A variational approach towards the $n-$dimensional stationary Navier-Stokes equations with a damping term. *

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Abstract

We analyze the stationary Navier-Stokes equations with a damping term in dimension n through a general minimax principle for which we first develop. Our minimax principle is broad enough and can be used in various ways to deal with the Stationary Navier-stokes equations. We shall provide existence results for linear and nonlinear dampings with no restriction on the damping constant.

Key words: Navier-Stokes equations, Variational Principles
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1 Introduction

In this paper we study the $n-$dimensional stationary Navier-Stokes equations with a damping term:

$$\begin{cases} -\Delta u + (u.\nabla)u + \mu|u|^{p-2}u + \nabla P = f(x), & \forall x \in \Omega, \\ \nabla.u = 0, & \forall x \in \Omega, \\ u = 0, & \forall x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is bounded, $p \geq 1$ and $\mu \in \mathbb{R}$. We address both linear and nonlinear dampings and we are allowing μ to take both positive and negative values. Here $u = (u_1, u_2, ..., u_n)$ is the velocity, P stand for scalar pressure and f is the external force.

The existence of global weak solutions of the classical evolutionary Navier–Stokes equations without damping were established by Leray [11] and Hopf [6]. Thereafter, the issue of uniqueness and regularity has received a lot of attention. However, the uniqueness of weak solutions and the global existence of strong solutions remain open so far.

Over the years, many authors turn to consider Navier-Stokes equations with damping term which in some cases is very advantageous from the mathematical point of view, as it allows to obtain solutions more regular than in the standard Navier-Stokes equations without damping. The damping comes from the resistance to the motion of the flow, to which describes various physical phenomena such as porous media flow, drag or friction effects, and some dissipative mechanisms (see [7, 23]).

Many authors obtained the long-time behavior of solutions for three dimensional evolutionary Navier–Stokes equations with damping term $\mu|u|^{p-2}u$ ($\mu > 0$), (see [2, 8, 9, 10, 20, 25, 24, 26]). For instance, in [2], has proved the existence of global weak solutions for $p \geq 2$, the existence of global strong solutions for $p \geq \frac{9}{2}$, and the uniqueness of strong solutions for $\frac{9}{2} \leq p \leq 6$ in the whole space, respectively. Due to this, the global

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attractor was studied in [22] and [21]. In [8], the L^2 decay of weak solutions, the optimal upper bounds of the higher-order derivative of the strong solution and asymptotic stability of the large solution were studied. The regularity criterion of the three-dimensional Navier–Stokes equations with nonlinear damping was studied in [26]. The author in [24] obtained the existence of strong solutions for $p \geq 4$, the uniqueness of strong-weak solutions for $2 \leq p$ and established two regularity criteria as $2 \leq p \leq 4$. The existence of weak solutions for the generalized Navier–Stokes equations with damping was provided in [17]. In [15], by using Fourier splitting method, the L_2 decay of weak solutions for three dimensional Navier–Stokes equations with damping was proved for $p > 3$.
For stationary Navier-Stokes equations with damping, the authors [13] obtained some partial results for existence and uniqueness of the weak solutions when $\mu > 0$. Lately, a lot of work has been done on numerical simulations of Navier-Stokes equations with damping (see [12, 14, 18, 19, 27]).

In this work, we first develop a minimax principle to deal with stationary Navier-Stokes equations with damping (see [16] for a comprehensive review on variational principles on convex subsets). Consider the Banach space

$$V = \{u \in H_0^1(\Omega) \cap L^p(\Omega) : \nabla.u = 0\},$$

equipped with the following norm

$$\|u\| := \|u\|_{H_0^1(\Omega)} + \|u\|_{L^p(\Omega)}.$$

Let Λu be the operator $\Lambda u := (u.\nabla)u$, and K be a convex and weakly closed subset of V. We shall define $M : K \times K \to \mathbb{R}$ as follows,

$$M(u,v) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{1}{2}\int_\Omega |\nabla v|^2\, dx + \int_\Omega (\Lambda u - f(x) - \frac{1}{p}|u|^{p-2}u)(u-v)\, dx, \tag{2}$$

where $f \in L^2(\Omega)$. Note that $\Lambda u - f(x) - \frac{1}{p}|u|^{p-2}u$ and $u - v$ are both vector functions in $\mathbb{R}^n$. In the definition of M, and also throughout the paper, the product of any two vectors is to be understood as the regular inner product in $\mathbb{R}^n$. The following variational principle on general convex sets K is a key component in our arguments. It is also broad enough to deal with various other cases by choosing a convex set K accordingly.

Theorem 1.1 *Let K be a convex and weakly closed subset of V. Assume that the following two assertions hold:*

(i) There exists $\bar{u} \in K$ such that

$$M(\bar{u}, v) \leq 0, \quad \forall v \in K,$$

where M is defined in (2).

(ii) There exists $\bar{v} \in K$ such that

$$-\Delta \bar{v} + \nabla P = f(x) + |\bar{u}|^{p-2}\bar{u} - \Lambda \bar{u},$$

in the weak sense, i.e.,

$$\int_\Omega \nabla \bar{v}.\nabla \eta\, dx = \int_\Omega (f(x) + |\bar{u}|^{p-2}\bar{u} - \Lambda \bar{u})\eta\, dx, \quad \forall \eta \in V.$$

Then $\bar{u} \in K$ is a weak solution of the equation

$$-\Delta u + \Lambda u + \nabla P = f(x) + |u|^{p-2}u\ .$$

It is worth noting that the primary consequence of this theorem centres on the choice of K, i.e., by choosing an appropriate K, one is able to establish the existence of a solution enjoying all the properties induced by the set K (see Remark 1.5 for an application where the problem (1) has some symmetry properties). Also, condition (i) in Theorem 1.1 is most of the time guarantied due to the the well-known Ky Fan's min-max principle by Brezis-Nirenberg-Stampacchia [1]. We provide more details of how to apply the above theorem in the sequel.
As an application of the above theorem we first prove the following result.

Theorem 1.2 *Let Ω be a bounded C^2 domain in $\mathbb{R}^n$ and $\mu < 0$. Then for $f \in L^2(\Omega)$ small enough, the following statements hold:*

(i) For $n \leq 4$ and $p > 2$, the Navier-Stokes equation (1) has a solution $u \in W^{2,2}(\Omega)$.

(ii) For $n = 5,\ 6$ and $2 < p \leq \dfrac{2n-4}{n-4}$, the Navier-Stokes equation (1) has a solution u in $W^{2,2}(\Omega)$.

Furthermore, for $n > 2$, there exists a scalar function $P : \Omega \to \mathbb{R}$ and a constant $C > 0$ such that

$$\|\Delta u\|_{L^2(\Omega)} + \|\nabla P\|_{L^2(\Omega)} \leq C\big(\|f\|_{L^2(\Omega)} + \|u\|^{p-1}_{L^{2(p-1)}(\Omega)} + \|u\|_{W^{1,2^*}(\Omega)}\|u\|_{L^n(\Omega)}\big), \tag{3}$$

where $2^* = 2n/(n-2)$.

When the constant μ in the damping term is non-negative we can cover higher values for p as shown in the following theorem.

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and $\mu > 0$. Suppose that $p \geq 1$ and $f \in L^2(\Omega)$. Then there exists $u \in V$ such that the following holds:*

(i) If $n \geq 2$, then

$$\int_\Omega \nabla u.\nabla \eta\, dx + \mu \int_\Omega |u|^{p-2} u\, \eta + \int_\Omega \Lambda u\, \eta\, dx = \int_\Omega f(x)\eta\, dx, \quad \forall \eta \in C^1_c(\Omega),\ \text{with } \nabla.\eta = 0.$$

(ii) If $n \leq 4$ or $p \geq 4$, then

$$\int_\Omega \nabla u.\nabla \eta\, dx + \mu \int_\Omega |u|^{p-2} u\, \eta\, dx + \int_\Omega \Lambda u\, \eta\, dx = \int_\Omega f(x)\eta\, dx, \quad \forall \eta \in V.$$

We would like to remark that the solution we are getting in part (i) of the above theorem is weaker than the one we are getting in part (ii). This is due to the fact that all the test functions η in part (i) are coming from $C^1_c(\Omega)$ on contrary to part (ii) where the test functions η live in a less regular space $H^1_0(\Omega) \cap L^p(\Omega)$.

We shall also deal with the linear damping term where $p = 2$ for positive and negative values of μ. To state our result we first recall the following standard fact about the first eigenfunction of the Laplacian on bounded domains. Recall that

$$\lambda_1 = \min_{\psi \in H^1_0(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla \psi|^2\, dx}{\int_\Omega \psi^2\, dx}.$$

where the minimum is taken over all $\psi : H^1_0(\Omega) \to \mathbb{R}$. Note that in Theorem 1.3 we have already covered the case $\mu > 0$. Here is our result for the linear case where we are allowing negative values for μ.

Theorem 1.4 *Let Ω be smooth bounded domain in $\mathbb{R}^n$ and $p = 2$. Assume that $-\lambda_1 \leq \mu < 0$, and $f \in L^2(\Omega)$. Then there exists $u \in V$ such that the following assertions hold:*

(i) If $n \geq 2$, then

$$\int_\Omega \nabla u.\nabla \eta\, dx + \mu \int_\Omega u\eta\, dx\ + \int_\Omega \Lambda u\, \eta\, dx = \int_\Omega f(x)\eta\, dx, \quad \forall \eta \in C^1_c(\Omega), \quad \text{with } \nabla.\eta = 0.$$

(ii) If $n \leq 4$, then

$$\int_\Omega \nabla u.\nabla \eta\, dx + \mu \int_\Omega u\, \eta\, dx + \int_\Omega \Lambda u\, \eta\, dx = \int_\Omega f(x)\eta\, dx, \quad \forall \eta \in V.$$

The highlight of the above theorem is the case where $\mu = -\lambda_1$ in which case one losses the coercivity required in most minimax arguments.

Remark 1.5 *Even though our main objective in this paper is to prove existence results having a damping term in mind, we would like emphasize that the applications of Theorem 1.1 goes well beyond this goal. In light of this remark, let us define the maps* $\pi_1, \pi_2, \pi_3 : \Omega \subset \mathbb{R}^3 \to \Omega$ *as follow:*

$$\pi_1(x_1, x_2, x_3) = (-x_1, x_2, x_3),$$
$$\pi_2(x_1, x_2, x_3) = (x_1, -x_2, x_3),$$
$$\pi_3(x_1, x_2, x_3) = (x_1, x_2, -x_3).$$

Consider the 3D case of the stationary Navier-Stokes equations with damping presented in equation (1). Assume that Ω *is invariant under the maps* $\pi_1, \pi_2, \pi_3 : \Omega \to \Omega$. *Moreover, assume that* K_S *is a subset of* V *containing all* $u \in V$ *with the following property:*

$$\begin{cases} u_1(x_1, x_2, x_3) = -u_1(-x_1, x_2, x_3), \\ u_2(x_1, x_2, x_3) = u_2(-x_1, x_2, x_3), \\ u_3(x_1, x_2, x_3) = u_3(-x_1, x_2, x_3). \end{cases} \tag{4}$$

Furthermore, assume that $f(x) \in L^2(\Omega)$ *also holds the same property; i.e.,*

$$\begin{cases} f_1(x_1, x_2, x_3) = -f_1(-x_1, x_2, x_3), \\ f_2(x_1, x_2, x_3) = f_2(-x_1, x_2, x_3), \\ f_3(x_1, x_2, x_3) = f_3(-x_1, x_2, x_3). \end{cases}$$

Then, the solution $u = (u_1, u_2, u_3)$ *obtained in Theorems 1.2, 1.3 and 1.4 is symmetric in the sense (4).*

The paper is organized as follows. In section 2, we shall provide the proof of Theorems 1.1 and 1.2 through a minimax principle. Section 3 is devoted to the proof of our results in Theorems 1.3 and 1.4.

2 A minimax principle and the proof of Theorem 1.2

In this section, we first prove the variational principle presented in Theorem 1.1 which is applicable specifically to our problem when $\mu < 0$, and $p > 2$. Afterwards, we proceed with the proof of Theorem 1.2.
We consider the Banach space $V = \{u \in H_0^1(\Omega) \cap L^p(\Omega) : \nabla.u = 0\}$ equipped with the following norm

$$\|u\| := \|u\|_{H_0^1(\Omega)} + \|u\|_{L^p(\Omega)}.$$

Let Λu be the operator $\Lambda u := (u.\nabla)u$, that is

$$< \Lambda u, v > = \int_\Omega (\Lambda u) v = \int_\Omega \sum_{j,k=1}^{n} u_k \frac{\partial u_j}{\partial x_k} v_j.$$

Let K be a convex and weakly closed subset of V. As stated in Theorem 1.1 we shall consider the functional $M : K \times K \to \mathbb{R}$ given in (2).

Proof of Theorem 1.1: It follows from condition (i) in the theorem that there exists $\bar{u} \in K$ such that

$$\frac{1}{2}\int_\Omega |\nabla \bar{u}|^2 \, dx - \frac{1}{2}\int_\Omega |\nabla v|^2 \, dx \leq \int_\Omega (f(x) + \frac{1}{p}|\bar{u}|^{p-2}\bar{u} - \Lambda \bar{u})(\bar{u} - v) \, dx, \quad \forall v \in K. \tag{5}$$

It also follows from (ii) that there exists $\bar{v} \in K$ such that

$$\int_\Omega \nabla \bar{v}.\nabla \eta \, dx = \int_\Omega (f(x) + |\bar{u}|^{p-2}\bar{u} - \Lambda \bar{u})\eta \, dx, \quad \forall \eta \in V. \tag{6}$$

Substituting $\eta = \bar{u} - \bar{v}$ in the latter equality gives

$$\int_\Omega \nabla \bar{v}.\nabla(\bar{u} - \bar{v}) \, dx = \int_\Omega (f(x) + |\bar{u}|^{p-2}\bar{u} - \Lambda \bar{u})(\bar{u} - \bar{v}) \, dx, \quad \forall \eta \in V. \tag{7}$$

Setting $v = \bar{v}$ in (5) and taking into account the equality (7) we obtain that

$$\int_\Omega \nabla \bar{v}.\nabla(\bar{u} - \bar{v})\, dx \geq \frac{1}{2}\int_\Omega |\nabla \bar{u}|^2\, dx - \frac{1}{2}\int_\Omega |\nabla \bar{v}|^2\, dx. \tag{8}$$

On the other hand, it follows from the convexity of $g(t) = \dfrac{1}{2}t^2$ that

$$\int_\Omega \nabla \bar{v}.\nabla(\bar{u} - \bar{v})\, dx \leq \frac{1}{2}\int_\Omega |\nabla \bar{u}|^2\, dx - \frac{1}{2}\int_\Omega |\nabla \bar{v}|^2\, dx. \tag{9}$$

Inequalities (8) and (9) together imply that

$$\int_\Omega \nabla \bar{v}.\nabla(\bar{u} - \bar{v})\, dx = \frac{1}{2}\int_\Omega |\nabla \bar{u}|^2\, dx - \frac{1}{2}\int_\Omega |\nabla \bar{v}|^2\, dx.$$

Therefore,

$$\int_\Omega |\nabla \bar{u} - \nabla \bar{v}|^2 = 0,$$

from which it follows that $\bar{v} = \bar{u}$ for a.e. $x \in \Omega$. Hence, the equality (6) proves the desired result. □

We shall apply Theorem 1.1 to prove the existence of a solution in Theorem 1.2. The convex subset K of V required in Theorem 1.1 is defined by

$$K := K(r) = \{u \in V : \|u\|_{W^{2,2}(\Omega)} \leq r\}, \tag{10}$$

for some $r > 0$ to be determined. To see that $K(r)$ is weakly closed, we present the proof of the statement in the following lemma.

Lemma 2.1 *Let $r > 0$ be fixed. The set*

$$K(r) = \{u \in V : \|u\|_{W^{2,2}(\Omega)} \leq r\},$$

is weakly closed in V.

Proof: Let $\{u_m\}$ be a sequence in $K(r)$ such that $u_m \rightharpoonup u$ weakly in V. Then there exists a subsequence of u_m, denoted by u_m again such that $u_m \to u$ a.e in Ω. On the other hand, $\|u_m\|_{W^{2,2}(\Omega)} \leq r$ for all $m \in \mathbb{N}$ and so $\{u_m\}$ is bounded in $W^{2,2}(\Omega)$. Going if necessary to a subsequence, there exists $\bar{u} \in W^{2,2}(\Omega)$ such that $u_m \rightharpoonup \bar{u}$ weakly in $W^{2,2}(\Omega)$ and $u_m(x) \to \bar{u}(x)$ for a.e. $x \in \Omega$. It follows then $u(x) = \bar{u}(x)$ for a.e. $x \in \Omega$. Thus $u_m \rightharpoonup u$ weakly in $W^{2,2}(\Omega)$. Now from the weak lower semi-continuity of the norm in $W^{2,2}(\Omega)$ follows that

$$\|u\|_{W^{2,2}(\Omega)} \leq \liminf_{m\to\infty} \|u_m\|_{W^{2,2}(\Omega)} \leq r,$$

which means that $u \in K(r)$. □

To apply Theorem 1.1, we shall need to verify both conditions (i) and (ii) in this theorem. To verify condition (i) we shall use the following version of the well-known Ky Fan's min-max principle [1]. We refer to Lemma 12.1 in [5] for a proof.

Lemma 2.2 *Let G be a closed convex subset of a reflexive Banach space X, and consider $M : G \times G \to \mathbb{R}$ to be a functional such that:*

1. *For each $y \in G$, the map $x \to M(x, y)$ is weakly lower semi-continuous on G.*
2. *For each $x \in G$, the map $y \to M(x, y)$ is concave on G.*
3. *There exists $\gamma \in \mathbb{R}$ such that $M(x, x) \leq \gamma$ for every $x \in G$.*
4. *There exists a $y_0 \in G$ such that $G_0 = \{x \in G : M(x, y_0) \leq \gamma\}$ is bounded.*

Then there exits $\bar{x} \in G$ such that $M(\bar{x}, y) \leq \gamma$ for all $y \in G$.

Without loss of generality, we can assume that $\mu = -1$ in Theorem 1.2. Recall that the functional $M : K \times K \to \mathbb{R}$ is defined as follows,

$$M(u,v) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{1}{2}\int_\Omega |\nabla v|^2\, dx + \int_\Omega (\Lambda u - f(x) - \frac{1}{p}|u|^{p-2}u)(u-v)\, dx. \tag{11}$$

One of the requirements in Lemma 2.2 is the lower semi-continuity of $M(.\ ,v)$ for a fixed v. In order to verify that, we begin by the following Lemma.

Lemma 2.3 *$\forall\ v \in K$, the map $u \to < \Lambda u, v >$ is weakly continuous on K for the values of n, p in Theorem 1.2.*

Proof: Fix $v \in K$, and let $\{u^m\}$ be a sequence in K such that $u^m \rightharpoonup u$ weakly in V. We have

$$\begin{aligned}\Big| < \Lambda u^m, v > - < \Lambda u, v > \Big| &= \Big| \sum_{j,k=1}^n \int_\Omega \big(u_k^m \frac{\partial u_j^m}{\partial x_k} v_j - u_k \frac{\partial u_j}{\partial x_k} v_j\big)\, dx \Big| \\ &= \Big| \sum_{j,k=1}^n \int_\Omega \big((u_k^m - u_k)\frac{\partial u_j^m}{\partial x_k} v_j + u_k \frac{\partial (u_j^m - u_j)}{\partial x_k} v_j\big)\, dx \Big| \\ &\leq \sum_{j,k=1}^n \int_\Omega \Big|(u_k^m - u_k)\frac{\partial u_j^m}{\partial x_k} v_j\Big| + \sum_{j,k=1}^n \int_\Omega \Big| u_k \frac{\partial (u_j^m - u_j)}{\partial x_k} v_j \Big|\, dx.\end{aligned}$$

On the other hand, by Hölder inequality we can conclude that

$$\begin{aligned}\sum_{j,k=1}^n \int_\Omega \Big|(u_k^m - u_k)\frac{\partial u_j^m}{\partial x_k} v_j\Big| &\leq C\|(u^m - u)v\|_{L^2(\Omega)} \|\nabla u^m\|_{L^2(\Omega)} \\ &\leq C\|u^m - u\|_{L^4(\Omega)} \|v\|_{L^4(\Omega)} \|\nabla u^m\|_{L^2(\Omega)},\end{aligned}$$

for some constant C. Therefore, we would have

$$\Big| < \Lambda u^m, v > - < \Lambda u, v > \Big| \leq C\|u^m - u\|_{L^4(\Omega)} \|v\|_{L^4(\Omega)} \|\nabla u^m\|_{L^2(\Omega)} + \sum_{j,k=1}^n \int_\Omega \Big| u_k \frac{\partial (u_j^m - u_j)}{\partial x_k} v_j \Big|\, dx.$$

Moreover, since the space $W^{2,2}(\Omega)$ is compactly imbedded into $L^4(\Omega)$, for all $n \leq 6$, it follows that $u^m \to u$ strongly in $L^4(\Omega)$. Furthermore, since u, v are in $W^{2,2}(\Omega)$, we deduce from Hölder's inequality that $u_k v_j \in L^2(\Omega)$. Finally, since $\nabla u^m \rightharpoonup \nabla u$ weakly in K, by definition of weak convergence the result follows. □

Lemma 2.4 *$\forall\ v \in K$, the map $u \to M(u, v)$ is weakly lower semi-continuous on K for the values of n, p in Theorem 1.2.*

Proof: Let $v \in K$ be fixed and, let $\{u^m\}$ be a sequence in K such that $u^m \rightharpoonup u$ weakly in V. Considering $< \Lambda u, u >= 0$, it follows from (2) that

$$\begin{aligned}M(u,v) = &\frac{1}{2}\int_\Omega |\nabla u|^2\, dx - < \Lambda u, v > - \int_\Omega f(x) u\, dx - \frac{1}{p}\int_\Omega |u|^p\, dx \\ &+ \frac{1}{p}\int_\Omega |u|^{p-2} uv\, dx - \frac{1}{2}\int_\Omega |\nabla v|^2\, dx + \int_\Omega f(x) v\, dx,\end{aligned} \tag{12}$$

Now we shall verify lower semi-continuity of every single term in (12) separately. Note that the last two terms in (12) are constant with respect to u.

- Since the function $g(u) = |u|^2$ is convex, it can be easily shown that

  $$\int_\Omega |\nabla u|^2\, dx \leq \liminf_{m\to\infty} \int_\Omega |\nabla u_m|^2\, dx.$$

  This implies that the map $u \to \int_\Omega |\nabla u|^2\, dx$ is weakly lower semi-continuous.

- The map $u \to -\int_\Omega \Lambda u.v \ \ dx$ is weakly lower semi-continuous by Lemma 2.3.
- Since $f \in L^2(\Omega)$, applying the definition of weak convergence leads to

$$\int_\Omega f(x)\ u\ dx = \liminf_{n\to\infty} \int_\Omega f(x) u_m\ dx.$$

- The map $u \to \int_\Omega |u|^p\ dx$ is weakly lower semi-continuous for n, p in Theorem 1.2 because

  (i) if $n \leq 4$, then $W^{2,2}(\Omega)$ is compactly imbedded into $L^p(\Omega)$ for all $p > 2$, and

  (ii) if $n = 5,\ 6$, then $W^{2,2}(\Omega)$ is compactly imbedded into $L^p(\Omega)$ for all $2 < p < \dfrac{2n}{n-4}$.

  It then follows for both of cases that

$$\lim_{n\to\infty} \int_\Omega |u_m|^p = \int_\Omega |u|^p\ dx,$$

- The map $u \to \int_\Omega |u|^{p-2} uv\ dx$ is weakly lower semi-continuous for n, p in Theorem 1.2 because

  (i) if $n \leq 4$, then $W^{2,2}(\Omega)$ is compactly imbedded into $L^{2(p-1)}(\Omega)$ for all $p > 2$, and

  (ii) if $n = 5,\ 6$, then $W^{2,2}(\Omega)$ is compactly imbedded into $L^{2(p-1)}(\Omega)$ for all $2 < p < \dfrac{2n-4}{n-4}$.

  In both cases, we have $|u|^{p-2}u \in L^2(\Omega)$ from which we deduce that the map $u \to \int_\Omega |u|^{p-2} uv\ dx$ is a continuous functional and

$$\lim_{n\to\infty} \int_\Omega |u_m|^{p-2} uv\ dx = \int_\Omega |u|^{p-2} uv\ dx.$$

This completes the proof. □

We are now in the position to state the following result addressing condition (i) in Theorem 1.1.

Lemma 2.5 *Let $K = K(r)$ be a convex and weakly closed subset of V defined in (10). Let $M : K \times K \to \mathbb{R}$ be defined as (2) and n, p as in Theorem 1.2. Then there exists $\bar{u} \in K$ such that*

$$M(\bar{u}, v) \leq 0, \quad \forall v \in K.$$

Proof: We shall show that the function M satisfies all the conditions of the Ky Fan's Min-Max Principle presented in Lemma 2.2. The condition (1) is provided by Lemma 2.4. For each $u \in K$, the map $v \to M(u, v)$ is concave on K since $M(u, v)$ is a linear functional with respect to v except $-\dfrac{1}{2}\int_\Omega |\nabla v|^2\ dx$, which is in fact concave . Also we have $M(u, u) = 0 = \gamma$ for every $u \in K$. Finally, since K is bounded, we can conclude that $\{u \in K : M(u, v) \leq 0\}$ is bounded. It now follows by Lemma 2.2 that there exists $\bar{u} \in K$ such that

$$M(\bar{u}, v) \leq 0, \quad \forall v \in K,$$

as desired. □

Our next task consists of verifying condition (ii) in Theorem 1.1. In order to do this, we start with the following two lemmas, which provide us the required estimates. Hereafter C will denote a positive constant, not necessarily the same one.

Lemma 2.6 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $1 < p$. Then there exists a constant C such that for any $u \in K(r)$ we have*

$$\big\|f + |u|^{p-2}u - \Lambda u\big\|_{L^2(\Omega)} \leq C\big(\|f\|_{L^2(\Omega)} + \|u\|^{p-1}_{L^{2(p-1)}(\Omega)} + \|u\|_{W^{1,2^*}(\Omega)}\|u\|_{L^n(\Omega)}\big), \tag{13}$$

when $n > 2$, and

$$\big\|f + |u|^{p-2}u - \Lambda u\big\|_{L^2(\Omega)} \leq C\big(\|f\|_{L^2(\Omega)} + \|u\|^{p-1}_{L^{2(p-1)}(\Omega)} + \|u\|_{W^{1,4}(\Omega)}\|u\|_{L^4(\Omega)}\big), \tag{14}$$

when $n = 2$.

Proof: Let $u \in K(r)$ and $n > 2$. By Hölder's inequality we have

$$\begin{aligned}
\big\|f + |u|^{p-2}u - \Lambda u\big\|_{L^2(\Omega)} &\leq \|f\|_{L^2(\Omega)} + \|u^{p-1}\|_{L^2(\Omega)} + \|\Lambda u\|_{L^2(\Omega)} \\
&\leq \|f\|_{L^2(\Omega)} + \|u\|^{p-1}_{L^{2(p-1)}(\Omega)} + C\|\nabla u\|_{L^{2^*}(\Omega)}\|u\|_{L^n(\Omega)}, \qquad (\text{where } 2^* = \frac{2n}{n-2}) \\
&\leq \|f\|_{L^2(\Omega)} + \|u\|^{p-1}_{L^{2(p-1)}(\Omega)} + C\|u\|_{W^{1,2^*}(\Omega)}\|u\|_{L^n(\Omega)},
\end{aligned}$$

as desired. For the case $n = 2$, one can proceed via the same argument considering the fact that

$$\|\Lambda u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^4(\Omega)}\|u\|_{L^4(\Omega)}.$$

□

Lemma 2.7 *Let $p > 2$ and $C > 0$ be given. If $\|f\|_{L^2(\Omega)}$ is small enough then there exists $0 < r \in \mathbb{R}$ which satisfies*

$$C(\|f\|_{L^2(\Omega)} + r^{p-1} + r^2) \leq r.$$

Proof: Since $p > 2$, we can choose r such that

$$C(r^{p-1} + r^2) \leq \frac{r}{2}.$$

Now if $C\|f\|_{L^2(\Omega)} \leq \frac{r}{2}$ then we have

$$C(\|f\|_{L^2(\Omega)} + r^{p-1} + r^2) \leq r,$$

as desired. □

Here is another useful results that we shall use in the sequel. See Theorem 1.2 in [3] for a more general version of the following theorem.

Lemma 2.8 *If $g \in L^2(\Omega)$, then there exists $u \in W^{2,2}(\Omega) \cap H^1_0(\Omega)$, a scalar function $P : \Omega \to \mathbb{R}$ and a constant C such that*

$$\Delta u + \nabla P = g, \quad \nabla . u = 0, \quad u|_{\partial\Omega} = 0,$$

and

$$\|\Delta u\|_{L^2(\Omega)} + \|\nabla P\|_{L^2(\Omega)} \leq C\|g\|_{L^2(\Omega)}.$$

The following result is proved in [4], Lemma 9.17.

Lemma 2.9 *Let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^n$ and let the operator $Lu = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u$ be strictly Elliptic in Ω with coefficients $a_{ij} \in C(\Omega)$, $b_i, c \in L^\infty(\Omega)$, with $i, j = 1, ..., n$ and $c \leq 0$. Then there exists a positive constant C (independent of u) such that*

$$\|u\|_{W^{2,p}(\Omega)} \leq C\|Lu\|_{L^p(\Omega)},$$

for all $u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$, $1 < p < \infty$.

Here comes a direct consequence of Lemma 2.9.

Corollary 2.10 *Let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^n$. Then there exists a constant C such that*

$$\|u\|_{W^{2,2}(\Omega)} \leq C\|\Delta u\|_{L^2(\Omega)},$$

for all $u \in W^{2,2}(\Omega) \cap H_0^1(\Omega)$.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Without loss of generality we may suppose that $\mu = -1$. Define $K = K(r)$ for $r > 0$ to be determined presently. By Lemma 2.5 we have the existence of a non-trivial $\bar{u} \in K$ such that

$$M(\bar{u}, v) \leq 0 \quad \forall v \in K.$$

Now we shall show the existence of $\bar{v}$ that satisfy condition (ii) in Theorem 1.1. Consider

$$g(x) = f + |\bar{u}|^{p-2}\bar{u} - \Lambda\bar{u}.$$

Thus we have to show there exists $\bar{v} \in K$ that the following equation holds in the weak sense,

$$-\Delta v + \nabla P = g(x), \tag{15}$$

where $P : \Omega \to \mathbb{R}$ is an appropriate scalar function. By Lemma 2.8 there exists $\bar{v} \in V$ which satisfies (15) and

$$\|\Delta\bar{v}\|_{L^2(\Omega)} + \|\nabla P\|_{L^2(\Omega)} \leq C\|g\|_{L^2(\Omega)}. \tag{16}$$

It is sufficient to show that $\bar{v} \in K$. For $n > 2$, the estimate (16) together with Lemma 2.6 imply that

$$\begin{aligned}\|\Delta\bar{v}\|_{L^2(\Omega)} + \|\nabla P\|_{L^2(\Omega)} &\leq C\|f + |\bar{u}|^{p-2}\bar{u} - \Lambda\bar{u}\|_{L^2(\Omega)}\\ &\leq C\big(\|f\|_{L^2(\Omega)} + \|\bar{u}\|^{p-1}_{L^{2(p-1)}(\Omega)} + \|\bar{u}\|_{W^{1,2^*}(\Omega)}\|\bar{u}\|_{L^n(\Omega)}\big).\end{aligned} \tag{17}$$

On the other hand, Corollary 2.10 together with (17) yield that

$$\begin{aligned}\|\bar{v}\|_{W^{2,2}(\Omega)} \leq C\|\Delta\bar{v}\|_{L^2(\Omega)} &\leq C\big(\|\Delta\bar{v}\|_{L^2(\Omega)} + \|\nabla P\|_{L^2(\Omega)}\big)\\ &\leq C\big(\|f\|_{L^2(\Omega)} + \|\bar{u}\|^{p-1}_{L^{2(p-1)}(\Omega)} + \|\bar{u}\|_{W^{1,2^*}(\Omega)}\|\bar{u}\|_{L^n(\Omega)}\big).\end{aligned} \tag{18}$$

Using the imbeddings of $W^{2,2}(\Omega) \hookrightarrow L^{2(p-1)}(\Omega)$, $W^{2,2}(\Omega) \hookrightarrow W^{1,2^*}(\Omega)$ and $W^{2,2}(\Omega) \hookrightarrow L^n(\Omega)$ for $2 < n \leq 6$, we obtain from (18) that

$$\|\bar{v}\|_{W^{2,2}(\Omega)} \leq C\big(\|f\|_{L^2(\Omega)} + \|\bar{u}\|^{p-1}_{W^{2,2}(\Omega)} + \|\bar{u}\|_{W^{2,2}(\Omega)}\|\bar{u}\|_{W^{2,2}(\Omega)}\big). \tag{19}$$

Let r be as in Lemma 2.7 for C given in the last inequality above. The inequality (19) and Lemmas 2.7 yield that

$$\|\bar{v}\|_{W^{2,2}(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + r^{p-1} + r^2) \leq r,$$

where $\|f\|_{L^2(\Omega)}$ is small enough. That means $\bar{v} \in K$ and so $\bar{v} = \bar{u}$ as in Theorem 1.1. This completes the proof of (i) for $n > 2$, and (ii). Now the inequality (17) gives that

$$\|\Delta u\|_{L^2(\Omega)} + \|\nabla P\|_{L^2(\Omega)} \leq C\big(\|f\|_{L^2(\Omega)} + \|u\|^{p-1}_{L^{2(p-1)}(\Omega)} + \|u\|_{W^{1,2^*}(\Omega)}\|u\|_{L^n(\Omega)}\big).$$

For $n = 2$, by inequality (14) in Lemma 2.6 and by repeating the same argument we have the desired result. □

3 Proof of Theorems 1.3 and 1.4

We shall need some preliminary results before proving our results in this section. We shall consider the same notation for the Banach space $V = \{u \in H_0^1(\Omega) \cap L^p(\Omega) : \nabla.u = 0\}$ with norm $\|u\| = \|u\|_{H_0^1(\Omega)} + \|u\|_{L^p(\Omega)}$. Where Ω is a bounded domain of $\mathbb{R}^n$. Note that the operator $\Lambda u = (u.\nabla)u$ may not be defined on whole space $H_0^1(\Omega)$. Although, there exists constant C such that

$$|<\Lambda u, v>| = \Big|\int_\Omega \sum_{j,k=1}^n u_k \frac{\partial u_j}{\partial x_k} v_j\Big| \leq C\|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}\|v\|_{C^1(\Omega)},$$

which means that for the dense linear subspace

$$E = \{u \in C_c^1(\Omega) : \nabla.u = 0\}$$

of V, we have that $<\Lambda u, v>$ is well defined for all $u \in V$ and $v \in E$. We shall define $\Phi : V \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \frac{1}{p}\int_\Omega |u|^p\, dx - \int_\Omega fu\, dx.$$

We also define $H : V \times V \to \mathbb{R}$ by

$$H(v, u) = \Phi(u) - \Phi(v).$$

For $r > 1$, set

$$K(r) = \{u \in V : \|u\| \leq r\},$$

that is convex and weakly closed in V by similar arguments as in the proof of Lemma 2.1. Let

$$K_0(r) = K(r) \cap E,$$

and define $M : K(r) \times K_0(r) \to \mathbb{R}$ by

$$M(u, v) = H(v, u) - <\Lambda u, v> . \tag{20}$$

In the case where $\mu > 0$ in the damping term, we shall make use of a different version of Ky-Fan minimax theorem (See Lemma 12.1 in [5]) for a proof). This version is more practical when one expects less regularity of the solution. For a set D, we denote its convex hull by $conv(D)$.

Lemma 3.1 *Let $\emptyset \neq D \subset G \subset X$ where G is a weakly compact convex set in a Banach space X, and consider $M : G \times conv(D) \to \mathbb{R}$ to be a function such that:*

1. *For each $y \in D$, the map $x \to M(x, y)$ is weakly lower semi-continuous on G.*
2. *For each $x \in G$, the map $y \to M(x, y)$ is concave on $conv(D)$.*
3. *$M(x, x) \leq 0$ for every $x \in conv(D)$.*

Then there exits $\bar{x} \in G$ such that $M(\bar{x}, y) \leq 0$ for all $y \in D$.

Proof of Theorem 1.3. Without loss of generality we may suppose that $\mu = 1$. By similar arguments as in Lemma 2.4 for M defined in (20) we obtain that

- For each $v \in K_0(r)$ the function $u \to M(u, v)$ is weakly lower semi-continuous.
- For each $u \in K(r)$ the function $v \to M(u, v)$ is concave.
- $M(u, u) = 0, \forall u \in K_0(r)$

Now we can apply Ky-Fan minimax principle (Lemma 3.1), which yields the existence of a $\bar{u}_r \in K(r)$ such that

$$M(\bar{u}_r, v) = H(v, \bar{u}_r) - < \Lambda\bar{u}_r, v > \leq 0, \quad \forall v \in K_0(r). \tag{21}$$

Substituting $v = 0$ in the latter inequality implies that $\Phi(\bar{u}_r) \leq 0$. Now the coercivity of the functional Φ follows that $\{\bar{u}_r\}_r$ is bounded in V and so there exists a sequence $r_n \to \infty$ and $\bar{u} \in V$ such that $\bar{u}_{r_n} \rightharpoonup \bar{u}$ weakly in V. If $v \in E$ is fixed, then from (21) and the weak lower semi-continuity of the functions involved, we get

$$H(v, \bar{u}) - < \Lambda\bar{u}, v > \leq 0, \qquad \forall v \in K_0(r). \tag{22}$$

Since $r > 1$, this indeed implies that

$$\sup_{v \in E, \|v\| \leq 1} < \Lambda\bar{u}, v > + \inf_{\|z\| \leq 1} H(z, \bar{u}) \leq 0. \tag{23}$$

Therefore,

$$\sup_{v \in E, \|v\| \leq 1} < \Lambda\bar{u}, v > \leq - \inf_{\|z\| \leq 1} H(z, \bar{u}) < \infty. \tag{24}$$

This implies that the linear functional $l : E + \mathbb{R}\bar{u} \to \mathbb{R}$ defined by $l(v + t\bar{u}) =< \Lambda\bar{u}, v >$ is continuous. It now follows from the bounded linear extension theorem that l can be extended to a bounded linear operator $L : V \to \mathbb{R}$ with the same operator norm as l. It then follows from the Riesz representation theorem that there exists $\hat{\Lambda}\bar{u} \in V^*$ such that

$$< \hat{\Lambda}\bar{u}, \bar{u} >= 0, \qquad \text{and} \qquad < \hat{\Lambda}\bar{u}, v >=< \Lambda\bar{u}, v >, \quad \forall v \in E. \tag{25}$$

This together with (22) yield that

$$H(v, \bar{u}) - < \hat{\Lambda}\bar{u}, v > \leq 0, \quad \forall v \in E. \tag{26}$$

But since E is dense in V and expression (26) is continuous with respect to v, we can conclude that

$$H(v, \bar{u}) - < \hat{\Lambda}\bar{u}, v > \leq 0, \quad \forall v \in V. \tag{27}$$

Now by substituting $v = \bar{u} + t\eta$, $\eta \in V$, into (27) we obtain that

$$H(\bar{u} + t\eta, \bar{u}) - < \hat{\Lambda}\bar{u}, \bar{u} + t\eta > \leq 0, \quad \forall t \in \mathbb{R}. \tag{28}$$

Dividing (28) by $t > 0$ and letting t converge to zero yield that

$$\int_\Omega \nabla\bar{u}.\nabla\eta \, dx + \int_\Omega |\bar{u}|^{p-2}\bar{u} \, \eta \, dx - \int_\Omega f\eta \, dx + < \hat{\Lambda}\bar{u}, \eta > \geq 0, \quad \forall \eta \in V. \tag{29}$$

Now substituting η by $-\eta$ in (29) we deduce the opposite inequality and thus

$$\int_\Omega \nabla\bar{u}.\nabla\eta \, dx + \int_\Omega |\bar{u}|^{p-2}\bar{u} \, \eta \, dx - \int_\Omega f\eta \, dx + < \hat{\Lambda}\bar{u}, \eta >= 0, \quad \forall \eta \in V. \tag{30}$$

This together with (25) follow that

$$\int_\Omega \nabla\bar{u}.\nabla\eta \, dx + \int_\Omega |\bar{u}|^{p-2}\bar{u} \, \eta \, dx - \int_\Omega f\eta \, dx + < \Lambda\bar{u}, \eta >= 0, \quad \forall \eta \in E.$$

This complete the proof of part (i).

For the proof of the second part we will consider two cases $n \leq 4$ and $p \geq 4$ separately.
Case 1, $(n \leq 4)$: Let $v \in V$. If $n \leq 4$, then $4 \leq \dfrac{2n}{n-2}$. From continuous imbedding of Sobolev space $H_0^1(\Omega)$ into $L^4(\Omega)$, and by the Hölder inequality we obtain for operator Λu

$$
\begin{aligned}
| < \Lambda u, v > | = \Big| \int_\Omega \sum_{j,k=1}^n u_k \frac{\partial u_j}{\partial x_k} v_j \Big| &\leq C \|uv\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \\
&\leq C \|u\|_{L^4(\Omega)} \|v\|_{L^4(\Omega)} \|u\|_{H_0^1(\Omega)} < \infty.
\end{aligned}
$$

This means, the operator Λu is well defined on V. Since E is a dense subspace of V, from uniqueness of the bounded linear extension theorem we have $< \hat{\Lambda}\bar{u}, v > = < \Lambda \bar{u}, v >$, $\forall v \in V$. Now the result follows from (30).
Case 2, $p \geq 4$: For $v \in V$, since $V \subset L^p(\Omega)$ we can deduce that

$$
| < \Lambda u, v > | \leq C \|uv\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \leq C \|u\|_{H_0^1(\Omega)} \|u\|_{L^p(\Omega)} \|v\|_{L^{\frac{2p}{p-2}}(\Omega)} < \infty
$$

where the last inequality follows from $\dfrac{2p}{p-2} \leq p$. Thus operator Λu is well defined on whole V and this completes the proof. □

As we have just seen, the case of $\mu > 0$ with the linear damping term was covered in the theorem 1.3. But for $\mu < 0$, due to an essential role of Lemma 2.7 in the proof of Theorem 1.2 we were not be able to deal with the linear damping term in this theorem. However, with a similar argument as in the proof of Theorem 1.3, we would be in a position to manage this separately. Note that when $p = 2$ we have that $V = \{u \in H_0^1(\Omega) : \nabla u = 0\}$, and

$$
\Phi(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{\mu}{2} \int_\Omega |u|^2 \, dx - \int_\Omega f \; u \, dx.
$$

Proof of Theorem 1.4. In the same way as in proof of Theorem 1.3, it follows from the Ky-Fan minimax principle (Lemma 3.1) that there exits $\bar{u}_r \in K(r)$ with

$$
M(\bar{u}_r, v) = H(v, \bar{u}_r) - < \Lambda \bar{u}_r, v > \leq 0, \qquad \forall v \in K_0(r). \tag{31}
$$

Now we claim that $\{\bar{u}_r\}_r$ is bounded in V and so there exists a sequence $r_n \to \infty$ and $\bar{u} \in V$ such that $\bar{u}_{r_n} \rightharpoonup \bar{u}$ weakly in V. Thus, by similar arguments as in the proof of Theorem 1.3 we obtain the result.
Now in order to complete the proof we have to show the claim. Assume, by contradiction, that $\{\bar{u}_r\}_r$ is unbounded. So there exists a sequence $r_m \to \infty$ such that $\{\bar{u}_{r_m}\}_m$ is unbounded. By substituting $v = 0$ in (31) we obtain that

$$
\Phi(\bar{u}_{r_m}) = \frac{1}{2} \int_\Omega |\nabla \bar{u}_{r_m}|^2 \, dx + \frac{\mu}{2} \int_\Omega |\bar{u}_{r_m}|^2 \, dx - \int_\Omega f \; \bar{u}_{r_m} \, dx \leq 0. \tag{32}
$$

Let $t_m^2 = \displaystyle\int_\Omega |\nabla \bar{u}_{r_m}|^2 \, dx$, and $w_m = \dfrac{\bar{u}_{r_m}}{t_m}$. Note that $\|w_m\|_{H_0^1(\Omega)} = 1$. Thus, there exists a $w = (z_1, ..., z_n) \in V$ such that $w_m \rightharpoonup w$ weakly in V. It follows that $w \neq 0$, because dividing (32) by t_m^2 we obtain

$$
\frac{1}{2} + \frac{\mu}{2} \int_\Omega |w_m|^2 \, dx \leq \frac{1}{t_m} \int_\Omega f \; w_m \, dx, \tag{33}
$$

and letting $m \to \infty$, due to the compact imbedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ we obtain that

$$
\frac{1}{2} + \frac{\mu}{2} \int_\Omega |w|^2 \, dx \leq 0, \tag{34}
$$

which implies $w \neq 0$. Also, we have

$$\frac{1}{2}\int_\Omega |\nabla w|^2\,dx + \frac{\mu}{2}\int_\Omega |w|^2\,dx \leq \liminf_{m\to\infty}\big(\frac{1}{2}\int_\Omega |\nabla w_m|^2\,dx + \frac{\mu}{2}\int_\Omega |w_m|^2\,dx\big)$$
$$\leq \frac{1}{2} + \frac{\mu}{2}\int_\Omega |w|^2\,dx.$$

This estimate together with (34) yield that

$$\int_\Omega |\nabla w|^2\,dx + \mu\int_\Omega |w|^2\,dx \leq 0.$$

Therefore,

$$\frac{\int_\Omega |\nabla w|^2\,dx}{\int_\Omega |w|^2\,dx} \leq -\mu, \tag{35}$$

from which together with hypothesis $-\lambda_1 \leq \mu$ in the theorem we obtain that

$$\frac{\int_\Omega |\nabla w|^2\,dx}{\int_\Omega |w|^2\,dx} \leq \lambda_1. \tag{36}$$

On the other hand, for the first eigenvalue λ_1 of $-\Delta$ we have

$$\frac{\int_\Omega |\nabla w|^2\,dx}{\int_\Omega |w|^2\,dx} = \frac{\sum_{i=1}^n \int_\Omega |\nabla z_i|^2\,dx}{\sum_{i=1}^n \int_\Omega z_i^2\,dx} \geq \frac{\sum_{i=1}^n \lambda_1 \int_\Omega z_i^2\,dx}{\sum_{i=1}^n \int_\Omega z_i^2\,dx} = \lambda_1, \tag{37}$$

where $w = (z_1, ..., z_n)$. It then follows from (36) and (37)

$$\lambda_1 = \frac{\int_\Omega |\nabla w|^2\,dx}{\int_\Omega |w|^2\,dx}, \tag{38}$$

from which we obtain that

$$\int_\Omega |\nabla z_i|^2\,dx = \lambda_1 \int_\Omega z_i^2\,dx, \qquad (i = 1, .., n).$$

Therefore,

$$-\Delta z_i = \lambda_1 z_i, \quad i = 1, ..., n. \tag{39}$$

Since the first eigenvalue of the $-\Delta$ is simple it follows that there exists $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$ such that

$$z_i = \alpha_i \psi_1, \quad i = 1, ..., n, \tag{40}$$

where $\psi_1 > 0$ is the unique eigenfunction of $-\Delta$ corresponding to λ_1 with $\|\psi_1\|_{L^2(\Omega)} = 1$, i.e.

$$-\Delta\psi_1 = \lambda_1\psi_1, \quad \psi_1|_{\partial\Omega} = 0.$$

Since $\nabla.w = 0$, it follows from (40) that

$$0 = \sum_{i=1}^n \frac{\partial z_i}{\partial x_i} = \sum_{i=1}^n \alpha_i \frac{\partial \psi_1}{\partial x_i} = \alpha.\nabla\psi_1. \tag{41}$$

Now let x be an interior point of Ω and $\bar{x}$ the closest point on $\partial\Omega$ to x such that $\bar{x} - x = C\alpha$ for some constant $C \in \mathbb{R}$, and the line joining x to $\bar{x}$ lies in $\bar{\Omega}$. Define

$$g : [0, 1] \to \mathbb{R}$$
$$g(t) = \psi_1(tx + (1-t)\bar{x}).$$

It can be easily deduced from (41) that

$$g'(t) = (x - \bar{x}).\nabla\psi_1(tx + (1-t)\bar{x}) = C\alpha.\nabla\psi_1(tx + (1-t)\bar{x}) = 0.$$

Thus, g is a constant function and since $\psi_1|_{\partial\Omega} = 0$ we have

$$g(t) = g(0) = \psi_1(\bar{x}) = 0, \quad \forall t \in [0, 1],$$

which implies that $\psi_1(x) = 0$. This is the contradiction we were looking for. □

Data availability statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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