(\theta)(e^{\alpha u(r,\theta)^2} - 1)\, r\, dr\, d\theta,\end{aligned}$$

and therefore

$$\int_0^{\frac{\pi}{2}} \int_{R_1}^{R_2} r^{N-2} \cos^{N-2}(\theta)(e^{\alpha u(r,\theta)^2} - 1)\, r\, dr\, d\theta \leq 2 \int_0^{\frac{\pi}{3}} \int_{R_1}^{R_2} r^{N-2} \cos^{N-2}(\theta)(e^{\alpha u(r,\theta)^2} - 1)\, r\, dr\, d\theta.$$

On the other hand,

$$\int_0^{\frac{\pi}{3}} \int_{R_1}^{R_2} r^{N-2} \cos^{N-2}(\theta)(e^{\alpha u(r,\theta)^2} - 1)\, r\, dr\, d\theta = \int_{\{\widehat{\Omega}, s \geq \delta\}} (e^{\alpha u(s,t)^2} - 1) s^{N-2}\, ds\, dt, \tag{26}$$

for some positive constant $\delta > 0$. By setting $\mathcal{O} = \{\widehat{\Omega}, s \geq \delta\}$ we have that

$$\begin{aligned} 1 \geq \|u\|^2_{H^1(\Omega)} &= C(m,n) \int_{\widehat{\Omega}} (u_t^2 + u_s^2 + u^2) s^{N-2}\, ds\, dt \\ &\geq C(m,n) \int_{\mathcal{O}} (u_t^2 + u_s^2 + u^2) s^{N-2}\, ds\, dt \\ &\geq C(m,n)\delta^{N-2} \int_{\mathcal{O}} (u_t^2 + u_s^2 + u^2)\, ds\, dt. \end{aligned}$$

Therefore, we have that

$$\int_{\mathcal{O}} (u_t^2 + u_s^2 + u^2)\, ds\, dt \leq \frac{1}{C(m,n)\delta^{N-2}}. \tag{27}$$

Looking at the term on the right hand side of (26), and applying Proposition 2.3, we have

$$\int_{\{\widehat{\Omega}, s \geq \delta\}} (e^{\alpha u(s,t)^2} - 1) s^{N-2}\, ds\, dt \leq C \int_{\mathcal{O}} (e^{\alpha u(s,t)^2} - 1)\, ds\, dt < \infty,$$

due to the inequality (27). This completes the proof. □

Theorem 2.9 *For $N \geq 3$, let Ω be an annulus in $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$. Assume that $n \leq m$ and*

$$1 \leq p < \frac{2(n+1)}{n-1}.$$

Then the imbedding $K(m,n) \subset L^p(\Omega)$ is compact with the obvious interpretation if $n = 1$.

Proof: We shall show that $\| u \|_{L^p} \leq C(\| u \|_{L^2} + \| \nabla u \|_{L^2})$ for all $u \in K$. We write (s,t) in terms of polar coordinates $s = r\cos\theta$ and $t = r\sin\theta$. Then for $u = u(s,t)$, we have

$$\int_{\widehat{\Omega}} |u(s,t)|^p s^{m-1} t^{n-1}\, ds\, dt = \int_0^{\frac{\pi}{2}} \int_{R_1}^{R_2} r^{m-1} \cos^{m-1}(\theta) r^{n-1} \sin^{n-1}(\theta) u(r,\theta)^p\, r\, dr\, d\theta.$$

If we choose $\theta \in [\frac{\pi}{3}, \frac{\pi}{2}]$ we see that there exist some constant C such that $\sin\theta \leq C\sin(\theta - \frac{\pi}{4})$. Since $\theta \to u(r,\theta)$ is monotone, we obtain that

$$\begin{aligned}
&\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_{R_1}^{R_2} r^{m-1}\cos^{m-1}(\theta) r^{n-1}\sin^{n-1}(\theta) u(r,\theta)^p \; r \; dr \; d\theta \\
&\leq C \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_{R_1}^{R_2} r^{m-1}\cos^{m-1}(\theta - \frac{\pi}{4}) r^{n-1}\sin^{n-1}(\theta - \frac{\pi}{4}) u(r,\theta - \frac{\pi}{4})^p \; r \; dr \; d\theta \\
&\leq C \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \int_{R_1}^{R_2} r^{m-1}\cos^{m-1}(\theta) r^{n-1}\sin^{n-1}(\theta) u(r,\theta)^p \; r \; dr \; d\theta,
\end{aligned}$$

and therefore

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \int_{R_1}^{R_2} r^{m-1}\cos^{m-1}(\theta) r^{n-1}\sin^{n-1}(\theta) u(r,\theta)^p \; r \; dr \; d\theta \\
&\leq C \int_0^{\frac{\pi}{3}} \int_{R_1}^{R_2} r^{m-1}\cos^{m-1}(\theta) r^{n-1}\sin^{n-1}(\theta) u(r,\theta)^p \; r \; dr \; d\theta.
\end{aligned}$$

On the other hand,

$$\int_0^{\frac{\pi}{3}} \int_{R_1}^{R_2} r^{m-1}\cos^{m-1}(\theta) r^{n-1}\sin^{n-1}(\theta) u(r,\theta)^p \; r \; dr \; d\theta = \int_{\{\widehat{\Omega}, s \geq \delta\}} |u(s,t)|^p s^{m-1} t^{n-1} \; ds \; dt,$$

for some positive constant $\delta > 0$. We can bound the right hand side above by

$$\int_{\{\widehat{\Omega}, s \geq \delta\}} |u(s,t)|^p s^{m-1} t^{n-1} \; ds \; dt \leq C \int_{\{\widehat{\Omega}, s \geq \delta\}} |u(s,t)|^p t^{n-1} \; ds \; dt.$$

Then by a change of variable $t = |y|$ we obtain

$$\int_{\{\widehat{\Omega}, s \geq \delta\}} |u(s,t)|^p t^{n-1} \; ds \; dt = \int_{\{\Omega_1, s \geq \delta\}} |u(s,y)|^p \; ds \; dy,$$

where $\Omega_1 = \{(s,y) : (s,|y|) \in \widehat{\Omega}\} \subset \mathbb{R}^{n+1}$. If $p \leq \dfrac{2(n+1)}{n-1}$ then

$$\begin{aligned}
(\int_{\{\Omega_1, s \geq \delta\}} |u(s,y)|^p \; ds \; dy)^{\frac{2}{p}} &\leq C \int_{\{\Omega_1, s \geq \delta\}} (|\nabla u(s,y)|^2 + u(s,y)^2) \; ds \; dy \\
&\leq C \int_{\{\widehat{\Omega}, s \geq \delta\}} (|\nabla u(s,t)|^2 + u(s,t)^2) \; t^{n-1} s^{m-1} ds \; dt \\
&\leq C \int_{\widehat{\Omega}} (|\nabla u(s,t)|^2 + u(s,t)^2) \; t^{n-1} s^{m-1} ds \; dt \\
&\leq C \int_{\Omega} (|\nabla u|^2 + u^2) \; dx \\
&= C\|u\|^2_{H^1(\Omega)}.
\end{aligned}$$

□

we can now proceed with verifying the validity of condition (i) in Theorem 2.7.

Lemma 2.10 *Let $K = K(N-1,1)$. The functional I_K defined in (18) fulfills both the mountain pass geometry and (PS) compactness condition.*

Proof: First recall that K is a convex cone in $H_1(\Omega)$ that is weakly closed. By applying Theorem 2.8, we can conclude that there exists a positive constant C such that

$$\|u\|_{H^1(\Omega)} \leq \|u\|_V \leq C\|u\|_{H^1(\Omega)}, \quad \forall u \in K. \tag{28}$$

Now, suppose that u_j is a sequence in K such that $I_K(u_j) \to c \in \mathbb{R}, \epsilon_j \to 0$ and

$$\langle D\Phi(u_j), u_j - v\rangle + \Psi(v) - \Psi(u_j) \geq -\epsilon_j \|v - u_j\|_V, \quad \forall v \in V. \tag{29}$$

Replacing v by ru_j $(r \in \mathbb{R})$ in (29), it becomes

$$\frac{1-r^2}{2}\|u_j\|^2_{H^1(\Omega)} + (r-1)\int_\Omega a(x)u_j^2(e^{{u_j}^2} - 1)\, dx \leq \epsilon_j(r-1)\|u_j\|_V. \tag{30}$$

On the other hand, since $I_K(u_j) \to c$, we have

$$\frac{1}{2}\|u_j\|^2_{H^1(\Omega)} - \frac{1}{2}\int_\Omega a(x)(e^{{u_j}^2} - u_j^2 - 1)\, dx \leq c + 1, \tag{31}$$

for large values of n. Now set $1 < r$ and $r^2 - 1 < 4(r-1)$. We can take $\zeta > 0$ such that

$$\frac{1}{4(r-1)} < \zeta < \frac{1}{r^2 - 1}.$$

Multiply (30) by ζ and adding up with (31) yields that

$$\begin{aligned}
&\frac{1+\zeta(1-r^2)}{2}\|u\|^2_{H^1(\Omega)} + \zeta(r-1)\int_\Omega a(x)u_j^2(e^{{u_j}^2} - 1)\, dx - \frac{1}{2}\int_\Omega a(x)(e^{{u_j}^2} - u_j^2 - 1)\, dx \\
&\leq c + 1 + \zeta\epsilon_j(r-1)\|u_j\|_V.
\end{aligned} \tag{32}$$

The choice of ζ implies that

$$\begin{aligned}
&\frac{1+\zeta(1-r^2)}{2}\|u_j\|^2_{H^1(\Omega)} + \frac{1}{4}\int_\Omega a(x)u_j^2(e^{{u_j}^2} - 1)\, dx - \frac{1}{2}\int_\Omega a(x)(e^{{u_j}^2} - u_j^2 - 1)\, dx \\
&\leq \frac{1+\zeta(1-r^2)}{2}\|u_j\|^2_{H^1(\Omega)} + \zeta(r-1)\int_\Omega a(x)u_j^2(e^{{u_j}^2} - 1)\, dx - \frac{1}{2}\int_\Omega a(x)(e^{{u_j}^2} - u_j^2 - 1)\, dx.
\end{aligned} \tag{33}$$

Also as a consequence of the inequality $x^2(e^{x^2} - 1) - 2(e^{x^2} - x^2 - 1) \geq 0$, we can deduce that

$$\frac{1}{4}\int_\Omega a(x)u_j^2(e^{{u_j}^2} - 1)\, dx - \frac{1}{2}\int_\Omega a(x)(e^{{u_j}^2} - u_j^2 - 1)\, dx \geq 0,$$

which, together with (32) and (33), gives

$$\frac{1+\zeta(1-r^2)}{2}\|u_j\|^2_{H^1(\Omega)} \leq c + 1 + \zeta\epsilon_j(r-1)\|u_j\|_V. \tag{34}$$

Since all the coefficients on the left-hand side of the inequality are positive due to the choice of ζ, we can conclude that

$$\|u_j\|^2_{H^1(\Omega)} \leq C(1 + \|u_j\|_V), \tag{35}$$

for some constant $C > 0$. Since $u_j \in K$, we can conclude, based on (28), that

$$\|u_j\|^2_{H^1(\Omega)} \leq C(1 + \|u_j\|_{H^1(\Omega)}). \tag{36}$$

Standard results in Sobolev spaces allow us to conclude, after possibly passing to a subsequence, that there exists a function $\bar{u} \in H^1(\Omega)$ such that $u_j \rightharpoonup \bar{u}$ weakly in $H^1(\Omega)$. This, in turn, implies that $u_j \to \bar{u}$ strongly in $L^2(\Omega)$. Also by Theorem 2.8 we have that $\bar{u} \in K$. By setting $v = \bar{u}$ in (29), and using Hölder's inequality we obtain

$$
\begin{aligned}
\frac{1}{2}(\|u_j\|^2_{H^1(\Omega)} - \|\bar{u}\|^2_{H^1(\Omega)}) \leq & \int_\Omega a(x) u_j (e^{{u_j}^2} - 1)(\bar{u} - u_j)\, dx + \epsilon_j \|u_j - \bar{u}\|_{H^1(\Omega)} \\
\leq & \|a(x)\|_{L^\infty} \|e^{u_j^2} - 1\|_{L^2} \|u_j(u_j - \bar{u})\|_{L^2} + \epsilon_j \|u_j - \bar{u}\|_{H^1(\Omega)} \\
\leq & \|a(x)\|_{L^\infty} \|e^{u_j^2} - 1\|_{L^2} \|u_j\|_{L^4} \|u_j - \bar{u}\|_{L^4} + \epsilon_j \|u_j - \bar{u}\|_{H^1(\Omega)}. \qquad (37)
\end{aligned}
$$

Furthermore, by the Trudinger-Moser inequality in $H^1(\Omega)$ presented in Theorem 2.8, along with the continuous embedding in Theorem 2.9, one can derive that

$$
\sup_{j \geq 1} \int_\Omega (e^{{u_j}^2} - 1)^2\, dx < \infty.
$$

Hence, from (37), we can conclude that

$$
\limsup_{j \to \infty} (\|u_j\|^2_{H^1(\Omega)} - \|\bar{u}\|^2_{H^1(\Omega)}) \leq 0 \qquad (38)
$$

Using the properties of weak convergence, we also have

$$
0 \leq \liminf (\|u_j\|^2_{H^1(\Omega)} - \|\bar{u}\|^2_{H^1(\Omega)})
$$

which together with (38) implies that $u_j \to \bar{u}$ strongly in $H^1(\Omega)$. This completes the proof of the (PS) compactness condition for the function I_k. We now verify the mountain pass geometry of the functional I_K. It is clear that $I_K(0) = 0$. Take $w \in K$. Then, for any $\lambda > 0$, we have

$$
I_K(\lambda w) = \frac{\lambda^2}{2} \int_\Omega (|\nabla w|^2 + w^2)\, dx - \frac{1}{2} \int_\Omega a(x)(e^{\lambda^2 w^2} - \lambda^2 w^2 - 1)\, dx.
$$

It is now obvious that $I_K(\lambda w) < 0$ for λ sufficiently large. Take $u \in K$ with $\|u\|_{H^1} = \rho > 0$. We have

$$
\begin{aligned}
I_K(u) &= \frac{1}{2} \int_\Omega (|\nabla u|^2 + u^2)\, dx - \frac{1}{2} \int_\Omega a(x)(e^{u^2} - u^2 - 1)\, dx \\
&\geq \frac{1}{4} \|u\|^2_{H^1} - \frac{1}{2} \nu \int_\Omega \left[e^{u^2} - (1 + \frac{1}{2\nu}) u^2 - 1 \right] dx,
\end{aligned}
$$

where $\nu = \|a(x)\|_{L^\infty}$. Note that

$$
\lim_{x \to 0} \frac{1}{x^2} \left[e^{x^2} - (1 + \frac{1}{2\nu}) x^2 - 1 \right] = -\frac{1}{2\nu},
$$

and also

$$
e^{x^2} - (1 + \frac{1}{2\nu}) x^2 - 1 \leq C e^{x^2},
$$

for some constant $C > 0$. Therefore we obtain

$$
e^{x^2} - (1 + \frac{1}{2\nu}) x^2 - 1 \leq C x^3 e^{x^2} - \frac{x^2}{4\nu}.
$$

As a consequence, it follows that

$$
\begin{aligned}
I_K(u) &\geq \frac{1}{4} \|u\|^2_{H^1} + \frac{1}{8} \|u\|^2_{L^2} - \frac{1}{2} \nu C \|u\|^3_{L^6} \Big(\int_\Omega e^{2u^2}\, dx \Big)^{\frac{1}{2}} \\
&\geq \frac{1}{4} \|u\|^2_{H^1} - \frac{1}{2} \nu C \|u\|^3_{L^6} \Big(\int_\Omega e^{2u^2}\, dx \Big)^{\frac{1}{2}}.
\end{aligned}
$$

We may now apply Theorem 2.8 and Sobolev imbedding, to conclude that

$$I_K(u) \geq \frac{1}{4}\|u\|_{H^1}^2 - C\|u\|_{H^1}^3 = \frac{1}{4}\rho^2 - C\rho^3 \ > 0,$$

provided ρ is small enough. If $u \notin K$, then clearly $I_K(u) > 0$. Therefore the mountain pass geometry holds for the functional I_k. □

In the following proposition, we shall prove the invariance property of the equation (11) with respect to the convex set $K = K(m,n)$.

Proposition 2.11 *Let $N = m+n$ with $m, n \geq 1$. Suppose $\Omega \subset \mathbb{R}^N$ is an annular domain, $a \in \mathcal{A}(m,n)$ and $u \in K(m,n)$. Then there exists $v \in K(m,n)$ satisfying*

$$\begin{cases} -\Delta v + \ v \ = \ a(x)u(e^{u^2}-1), & x \in \Omega \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{39}$$

in the weak sense.

Proof: Let $u \in K$ be fixed. By setting $u_k(x) = min\{u(x), k\}$ for $k \geq 1$, we have $u_k \in H^1(\Omega)$. Observe that the cut off does not effect the symmetry and also preserves the monotonicity of u. Therefore since $u \in H^1_G(\Omega)$ has symmetry, we obtain that $u_k \in H^1_G(\Omega)$ and the monotonicity of u and u_k should be the same. Now we shall consider the following problem

$$\begin{cases} -\Delta v + \ v \ = \ a(x)u_k(e^{u_k^2}-1), & x \in \Omega \\ \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{40}$$

Taking into account the associated energy on $H^1_G(\Omega)$ and using the standard regularity theory we can deduce the existence of a unique $0 \leq v^k \in H^3_G(\Omega) \cap C^{1,\alpha}(\Omega)$ which solves (40). We want to show that $v^k \in K(m,n)$. Note that $v^k = v^k(s,t)$ satisfies the equation

$$-v^k_{ss} - v^k_{tt} - \frac{m-1}{s}v^k_s - \frac{n-1}{t}v^k_t + v^k = a(s,t)u_k(e^{u_k^2}-1) \quad in \ \widehat{\Omega}, \tag{41}$$

with $\dfrac{\partial v^k}{\partial \nu} = sv^k_s + tv^k_t = 0$ on $(s,t) \in \partial\widehat{\Omega} \setminus \big(\{s=0\} \cup \{t=0\}\big)$. We set $w^k = sv^k_t - tv^k_s$. Then differentiating w^k with respect to s and t, we obtain

$$w^k_s = v^k_t + sv^k_{ts} - tv^k_{ss} \quad , \quad w^k_t = -v^k_s - tv^k_{st} + sv^k_{tt}$$

and

$$w^k_{ss} = 2v^k_{st} + sv^k_{tss} - tv^k_{sss} \quad , \quad w^k_{tt} = -2v^k_{st} - tv^k_{stt} + sv^k_{ttt}$$

These together with a computation from the equation (41) imply that for all $(s,t) \in \widehat{\Omega}$

$$-w^k_{ss} - w^k_{tt} - \frac{m-1}{s}w^k_s - \frac{n-1}{t}w^k_t + \frac{m-1}{s^2}w^k + \frac{n-1}{t^2}w^k + w^k = H \tag{42}$$

where

$$H = u_k(e^{u_k^2}-1)(sa_t - ta_s) + a(s,t)(2u_k^2e^{u_k^2} + e^{u_k^2} - 1)(s(u_k)_t - t(u_k)_s).$$

This problem behaves like a two dimensional problem away from the sets $\{s=0\}$ and $\{t=0\}$. Since $u_k \geq 0$ and it has the same monotonicity as u, it follows that $H \leq 0$ in $\widehat{\Omega}$. We now turn our attention to what happens along the $\partial\widehat{\Omega}$. On the portions of $\partial\widehat{\Omega}$ that correspond to $\{s=0\}$ and $\{t=0\}$ we have $v^k_s = 0$ and $v^k_t = 0$, respectively. This is enough to ensure $w^k = 0$ on these portions of the boundary. Furthermore, on the remaining portion of $\partial\widehat{\Omega}$ we have

$$\begin{aligned} \frac{\partial w^k}{\partial \nu} &= sw^k_s + tw^k_t = sv^k_t + s^2v^k_{ts} - stv^k_{ss} - tv^k_s - t^2v^k_{st} + stv^k_{tt} \\ &= s(v^k_t + sv^k_{ts} + tv^k_{tt}) - t(sv^k_{ss} + v^k_s + tv^k_{ts}) \\ &= s(\frac{\partial v^k}{\partial \nu})_t - t(\frac{\partial v^k}{\partial \nu})_s = 0. \end{aligned}$$

For $\epsilon > 0$ small consider $\varphi(s,t) := (w^k(s,t) - \epsilon)^+$. Let $d\mu(s,t) = s^{m-1}t^{n-1}\,ds\,dt$ and note that

$$\begin{aligned}
\int_{\widehat{\Omega}} |\nabla_{s,t}(w^k - \epsilon)^+|^2 \, d\mu(s,t) &= \int_{\widehat{\Omega}} \nabla_{s,t}(w^k).\nabla_{s,t}(w^k - \epsilon)^+ \, d\mu(s,t) \\
&= \int_{\widehat{\Omega}} -\Delta_{s,t}(w^k)(w^k - \epsilon)^+ \, d\mu(s,t) \\
&- \int_{\widehat{\Omega}} (w^k - \epsilon)^+ \big(\frac{m-1}{s} w^k_s + \frac{n-1}{t} w^k_t\big) \, d\mu(s,t) \\
&+ \int_{\partial\widehat{\Omega}} (w^k - \epsilon)^+ \frac{\partial w^k}{\partial \nu} \, d\mu(s,t), \qquad (43)
\end{aligned}$$

from which together with the fact that $(w^k(s,t) - \epsilon)^+ = 0$ near $\{s = 0\} \cup \{t = 0\}$ and $\dfrac{\partial w^k}{\partial \nu} = 0$ on the remaining portions of $\partial\widehat{\Omega}$, we obtain

$$\begin{aligned}
&\int_{\widehat{\Omega}} |\nabla_{s,t}(w^k - \epsilon)^+|^2 \, d\mu(s,t) \\
&= \int_{\widehat{\Omega}} -\Delta_{s,t}(w^k)(w^k - \epsilon)^+ \, d\mu(s,t) - \int_{\widehat{\Omega}} (w^k - \epsilon)^+ \big(\frac{m-1}{s} w^k_s + \frac{n-1}{t} w^k_t\big) \, d\mu(s,t). \qquad (44)
\end{aligned}$$

Multiplying inequality (42) by φ and integrating it over $\widehat{\Omega}$ yields that

$$\begin{aligned}
\int_{\widehat{\Omega}} -\Delta_{s,t}(w^k)(w^k - \epsilon)^+ \, d\mu(s,t) - \int_{\widehat{\Omega}} (w^k - \epsilon)^+ \big(\frac{m-1}{s} w^k_s + \frac{n-1}{t} w^k_t\big) \, d\mu(s,t) \\
+ \int_{\widehat{\Omega}} (w^k - \epsilon)^+ \big(\frac{m-1}{s^2} w^k + \frac{n-1}{t^2} w^k + w^k\big) \, d\mu(s,t) \\
= \int_{\widehat{\Omega}} (w^k - \epsilon)^+ H \, d\mu(s,t) \leq 0.
\end{aligned}$$

Therefore, we have

$$\int_{\widehat{\Omega}} |\nabla_{s,t}(w^k - \epsilon)^+|^2 \, d\mu(s,t) + \int_{\widehat{\Omega}} (w^k - \epsilon)^+ \big(\frac{m-1}{s^2} w^k + \frac{n-1}{t^2} w^k + w^k\big) \, d\mu(s,t) \leq 0,$$

thereby giving that

$$\int_{\widehat{\Omega}} |\nabla_{s,t}(w^k - \epsilon)^+|^2 \, d\mu(s,t) + \int_{\widehat{\Omega}} |(w^k - \epsilon)^+|^2 \big(\frac{m-1}{s^2} + \frac{n-1}{t^2} + 1\big) \, d\mu(s,t) \leq 0.$$

We can conclude that $(w^k - \epsilon)^+ = 0$, and hence $w^k \leq \epsilon$ holds for all $\epsilon > 0$ on $\widehat{\Omega}$. As a result, we have $w^k \leq 0$ in $\widehat{\Omega}$, which implies that $v^k \in K(m,n)$. We now proceed to show that v^k is bounded in $H^1(\Omega)$. Using v^k as a test function, it follows from (40) that

$$\int_{\Omega} |\nabla v^k|^2 \, dx + \int_{\Omega} |v^k|^2 \, dx = \int_{\Omega} a(x) u_k (e^{u_k^2} - 1) v^k \, dx. \qquad (45)$$

On the other hand,

$$\begin{aligned}
\int_{\Omega} a(x) u_k (e^{u_k^2} - 1) v^k \, dx &\leq \| a(x) \|_{L^\infty} \big(\int_{\Omega} u_k^2 (e^{u_k^2} - 1)^2 \, dx\big)^{\frac{1}{2}} \| v^k \|_{L^2} \\
&\leq \| a(x) \|_{L^\infty} \|u_k\|_{L^4} \|e^{u_k^2} - 1\|_{L^4} \| v^k \|_{L^2} \\
&\leq C \| v^k \|_{H^1}, \qquad (46)
\end{aligned}$$

where we have used Theorem 2.8 in the last inequality together with the fact that $v^k \in K$. From (45) and (46) we deduce that

$$\| v^k \|^2_{H^1} \leq C \| v^k \|_{H^1},$$

which implies that v^k is bounded. After passing to a subsequence (still denoted by v^k), we may assume that $v^k \rightharpoonup v$ weakly in $H^1(\Omega)$ for some v. More precisely, we have $v \in H^1_G(\Omega)$. On the other hand, since $u_k^2(e^{u_k^2}-1)^2 \to u^2(e^{u^2}-1)^2$ pointwise and the function $y \to y^2(e^{y^2}-1)^2$ is increasing, we can apply the monotone convergence theorem to conclude that $u_k(e^{u_k^2}-1) \to u(e^{u^2}-1)$ in $L^2(\Omega)$. Passing to a subsequence deduce that $v^k \to v$ in $W^{2,2}(\Omega)$. It follows that $\nabla v^k \to \nabla v$ in $L^2(\Omega)$, and therefore $v_s^k \to v_s$ a.e. in Ω and $v_t^k \to v_t$ a.e. in Ω. Hence, $v_s^k \to v_s$ a.e. $(s,t) \in \widehat{\Omega}$ and $v_t^k \to v_t$ a.e. $(s,t) \in \widehat{\Omega}$. Indeed, setting $w := sv_t - tv_s$, we have $w^k \to w$ a.e. $(s,t) \in \widehat{\Omega}$, and consequently $w \le 0$ in $\widehat{\Omega}$. This implies that $v \in K(m,n)$, as desired. □

Proof of Theorem 1.1. Let $K = K(N-1,1)$. From Lemma 2.10, we can conclude that the function I_k satisfies both the mountain pass geometry and the (PS) compactness condition. Consequently, by applying Theorem 2.6, we obtain that I_K possesses a non-trivial critical point $\bar{u} \in K$. Therefore, condition (i) of Theorem 2.7 is satisfied. Furthermore, condition (ii) in Theorem 2.7 can be verified using Proposition 2.11. This completes the proof of the existence of a weak solution for (11). □

Now we shall prove that the solution obtained in Theorem 1.3 is nonradial provided $a(x) = a(|x|)$ is a radial function, that is

$$\begin{cases} -\Delta u + u = a(|x|)u(e^{u^2}-1), & x \in \Omega \\ u > 0, & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{47}$$

Before proceeding with the proof, we need to cover some preliminaries. Consider the variational formulation of an eigenvalue problem given by

$$\mu_1 = \inf_{\psi \in H^1_{loc}(0,\frac{\pi}{2})} \left\{ \int_0^{\frac{\pi}{2}} |\psi'(\theta)|^2 w(\theta)\, d\theta;\ \int_0^{\frac{\pi}{2}} |\psi(\theta)|^2 w(\theta)\, d\theta = 1,\ \int_0^{\frac{\pi}{2}} \psi(\theta) w(\theta)\, d\theta = 0 \right\}, \tag{48}$$

where $w(\theta) = \cos^{m-1}(\theta)\sin^{n-1}(\theta)$ and suppose ψ_1 satisfies the minimization problem. Then (μ_1, ψ_1) satisfies

$$\begin{cases} -\partial_\theta(w(\theta)\psi_1'(\theta)) = \mu_1 w(\theta)\psi_1(\theta), & \theta \in (0, \frac{\pi}{2}) \\ \psi_1'(\theta) > 0, & \theta \in (0, \frac{\pi}{2}) \\ \psi_1'(0) = \psi_1'(\dfrac{\pi}{2}) = 0, \end{cases} \tag{49}$$

and note (μ_1, ψ_1) is the second eigenpair, the first eigenpair is given by $(\mu_0, \psi_0) = (0, 1)$. An easy computation shows that

$$\mu_1 = 2N, \qquad \psi_1(\theta) = \frac{m-n}{N} - \cos(2\theta).$$

Let us recall the definition of the best constant in Hardy inequality for the domain Ω,

$$\beta = \inf_{\psi \in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla\psi|^2\, dx + \int_\Omega \psi^2\, dx}{\int_\Omega \frac{\psi^2}{|x|^2}\, dx}.$$

We are now ready to prove the existence of a non-radial solution.

Proof of Theorem 1.2. Assuming by contradiction, the solution u of (47) obtained in Theorem 1.1 is a radial function. Note that $I_K(u) = c > 0$ where the critical value c is characterized by

$$c = \inf_{\gamma\in\Gamma} \sup_{\tau\in[0,1]} I(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0,1], V) : \gamma(0) = 0 \neq \gamma(1),\ I_K(\gamma(1)) \le 0\}$. We claim that there exists some element $\gamma \in \Gamma$ such that

$$I_K(\gamma(\tau)) < I_K(u), \quad \forall\, \tau \in [0,1].$$

This implies that

$$c \le \max_{\tau\in[0,1]} I_K(\gamma(\tau)) < I_K(u),$$

which contradicts $I_K(u) = c$. Now to prove our claim, set $v(r, \theta) = u(r)\psi(\theta)$ where $\psi(\theta) = \dfrac{m-n}{N} - \cos(2\theta)$ being the solution of (49) with $\mu_1 = 2N$. Let $l > 0$ be such that $I_K((u + \sigma v)l) \leq 0$ for all $|\sigma| < 1$. Consider

$$\gamma_\sigma(\tau) = \tau(u + \sigma v)l.$$

We have $\gamma_\sigma \in \Gamma$ for all $|\sigma| < 1$. Moreover, there exists a unique twice differentiable real function g on a small neighbourhood of zero with $g'(0) = 0$ and $g(0) = 1/l$ such that

$$\max_{\tau \in [0,1]} I_K(\gamma_\sigma(\tau)) = I_K(g(\sigma)(u + \sigma v)l).$$

Now we define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(\sigma) = I_K(g(\sigma)(u + \sigma v)l) - I_K(u).$$

We already know that $h(0) = 0$ and $h'(0) = \langle I'_K(u), v\rangle = 0$. If we prove that $h''(0) < 0$, then there exists σ sufficiently small such that $h(\sigma) < 0$, or equivalently,

$$\max_{\tau \in [0,1]} I_K(\gamma_\sigma(\tau)) = I_K(g(\sigma)(u + \sigma v)l) < I_K(u).$$

In this way the desired conclusion follows by taking $\gamma = \gamma_\sigma$. The only remaining condition that needs to be checked is

$$h''(0) = \langle I''_K(u); v, v\rangle = \int_\Omega (|\nabla v|^2 + v^2)\, dx - \int_\Omega a(|x|)(e^{u^2} - 1 + 2u^2 e^{u^2})v^2\, dx < 0.$$

To do this, we aim to show that $M(u, v) < 0$, where $M(u, v)$ is defined as

$$M(u,v) = \int_{\widehat{\Omega}} (v_s^2 + v_t^2 + v^2) s^{m-1} t^{n-1}\, ds\, dt - \int_{\widehat{\Omega}} a(s,t)(e^{u^2} - 1 + 2u^2 e^{u^2})v^2 s^{m-1} t^{n-1}\, ds\, dt.$$

By writing $M(u, v)$ in polar coordinates, we get

$$M(u,v) = \int_0^{\frac{\pi}{2}} \int_{R_1}^{R_2} \left(\psi^2(u_r^2 + u^2) + \frac{u^2 \psi'^2}{r^2} - a(r)(e^{u^2} - 1 + 2u^2 e^{u^2})u^2\psi^2\right) r^{N-1} w(\theta)\, dr\, d\theta.$$

where $w(\theta) = \cos^{m-1}(\theta)\sin^{n-1}(\theta)$. If we consider the equation $-\Delta u + u = a(r)u(e^{u^2} - 1)$, then we have

$$\int_{R_1}^{R_2} (u_r^2 + u^2) r^{N-1}\, dr = \int_{R_1}^{R_2} a(r) u^2 (e^{u^2} - 1) r^{N-1}\, dr. \tag{50}$$

and therefore

$$M(u,v) = \int_0^{\frac{\pi}{2}} \int_{R_1}^{R_2} \Big(\frac{u^2\psi'^2}{r^2} - 2a(r)u^4 e^{u^2}\psi^2\Big) r^{N-1} w(\theta)\, dr\, d\theta.$$

By definition of $\mu_1 = 2N$ in (48), we can simplify the above expression to obtain

$$\begin{aligned} M(u,v) &= \int_0^{\frac{\pi}{2}} \int_{R_1}^{R_2} \Big(2N\frac{u^2\psi^2}{r^2} - 2a(r)u^4 e^{u^2}\psi^2\Big) r^{N-1} w(\theta)\, dr\, d\theta \\ &\leq \int_0^{\frac{\pi}{2}} \int_{R_1}^{R_2} \Big(\frac{2N}{\beta}(u_r^2 + u^2)\psi^2 - 2a(r)u^4 e^{u^2}\psi^2\Big) r^{N-1} w(\theta)\, dr\, d\theta, \end{aligned} \tag{51}$$

where β is the best constant in Hardy inequality. Putting this together with (50) gives

$$M(u,v) \leq \int_0^{\frac{\pi}{2}} \int_{R_1}^{R_2} a(r)u^2\psi^2 \left(\frac{2N}{\beta}(e^{u^2} - 1) - 2u^2 e^{u^2}\right) r^{N-1} w(\theta)\, dr\, d\theta. \tag{52}$$

It should be noted that the inequality $A(e^{u^2}-1) < 2u^2e^{u^2}$ holds for every $A < 2$. Thus, the assumption $N < \beta$ implies $M(u,v) < 0$, which completes the proof that the solution is non-radial. We will now show that β can be sufficiently large for large values of R when $R_1 = R$, $R_2 = R+1$. Set $\Omega_R := \{x \in \mathbb{R}^N : R < |x| < R+1\}$ and

$$\beta_R = \inf_{\psi \in H^1(\Omega_R)\setminus\{0\}} \frac{\int_{\Omega_R} |\nabla\psi|^2\, dx + \int_{\Omega_R} \psi^2\, dx}{\int_{\Omega_R} \frac{\psi^2}{|x|^2}\, dx}.$$

It suffices to show that $\dfrac{\beta_R}{R^2} \to C$ as $R \to \infty$, for some constants C. Note that β_R is attained at some w_R that is radial. More precisely, w_R solves the equation

$$\begin{cases} -\Delta(w_R(x)) + w_R(x) \;=\; \frac{\beta_R w_R(x)}{|x|^2}, & in\;\; \Omega_R \\ \dfrac{\partial w_R}{\partial\nu} = 0, & in\;\; \partial\Omega_R. \end{cases} \tag{53}$$

Setting $v_R(r) = w_R(R+r)$, we obtain

$$-v_R''(r) - \frac{N-1}{R+r}v_R'(r) = [\frac{\beta_R}{(R+r)^2} - 1]v_R(r), \quad in \quad 0 < r < 1, \tag{54}$$

and

$$v_R'(0) = v_R'(1) = 0.$$

We may assume that $\max_{[0,1]} v_R = 1$, by normalizing, which implies that v_R is a bounded sequence. Let $\varphi \in C_c^\infty(0,1)$ be fixed. Set $\bar\varphi(r) = \varphi(r-R)$. Then

$$\beta_R \le \frac{\int_R^{R+1}[|\bar\varphi'|^2 + \bar\varphi^2]r^{N-1}\, dr}{\int_R^{R+1} \frac{\bar\varphi^2}{r^2} r^{N-1}\, dr} = \frac{\int_0^1[|\varphi'|^2 + \varphi^2](R+t)^{N-1}\, dt}{\int_0^1 \frac{\varphi^2}{(R+t)^2}(R+t)^{N-1}\, dt} \le C(R+1)^2,$$

for some constant C. This implies that, there exists a subsequence β_{R_m} and $\bar\beta \in \mathbb{R}$ such that $\frac{\beta_{R_m}}{R^2} \to \bar\beta$. On the other hand, $v_m = v_{R_m}$ is bounded, so that by passing to a subsequence if necessary, there is some $v \ge 0$ such that $v_m \to v$ in $C^{0,\delta}[0,1]$. Therefore, by passing to the limit in (54) we get that v satisfies $-v''(r) = (\bar\beta - 1)v(r) \quad in \quad (0,1)$ with $v(0) = v(1) = 0$ and $\sup_{(0,1)} v = 1$. It is now obvious that $v(r) = \cos(\pi r - \frac{\pi}{2})$ and $\bar\beta = \pi^2 + 1$. This in fact shows that β is sufficiently large for large values of R, as desired. □

3 Supercritical elliptic problems

In this section we examine the equation

$$\begin{cases} -\Delta u + \; u \;=\; a(x)|u|^{p-2}u, & x \in \Omega \\ u > 0, & x \in \Omega \\ \dfrac{\partial u}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{55}$$

where Ω is a bounded annulus in $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$ and $p > 2$. Here we assume that $a \in \mathcal{A}(m,n)$ and $n \le m$. We shall consider the Banach space $V = H^1(\Omega) \cap L^p(\Omega)$ equipped with the following norm

$$\|u\| = \|u\|_{H^1(\Omega)} + \|u\|_{L^p(\Omega)}.$$

Let $I : V \to \mathbb{R}$ be the Euler-Lagrange functional corresponding to problem (55), i.e.,

$$I(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + u^2)\, dx - \frac{1}{p}\int_\Omega a(x)|u|^p\, dx.$$

Let

$$K = K(m,n) = \{0 \le u \in H^1_G \cap L^p(\Omega) : su_t - tu_s \le 0 \ \ a.e. \ \ in \ \widehat{\Omega}\}. \tag{56}$$

In this case the Euler-Lagrange functional corresponding to (55) restricted to K is

$$I_K = \Psi_K - \Phi, \tag{57}$$

where

$$\Psi = \frac{1}{2}\int_\Omega (|\nabla u|^2 + u^2)\,dx \quad , \quad \Phi = \frac{1}{p}\int_\Omega a(x)|u|^p\,dx,$$

and Ψ_K is the restriction of Ψ to K defined by

$$\Psi_K(u) = \begin{cases} \Psi(u), & u \in K \\ +\infty, & u \notin K. \end{cases}$$

To prove Theorem 1.3, we will utilize a modified version of Theorem 2.7 applicable specifically to our problem (55).

Theorem 3.1 *Let $V = H^1(\Omega) \cap L^p(\Omega)$ and K be a convex and closed subset of V defined in (56). Let a be a non-negative continuously differentiable function that is not identically zero. Assume that the following two assertions hold:*

(i) The functional $I_K : V \to \mathbb{R}$ defined in (57) has a critical point $\bar{u} \in V$ as in Definition 2.4, and;

(ii) There exists $\bar{v} \in K$ with $\frac{\partial \bar{v}}{\partial \nu} = 0$ on the boundary of Ω such that

$$-\Delta \bar{v} + \bar{v} \; = \; a(x)|\bar{u}|^{p-2}\bar{u},$$

in the weak sense.

Then there exist $\hat{u} \in K$ such that $I(\hat{u}) > 0$ and $\hat{u}$ is a weak solution of the equation (55).

Since the proof follows by a similar strategy as that of Theorem 2.7, we omit it for brevity. We employ Theorem 2.9 to verify condition (i) in Theorem 3.1 and establish the existence of a critical point for the non-smooth functional I_K in the following lemma.

Lemma 3.2 *The functional I_k satisfies the mountain pass geometry and (PS) compactness condition. Moreover, I_K has a non-trivial critical point in K.*

Proof: Suppose that u_j is a sequence in K such that $I_K(u_j) \to c \in \mathbb{R}, \epsilon_j \to 0$ and

$$\langle D\Phi(u_j), u_j - v\rangle - \Psi(v) - \Psi(u_j) \ge -\epsilon_j \|v - u_j\|_V, \quad \forall v \in V. \tag{58}$$

Replacing v by ru_j ($r \in \mathbb{R}$) in (58), it becomes

$$\frac{1-r^2}{2}\|u_j\|^2_{H^1(\Omega)} + (r-1)\int_\Omega a(x)|u_j|^p\,dx \le \epsilon_j(r-1)\|u_j\|_V. \tag{59}$$

On the other hand, since $I_K(u_j) \to c$, we have

$$\frac{1}{2}\|u_j\|^2_{H^1(\Omega)} - \frac{1}{p}\int_\Omega a(x)|u_j|^p\,dx \le c+1, \tag{60}$$

for large values of n. Now set $1 < r$ and $r^2 - 1 < p(r-1)$. We can take $\zeta > 0$ such that

$$\frac{1}{p(r-1)} < \zeta < \frac{1}{r^2-1}.$$

Multiply (59) by ζ and adding up with (60) yields that

$$\frac{1+\zeta(1-r^2)}{2}\|u_j\|^2_{H^1(\Omega)}+[\zeta(r-1)-\frac{1}{p}]\int_\Omega a(x)|u_j|^p\,dx\leq c+1+\zeta\epsilon_j(r-1)\|u_j\|_V. \tag{61}$$

Now from the choice of ζ, all the coefficients in the left-hand side of the latter inequality are positive, thus we have

$$\|u_j\|^2_{H^1(\Omega)}\leq C(1+\|u_j\|_V), \tag{62}$$

for some constant $C>0$. Moreover, since K is a weakly closed convex subset in $H^1(\Omega)$ and also according to Theorem 2.9,compactly embedded in L^p, there exists a constant C such that

$$\|u\|_{H^1(\Omega)}\leq\|u\|_V\leq C\|u\|_{H^1(\Omega)},\quad\forall u\in K,$$

which, together with (62), imply that u_j is bounded in H^1. It follows, by a standard result in Sobolev spaces, after passing to a subsequence (still denoted by u_j), that there exists $\bar{u}\in H^1(\Omega)$ such that $u_j\rightharpoonup\bar{u}$ weakly in $H^1(\Omega)$ and $u_j\to\bar{u}$ a.e.. Also, again by Theorem 2.9, from boundedness of $\{u_j\}\subset K$ in $H^1(\Omega)$, one can deduce that the strong convergence of u_j to $\bar{u}$ in L^p. By setting $v=\bar{u}$ in (58), and using Hölder's inequality we obtain

$$\begin{aligned}\frac{1}{2}(\|u_j\|^2_{H^1(\Omega)}-\|\bar{u}\|^2_{H^1(\Omega)})&\leq\int_\Omega a(x)|u_j|^{p-1}(u_j-\bar{u})\,dx+\epsilon_j\|u_j-\bar{u}\|_V\\&\leq\|a(x)\|_{L^\infty}\|u_j\|^{p-1}_{L^p}\|u_j-\bar{u}\|_{L^p}+\epsilon_j\|u_j-\bar{u}\|_V.\end{aligned}$$

Therefore,

$$\limsup_{n\to\infty}(\|u_j\|^2_{H^1(\Omega)}-\|\bar{u}\|^2_{H^1(\Omega)})\leq 0. \tag{63}$$

Also, the properties of weak convergence implies that

$$0\leq\liminf(\|u_j\|^2_{H^1(\Omega)}-\|\bar{u}\|^2_{H^1(\Omega)}),$$

which together with (63) yield that $u_j\to\bar{u}$ strongly in V. This completes the proof of the (PS) compactness condition for the function I_k. Verifying that the mountain pass geometry holds for the functional I_k is a straightforward process. Now by virtue of Theorem 2.6, we can deduce that the functional I_K possesses a non-trivial critical point $\bar{u}\in K$. □

In the following proposition, we demonstrate that condition (ii) of Theorem 2.7 holds, which is necessary to establish the effectiveness of the variational approach.

Proposition 3.3 *Let Ω be an annulus and a satisfies condition ($\mathcal{A}(m,n)$). Suppose $u\in K(m,n)$. Then there exists $v\in K(m,n)$ satisfying*

$$\begin{cases}-\Delta v+\ v\ =\ a(x)u^{p-1}, & x\in\Omega\\ \dfrac{\partial v}{\partial\nu}=0, & x\in\partial\Omega,\end{cases} \tag{64}$$

in the weak sense.

Proof: By setting $u_k(x)=min\{u(x),k\}$ for $k\geq 1$, we obtain $u_k\in H^1(\Omega)$. It should be observed that $u_k\in H^1_G(\Omega)$ and the monotonicity of u and u_k are the same. Now we shall consider the following problem

$$\begin{cases}-\Delta v+\ v\ =\ a(x){u_k}^{p-1}, & x\in\Omega\\ \dfrac{\partial v}{\partial\nu}=0, & x\in\partial\Omega.\end{cases} \tag{65}$$

Taking into account the associated energy on $H^1_G(\Omega)$ and using the standard regularity theory, we can deduce the existence of a unique $0\leq v^k\in H^3_G(\Omega)\cap C^{1,\alpha}(\Omega)$ which solves the equation (65). Following the same argument as in the proof of Proposition 3.3, it is easy to see that that $v^k \in K(m,n)$ and possesses boundedness. Passing to a subsequence (still denoted by v^k), we may assume that $v^k \rightharpoonup v$ weakly in $H^1(\Omega)$ for some $v \in H^1_G(\Omega)$. On the other hand, since $p'(p-1) = p$, we may use $u_k^{p-1} \to u^{p-1}$ in $L^{p'}(\Omega)$ and, after passing to a subsequence, deduce that $v^k \to v$ in $W^{2,p'}(\Omega)$. It follows that $\nabla v^k \to \nabla v$ in $L^{p'}(\Omega)$, and therefore $v_s^k \to v_s$ a.e. in Ω and $v_t^k \to v_t$ a.e. in Ω. Hence, $v_s^k \to v_s$ a.e. $(s,t) \in \widehat{\Omega}$ and $v_t^k \to v_t$ a.e. $(s,t) \in \widehat{\Omega}$. By defining $w := sv_t - tv_s$, we have $sv_t^k - tv_s^k \to w$ a.e. $(s,t) \in \widehat{\Omega}$, and thus $w \le 0$ in $\widehat{\Omega}$. This gives the desired monotonicity of v and hence $v \in K(m,n)$. □

Proof of Theorem 1.3. Note that conditions (i) and (ii) in Theorem 2.7 follows from Theorem 2.9 and Proposition 3.3 respectively. This proves the existence of a weak solution for (55). □

Now we discuss the case when $a(x) = a(|x|)$ is radial, that is

$$\begin{cases} -\Delta u + \ u \ = \ a(|x|)|u|^{p-2}u, & x \in \Omega \\ u > 0, & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{66}$$

We shall show that the solution obtained in Theorem 1.3 is nonradial.

Proof of Theorem 1.4. Assuming, by contradiction, that the solution u of (66) obtained in Theorem 1.3 is a radial function, we can follow a similar approach as in the proof of Theorem 1.2. The only remaining condition that needs to be checked is

$$h''(0) = \langle I_K''(u); v, v\rangle = \int_\Omega (|\nabla v|^2 + v^2)\, dx - (p-1)\int_\Omega \left|a(|x|)u\right|^{p-2} v^2\, dx < 0.$$

We proceed to do this by showing $M(u,v) < 0$, where

$$M(u,v) = \int_{\widehat{\Omega}} (v_s^2 + v_t^2 + v^2) s^{m-1} t^{n-1}\, ds\, dt - (p-1)\int_{\widehat{\Omega}} a(s,t) u^{p-2} v^2 s^{m-1} t^{n-1}\, ds\, dt.$$

By writing $M(u,v)$ in polar coordinates, we get

$$M(u,v) = \int_0^{\frac{\pi}{2}} \int_{R_1}^{R_2} \left(\psi^2(u_r^2 + u^2) + \frac{u^2\psi'^2}{r^2} - (p-1)a(r)u^p\psi^2\right) r^{N-1} w(\theta)\, dr\, d\theta,$$

where $w(\theta) = \cos^{m-1}(\theta)\sin^{n-1}(\theta)$. If we consider the equation $-\Delta u + u = a(r)u^{p-1}$, then we have

$$\int_{R_1}^{R_2} (u_r^2 + u^2) r^{N-1}\, dr = \int_{R_1}^{R_2} a(r) u^p r^{N-1}\, dr.$$

and therefore

$$M(u,v) = \int_0^{\frac{\pi}{2}} \int_{R_1}^{R_2} \frac{u^2\psi'^2}{r^2} r^{N-1} w(\theta)\, dr\, d\theta - (p-2)\int_0^{\frac{\pi}{2}} \int_{R_1}^{R_2} \psi^2 \big(u_r^2 + u^2\big) r^{N-1} w(\theta)\, dr\, d\theta.$$

Combining this with the definition of $\mu_1 = 2N$ in (48) implies that

$$\begin{aligned} M(u,v) &= \mu_1 \int_0^{\frac{\pi}{2}} \int_{R_1}^{R_2} \frac{u^2\psi^2}{r^2} r^{N-1} w(\theta)\, dr\, d\theta - (p-2)\int_0^{\frac{\pi}{2}} \int_{R_1}^{R_2} \psi^2 \big(u_r^2 + u^2\big) r^{N-1} w(\theta)\, dr\, d\theta \\ &= \int_0^{\frac{\pi}{2}} |\psi(\theta)|^2 w(\theta)\, d\theta \left(\mu_1 \int_{R_1}^{R_2} \frac{u^2}{r^2} r^{N-1}\, dr - (p-2)\int_{R_1}^{R_2} \big(u_r^2 + u^2\big) r^{N-1}\, dr\right). \end{aligned} \tag{67}$$

We deduce from definition of β, the best constant in Hardy inequality, that

$$\int_{R_1}^{R_2} \beta \frac{u^2}{r^2} r^{N-1}\, dr < \int_{R_1}^{R_2} (u_r^2 + u^2) r^{N-1}\, dr.$$

Putting this together with (67) gives

$$\begin{aligned} M(u,v) &\leq \int_0^{\frac{\pi}{2}} |\psi(\theta)|^2 w(\theta)\, d\theta \Big(\frac{\mu_1}{\beta} \int_{R_1}^{R_2} u_r^2 r^{N-1}\, dr - (p-2) \int_{R_1}^{R_2} \big(u_r^2 + u^2\big) r^{N-1}\, dr \Big) \\ &= \int_0^{\frac{\pi}{2}} |\psi(\theta)|^2 w(\theta)\, d\theta \int_{R_1}^{R_2} \big(u_r^2 + u^2\big) r^{N-1}\, dr \Big(\frac{2N}{\beta} - (p-2) \Big) < 0. \end{aligned}$$

This completes the proof. □

The existence of distinct solutions in Theorem 1.5 can be demonstrated by selecting different decompositions of $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$

Proof of Theorem 1.5. Let $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$ with $n \leq k$. By doing so, we obtain $\dfrac{2k+2}{k-1} \leq \dfrac{2n+2}{n-1}$ which implies that

$$p < \frac{2n+2}{n-1}.$$

Now, we can apply Theorem 1.3 and deduce that for each $n \in \{1, ..., k\}$ there exists a solution of the form $u_{m,n} = u_{m,n}(s,t)$ where $s = \sqrt{x_1^2 + \cdots + x_m^2}$ and $t = \sqrt{x_{m+1}^2 + \cdots + x_N^2}$. Furthermore, since $p - 2 > \frac{2N}{\beta}$, we obtain from Theorem 1.4 that the solutions $u_{m,n}$ are non-radial. Now we shall prove the distinction between non-radial solutions for different values of n and m. Let $1 \leq n < n' \leq \lfloor \frac{N}{2} \rfloor$ and set $m = N - n, m' = N - n'$. Suppose $u_{m,n} \in K(m,n)$ and $u_{m',n'} \in K(m',n')$ are the non-radial solutions obtained in Theorem 1.3, corresponding to the decomposition of $\mathbb{R}^N$ into $\mathbb{R}^n \times \mathbb{R}^m$ and $\mathbb{R}^{n'} \times \mathbb{R}^{m'}$, respectively. Assume by contradiction that $u_{m,n}(s,t) = u_{m',n'}(s',t')$ where

$$s = \sqrt{x_1^2 + \cdots + x_m^2}, \quad t = \sqrt{x_{m+1}^2 + \cdots + x_N^2},$$

$$s' = \sqrt{x_1^2 + \cdots + x_m'^2}, \quad t' = \sqrt{x_{m'+1}^2 + \cdots + x_N^2}.$$

Assuming $x_i = 0$ for $i \neq 1, m'+1$, we have $s = \sqrt{x_1^2 + x_{m'+1}^2}$, $t = 0$, $s' = |x_1|$ and $t' = |x_{m'+1}|$. Therefore,

$$u_{m,n}(\sqrt{x_1^2 + x_{m'+1}^2}, 0) = u_{m',n'}(|x_1|, |x_{m'+1}|)$$

It follows that $u_{m',n'}$ is a radial function, which is a contradiction. This completes the proof of the fact that the equation (9) has k positive distinct nonradial solutions. By using similar arguments to those in the proof of Theorem 1.2, it is possible to observe that β can be made sufficiently large for large values of R when $R_1 = R$ and $R_2 = R+1$. We now assume that $k = \lfloor \frac{N}{2} \rfloor$. We deduce from the above discussion that there are $\lfloor \frac{N}{2} \rfloor$ distinct positive nonradial solutions of (9) when

$$2 + \frac{2N}{\beta} < p < \frac{2\lfloor \frac{N}{2} \rfloor + 2}{\lfloor \frac{N}{2} \rfloor - 1}.$$

Using the fact that β can be sufficiently large, we have $\lfloor \frac{N}{2} \rfloor$ distinct positive nonradial solutions provided

$$2 < p < \frac{2\lfloor \frac{N}{2} \rfloor + 2}{\lfloor \frac{N}{2} \rfloor - 1}.$$

□

References

[1] C. O. Alves, A. Moameni, *Super-critical Neumann problems on unbounded domains.* Nonlinearity 33(9) (2020), 4568-4589.

[2] V. Barutello, S. Secchi, and E. Serra, *A note on the radial solutions for the supercritical Hénon equation*, J. Math. Anal. Appl. 341(1) (2008), 720-728.

[3] D. Bonheure, M. Grossi, B. Noris and S. Terracini,*Multi-layer radial solutions for a supercritical Neumann problem*, J. Differential Equations, 261(1) (2016), 455-504.

[4] D. Bonheure, B. Noris and T. Weth, *Increasing radial solutions for Neumann problems without growth restrictions*, Ann. Inst. H. Poincaré Anal. Non Lin´eaire 29(4) (2012), 573-588.

[5] D. Bonheure and E. Serra, *Multiple positive radial solutions on annuli for nonlinear Neumann problems with large growth*, NoDEA 18(2) (2011), 217-235.

[6] F. Colasuonno and B. Noris.*A p-Laplacian supercritical Neumann problem. Discrete and Continuous Dynamical Systems*, 37(6) (2017), 3025-305

[7] C. Cowan and A. Moameni, *A new variational principle, convexity, and supercritical Neumann problems*, Transactions of the American Mathematical Society 371(9) (2019), 5993-6023.

[8] C. Cowan, A. Moameni and L. Salimi, *Supercritical Neumann problems via a new variational principle*, Electron. J. Differential Equations (213) (2017), 1-19.

[9] M. del Pino, M. Musso and A. *Pistoia, Super-critical boundary bubbling in a semilinear Neumann problem*, Annales de l'Institut Henri Poincare (C) Non Linear Analysis Volume 22, Issue 1 (2005), 45-82.

[10] N. Ghoussoub and C. Gui, *Multi-peak solutions for a semilinear Neumann problem involving the critical Sobolev exponent* Math. Z., 229(3) (1998), 443-474.

[11] M. Grossi, *A class of solutions for the Neumann problem* $-\Delta+\lambda u = u^{\frac{N+2}{N-2}}$, Duke Math. J., 79(2) (1995), 309-334.

[12] M. Grossi and B. Noris. *Positive constrained minimizers for supercritical problems in the ball.* Proc. Amer. Math. Soc., 140(6) (2012), 2141-2154.

[13] C. Gui, *Multi-peak solutions for a semilinear Neumann problem*, Duke Math. J., 84 (1996), 739-769.

[14] C. Gui and C.-S. Lin, *Estimates for boundary-bubbling solutions to an elliptic Neumann problem*, J. Reine Angew. Math., 546 (2002), 201-235.

[15] C. Gui and J. Wei, *Multiple interior peak solutions for some singularly perturbed Neumann problems*, J. Differential Equations, 158(1) (1999), 1-27.

[16] A. Moameni, *Critical point theory on convex subsets with applications in differential equations and analysis.* J. Math. Pures Appl. (9), 141 (2020), 266-315.

[17] Y. Lu, T. Chen, and R. Ma, *On the Bonheure-Noris-Weth conjecture in the case of linearly bounded nonlinearities*, Discrete Contin. Dyn. Syst. Ser. B, 21(8) (2016), 2649-2662.

[18] O. Rey and J. Wei, *Blowing up solutions for an elliptic Neumann problem with sub- or supercritical nonlinearity Part I: N = 3*, Journal of Functional Analysis Volume 212, Issue 2, 15 (2004), 472-499.

[19] B. Ruf: *A sharp Trudinger-Moser type inequality for unbounded domains in* $\mathbb{R}^2$. J. Funct. Anal. 219(2) (2005), 340-367.

[20] E. Serra and P. Tilli. *Monotonicity constraints and supercritical Neumann problems.* Ann. Inst. H. Poincaré Anal. Non Lin´eaire, 28(1) (2011), 63-74.

[21] S. Secchi. *Increasing variational solutions for a nonlinear p-Laplace equation without growth conditions.* Ann. Mat. Pura Appl. (4), 191(3) (2012), 469-485.

[22] A. Szulkin, *Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems.* Ann. Inst. H. Poincaré Anal. Non Linéaire 3(2) (1986), 77-109.

[23] Y. Yang, *Moser-Trudinger trace inequalities on a compact Riemannian surface with boundary.* Pacific J. Math. 227 (2006), 177-200.

[24] J. Wei, *On the boundary spike layer solutions to a singularly perturbed Neumann problem*, J. Differential Equations, 134(1) (1997), 104-133.