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On the topological centre of the algebra $\mathrm{LUC}(\mathcal{G})^*$ for general topological groups ☆

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Abstract

We consider the Banach algebra $\mathrm{LUC}(\mathcal{G})^*$ for a *not necessarily locally compact* topological group $\mathcal{G}$. Our goal is to characterize the topological centre $Z_t(\mathrm{LUC}(\mathcal{G})^*)$ of $\mathrm{LUC}(\mathcal{G})^*$. For locally compact groups $\mathcal{G}$, it is well known that $Z_t(\mathrm{LUC}(\mathcal{G})^*)$ equals the measure algebra $\mathrm{M}(\mathcal{G})$. We shall prove that for every second countable (not precompact) group $\mathcal{G}$, we have $Z_t(\mathrm{LUC}(\mathcal{G})^*) = \mathrm{M}(\widehat{\mathcal{G}})$, where $\widehat{\mathcal{G}}$ denotes the completion of $\mathcal{G}$ with respect to its right uniform structure (if $\mathcal{G}$ is precompact, then $Z_t(\mathrm{LUC}(\mathcal{G})^*) = \mathrm{LUC}(\mathcal{G})^*$, of course). In fact, this will follow from our more general result stating that for *any* separable (or any precompact) group $\mathcal{G}$, we have $Z_t(\mathrm{LUC}(\mathcal{G})^*) = \mathrm{Leb}(\mathcal{G})$, where $\mathrm{Leb}(\mathcal{G})$ denotes the algebra of uniform measures. The latter result also partially answers a conjecture made by I. Csiszár 35 years ago [I. Csiszár, On the weak* continuity of convolution in a convolution algebra over an arbitrary topological group, Studia Sci. Math. Hungar. 6 (1971) 27–40]. We shall give similar results for the topological centre $\Lambda(\mathcal{G}^{\mathrm{LUC}})$ of the LUC-compactification $\mathcal{G}^{\mathrm{LUC}}$ of $\mathcal{G}$. In particular, we shall prove that for *any* second countable (not precompact) group $\mathcal{G}$ admitting a group completion, we have $\Lambda(\mathcal{G}^{\mathrm{LUC}}) = \widehat{\mathcal{G}}$ (if $\mathcal{G}$ is precompact, then $\Lambda(\mathcal{G}^{\mathrm{LUC}}) = \mathcal{G}^{\mathrm{LUC}}$). Finally, we shall show that every linear (left) $\mathrm{LUC}(\mathcal{G})^*$-module map on $\mathrm{LUC}(\mathcal{G})$ is automatically continuous whenever $\mathcal{G}$ is, e.g., separable and not precompact.

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1. Introduction

In this paper we are concerned with the problem of describing the topological centre $Z_t(\mathrm{LUC}(\mathcal{G})^*)$ of the Banach algebra $\mathrm{LUC}(\mathcal{G})^*$, and the topological centre $\Lambda(\mathcal{G}^{\mathrm{LUC}})$ of the LUC-compactification of $\mathcal{G}$, for a *not necessarily locally compact* topological group $\mathcal{G}$. Both problems have been extensively studied in the locally compact case. The first problem was originally considered by A. Zappa in [38] for $\mathcal{G} = (\mathbb{R}, +)$ and for discrete subgroups of $(\mathbb{R}, +)$. It was solved for locally compact abelian groups by M. Grosser and V. Losert in [14], and without the assumption of commutativity by A.T.-M. Lau in [20], where it was proved that, if $\mathcal{G}$ is a locally compact group, then $Z_t(\mathrm{LUC}(\mathcal{G})^*)$ equals the measure algebra $\mathrm{M}(\mathcal{G})$. The second problem was solved by A.T.-M. Lau and J.S. Pym in [21], where it was established that, for locally compact groups, $\Lambda(\mathcal{G}^{\mathrm{LUC}})$ coincides with $\mathcal{G}$; see also the paper [22] by A.T.-M. Lau, P. Milnes and J.S. Pym. A simpler proof of the slightly stronger assertion $\Lambda(\mathcal{G}^{\mathrm{LUC}} \setminus \mathcal{G}) = \emptyset$ was given by I. Protasov and J.S. Pym in [35]. The second author of the present paper considered in [26] the topological centre problem for the algebra $L_1(\mathcal{G})^{**}$ endowed with the first Arens product, as well as for its quotient $\mathrm{LUC}(\mathcal{G})^*$ and, by introducing a new, unified approach to these problems, proved that $Z_t(L_1(\mathcal{G})^{**}) = L_1(\mathcal{G})$ and $Z_t(\mathrm{LUC}(\mathcal{G})^*) = \mathrm{M}(\mathcal{G})$. He was then able to show in [27] by a similar technique that, when $\mathcal{G}$ is a locally compact group, $\Lambda(\mathcal{G}^{\mathrm{LUC}} \setminus \mathcal{G}) = \emptyset$.

We show that for every second countable (not precompact) group $\mathcal{G}$, the equality

$$Z_t\big(\mathrm{LUC}(\mathcal{G})^*\big) = \mathrm{M}(\widehat{\mathcal{G}})$$

holds, where $\widehat{\mathcal{G}}$ denotes the completion of $\mathcal{G}$ with respect to its right uniform structure (generated by the sets $\{(x, y) \in \mathcal{G} \times \mathcal{G}\colon xy^{-1} \in U\}$, where U is a neighbourhood of the identity e of $\mathcal{G}$); note that if $\mathcal{G}$ is precompact, then obviously $Z_t(\mathrm{LUC}(\mathcal{G})^*) = \mathrm{LUC}(\mathcal{G})^*$. We even prove a more general result which has the above as a consequence: for *any* separable (or any precompact) group $\mathcal{G}$, we have

$$Z_t\big(\mathrm{LUC}(\mathcal{G})^*\big) = \mathrm{Leb}(\mathcal{G}), \tag{1.1}$$

where $\mathrm{Leb}(\mathcal{G})$ denotes the algebra of uniform measures. This also gives a partial—affirmative—answer to a conjecture made by I. Csiszár in 1971 (see [8, Remark (ii), p. 33]). Moreover, as we shall see, the inclusions

$$\mathrm{M}(\widehat{\mathcal{G}}) \subseteq \mathrm{Leb}(\mathcal{G}) \subseteq Z_t\big(\mathrm{LUC}(\mathcal{G})^*\big) \tag{1.2}$$

hold for *any* topological group $\mathcal{G}$. The main tool for the proof of (1.1) is our Theorem 3.4, an analogue for non-locally compact groups of [25, Satz 3.6.2] (see also [28]).

Moreover, we shall derive similar results for the topological centre $\Lambda(\mathcal{G}^{\mathrm{LUC}})$ of the LUC-compactification $\mathcal{G}^{\mathrm{LUC}}$ of $\mathcal{G}$. Namely, for *any* second countable (not precompact) group $\mathcal{G}$ admitting a group completion, we have $\Lambda(\mathcal{G}^{\mathrm{LUC}}) = \widehat{\mathcal{G}}$; analogously to the above situation, if $\mathcal{G}$ is precompact, then $\Lambda(\mathcal{G}^{\mathrm{LUC}}) = \mathcal{G}^{\mathrm{LUC}}$. As shown in [12], the inclusion $\widehat{\mathcal{G}} \subseteq \Lambda(\mathcal{G}^{\mathrm{LUC}})$ holds for *any* topological group $\mathcal{G}$. In fact, we shall consider a natural subsemigroup $\mathrm{DL}(\mathcal{G})$ of $\mathcal{G}^{\mathrm{LUC}}$ defined by a double limit criterion, which plays the role of the algebra $\mathrm{Leb}(\mathcal{G})$ of uniform measures in the setting of semigroup compactifications; we then obtain the precise analogue of (1.2):

$$\widehat{\mathcal{G}} \subseteq \mathrm{DL}(\mathcal{G}) \subseteq \Lambda\big(\mathcal{G}^{\mathrm{LUC}}\big) \tag{1.3}$$

for *any* topological group $\mathcal{G}$.

Finally, we shall prove a result on automatic continuity: every linear (left) $\mathrm{LUC}(\mathcal{G})^*$-module map on $\mathrm{LUC}(\mathcal{G})$ is automatically bounded whenever $\mathcal{G}$ is, for instance, separable and not precompact.

2. Preliminaries

Given any group $\mathcal{G}$ and $s \in \mathcal{G}$ we denote by r_s (ℓ_s) the right (left) translation by s, defined by $(r_s f)(t) := f(ts)$ ($(\ell_s f)(t) := f(st)$) for $t \in \mathcal{G}$ and $f : \mathcal{G} \to \mathbb{C}$. We denote the identity of $\mathcal{G}$ by e.

We write $C_b(\mathcal{G})$ for the space of complex-valued, bounded, continuous functions on $\mathcal{G}$. We denote by $\mathrm{LUC}(\mathcal{G})$ the subspace of $C_b(\mathcal{G})$ consisting of left uniformly continuous functions on $\mathcal{G}$, i.e., the functions $f \in C_b(\mathcal{G})$ for which the map $\mathcal{G} \ni s \mapsto \ell_s f \in (C_b(\mathcal{G}), \|\cdot\|_\infty)$ is continuous. Note that there is no common notation in the literature; e.g., our space $\mathrm{LUC}(\mathcal{G})$ is written as $U_r(\mathcal{G})$ in [8].

If $n \in \mathrm{LUC}(\mathcal{G})^*$ and $f \in \mathrm{LUC}(\mathcal{G})$, then the function $n \cdot f$ defined by

$$(n \cdot f)(x) := \langle n, \ell_x f \rangle \quad (x \in \mathcal{G})$$

belongs to $\mathrm{LUC}(\mathcal{G})$ (see, for example, [6, Theorem 4.4.3]), i.e., $\mathrm{LUC}(\mathcal{G})$ is left introverted.

This operation induces a natural product on $\mathrm{LUC}(\mathcal{G})^*$ defined by

$$\langle m \cdot n, f \rangle := \langle m, n \cdot f \rangle \quad \big(m, n \in \mathrm{LUC}(\mathcal{G})^*,\ f \in \mathrm{LUC}(\mathcal{G})\big),$$

which turns $\mathrm{LUC}(\mathcal{G})^*$ into a Banach algebra and $\mathrm{LUC}(\mathcal{G})$ into a left $\mathrm{LUC}(\mathcal{G})^*$-module with the action introduced above.

Definition 2.1. The *topological centre* $Z_t(\mathrm{LUC}(\mathcal{G})^*)$ of $\mathrm{LUC}(\mathcal{G})^*$ is defined as the set of elements $m \in \mathrm{LUC}(\mathcal{G})^*$ such that left multiplication by m is w^*–w^*-continuous.

Remark 2.2. Let us briefly note that the question of determining $Z_t(\mathrm{LUC}(\mathcal{G})^*)$ is only interesting for groups $\mathcal{G}$ that are not precompact. Indeed, in the precompact case, we have $\mathrm{LUC}(\mathcal{G}) = \mathrm{WAP}(\mathcal{G})$, the algebra of weakly almost periodic functions on $\mathcal{G}$ (cf., e.g., [6, Corollary 4.4.11]), and it is well known that multiplication in $\mathrm{WAP}(\mathcal{G})^*$ (defined as in $\mathrm{LUC}(\mathcal{G})^*$) is separately w^*–w^*-continuous. Hence $Z_t(\mathrm{LUC}(\mathcal{G})^*) = \mathrm{LUC}(\mathcal{G})^*$ whenever $\mathcal{G}$ is precompact.

If we denote by δ_x the point evaluation at x ($x \in \mathcal{G}$) and consider the w^*-closure $\overline{\delta_{\mathcal{G}}}^{w^*} \subseteq \mathrm{LUC}(\mathcal{G})^*$ of the set of all point evaluations, then (cf. [6]) this set with the induced product (and the topology inherited from the w^*-topology on $\mathrm{LUC}(\mathcal{G})^*$) is a semigroup compactification of $\mathcal{G}$; this compactification is denoted by $\mathcal{G}^{\mathrm{LUC}}$. It equals the spectrum of the commutative C^*-algebra $\mathrm{LUC}(\mathcal{G})$. It can also be shown that $\mathcal{G}^{\mathrm{LUC}}$ is the largest semigroup compactification, in the sense that any other semigroup compactification of $\mathcal{G}$ is a natural quotient of $\mathcal{G}^{\mathrm{LUC}}$. Moreover, $\mathcal{G}^{\mathrm{LUC}}$

can be characterized as the greatest ambit of $\mathcal{G}$, i.e., the greatest $\mathcal{G}$-flow which has a point with dense orbit (see [19] or [32]).

Definition 2.3. The *topological centre* $\Lambda(\mathcal{G}^{\mathrm{LUC}})$ of $\mathcal{G}^{\mathrm{LUC}}$ is defined as the set of all points $x \in \mathcal{G}^{\mathrm{LUC}}$ such that left multiplication by x is continuous.

We now recall the definition and basic properties of *uniform measures* which—under different names—have first been introduced and studied by Berezanskii [5], Fedorova [11] and LeCam [23,24]; Csiszár [8] investigated an equivalent property for positive functionals on $\mathrm{LUC}(\mathcal{G})$ which he called ρ-continuity. Below we shall use Csiszár's terminology (extended to not necessarily positive functionals). Caution is advised with the notion of "measure": uniform measures are generally *not* measures in the usual sense, but merely functionals on $\mathrm{LUC}(\mathcal{G})$ with an additional continuity property (which they share in particular with ordinary measures). For very recent interesting developments in this area we recommend Pachl's work [30] which moreover gives an excellent survey of the subject—and in fact also refers to (an earlier version of) the present paper.

Definition 2.4. Let $\mathcal{G}$ be a topological group.

(i) A family of functions $\{f_i \mid i \in I\} \subseteq \mathrm{LUC}(\mathcal{G})$ is *equi-LUC* if for all $\varepsilon > 0$ there exists a neighbourhood U of e such that $\|\ell_x f_i - f_i\|_\infty < \varepsilon$ for all $i \in I$ and $x \in U$.
(ii) The algebra $\mathrm{Leb}(\mathcal{G})$ of *ρ-continuous functionals on* $\mathrm{LUC}(\mathcal{G})$ (or of *uniform measures*) is defined to be the subalgebra of $\mathrm{LUC}(\mathcal{G})^*$ consisting of all functionals $m \in \mathrm{LUC}(\mathcal{G})^*$ such that, if $(f_\alpha) \subseteq \mathrm{Ball}(\mathrm{LUC}(\mathcal{G}))$ is an equi-LUC net of functions with $f_\alpha \to 0$ pointwise, then $\langle m, f_\alpha\rangle \to 0$. The algebra consisting of the functionals $m \in \mathrm{LUC}(\mathcal{G})^*$ which satisfy the above property only for equi-LUC sequences will be denoted by $\mathrm{Leb}^s(\mathcal{G})$.

Remark 2.5. Let $\mathcal{G}$ be a topological group.

(i) Clearly, $\mathrm{Leb}(\mathcal{G})$ and $\mathrm{Leb}^s(\mathcal{G})$ are norm-closed subalgebras of $\mathrm{LUC}(\mathcal{G})^*$.
(ii) $\mathrm{Leb}(\mathcal{G})$ and $\mathrm{Leb}^s(\mathcal{G})$ are bands in $\mathrm{LUC}(\mathcal{G})^*$; this follows from [24, Lemma 3].
(iii) If $\mathcal{G}$ is separable, we have $\mathrm{Leb}(\mathcal{G}) = \mathrm{Leb}^s(\mathcal{G})$. Indeed, if $m \in \mathrm{Leb}(\mathcal{G})$ (respectively, $m \in \mathrm{Leb}^s(\mathcal{G})$), then on every bounded equi-LUC subset of $\mathrm{LUC}(\mathcal{G})$, the functional m is (sequentially) continuous for the topology of pointwise convergence. But on such a set pointwise convergence is equivalent to pointwise convergence on a dense subset. Since the topology of pointwise convergence on a countable set is metrizable, sequential continuity implies continuity.
(iv) If $\mathcal{G}$ is precompact, then $\mathrm{Leb}(\mathcal{G}) = \mathrm{LUC}(\mathcal{G})^*$; see [24, Note, p. 18].

Finally, we recall various notions of boundedness for a topological group. These concepts are studied in detail in [1–3,15].

Definition 2.6. A topological group $\mathcal{G}$ is called:

(i) *bounded* if, given any neighbourhood V of e, there exist $n \in \mathbb{N}$ and a finite subset $F \subseteq \mathcal{G}$ such that $\mathcal{G} = V^n F$;

(ii) *totally bounded* (or *precompact*) if, given any neighbourhood V of e, there exists a finite subset $F \subseteq \mathcal{G}$ such that $\mathcal{G} = VF$;
(iii) *ω-bounded* if, given any neighbourhood V of e, there exists a countable subset $C \subseteq \mathcal{G}$ such that $\mathcal{G} = VC$.

For the following, let us recall a very useful combinatorial result on partitions of groups.

Theorem 2.7. *If $\mathcal{G} = A_1 \cup \cdots \cup A_m$ is a finite partition of an arbitrary group $\mathcal{G}$, then there exists a subset $A = A_i$ of the partition such that $\mathcal{G} = A^{-1}AF$ for some finite subset $F \subseteq \mathcal{G}$.*

Proof. This is [34, Theorem 11.5.1]. □

Remark 2.8. Let $\mathcal{G}$ be a topological group.

(i) Suppose that, for every neighbourhood V of e, there exist finite subsets $A, B \subseteq \mathcal{G}$ and $n \in \mathbb{N}$ such that

$$\mathcal{G} = \bigcup_{\substack{x \in A \\ y \in B}} x V^n y,$$

then $\mathcal{G}$ is bounded.
(ii) Suppose that, for every neighbourhood V of e, there exist finite subsets $A, B \subseteq \mathcal{G}$ such that

$$\mathcal{G} = \bigcup_{\substack{x \in A \\ y \in B}} x V y,$$

then $\mathcal{G}$ is totally bounded.

Proof. We shall only prove (i) since (ii) is obtained similarly.

We can obviously assume V to be symmetric. By Theorem 2.7, there exist points $\tilde{x} \in A$ and $\tilde{y} \in B$ and a finite subset $F \subseteq \mathcal{G}$ such that

$$\mathcal{G} = \bigcup_{x \in F} (\tilde{x} V^n \tilde{y})^{-1} (\tilde{x} V^n \tilde{y}) x.$$

Hence, $\mathcal{G} = \bigcup_{x \in F} \tilde{y}^{-1} V^{2n} \tilde{y} x$, which implies that $\mathcal{G} = \tilde{y}\mathcal{G} = \bigcup_{x \in F} V^{2n} \tilde{y} x$. □

When $\mathcal{G}$ is locally compact, the concepts of boundedness and compactness coincide. In the realm of non-locally compact groups the class of unbounded, separable (in particular, ω-bounded) groups is very large indeed. Here we recall a few examples of such groups.

- The abelian groups $(\mathbb{Q}^n, +)$, $(\mathbb{Q} \setminus \{0\}, \cdot)$, $(\mathbb{A}^n, +)$ and $(\mathbb{A} \setminus \{0\}, \cdot)$ are separable and unbounded. Here by $\mathbb{A}$ we denote the set of algebraic numbers over $\mathbb{Z}$ with the topology induced by the standard topology of $\mathbb{R}$.
- Any separable locally convex (in particular, Banach) space is an unbounded, separable abelian group.

- Let X be a path-connected, compact, metric space. Denote by $C_b(X, \mathbb{T})$ the abelian group (with pointwise multiplication) of continuous functions $f : X \to \mathbb{T}$, endowed with the topology of uniform convergence on bounded subsets. Then the connected component $C_b^0(X, \mathbb{T})$ of the identity is a separable, unbounded group (see [31, Lemma 7]).
- Any closed subgroup of the infinite permutation group S_∞ with the pointwise convergence topology is a group admitting "small open subgroups" (i.e., it has a countable neighbourhood basis at e consisting of open subgroups), and is Polish (see [4, 1.5]); hence it is unbounded if and only if it is not precompact.
- If $\mathcal{F}$ is a Fraïssé structure (see for instance [33] or [19]), then its automorphism group $\mathrm{Aut}(\mathcal{F})$ is a separable, unbounded group. This class in particular contains the automorphism group $\mathrm{Aut}(\Gamma)$ of the infinite, countable random graph Γ and the automorphism group of $(\mathbb{Q}, \leqslant)$.

3. The factorization theorem

We start by recalling a well-known concept.

Definition 3.1. A family $\{A_i \mid i \in I\}$ of subsets of $\mathcal{G}$ is *left uniformly separated* if there exists a neighbourhood V of e such that $VA_i \cap VA_j = \emptyset$ whenever $i, j \in I$ with $i \neq j$.

The following lemma shows that pointwise sums of LUC-functions still belong to $\mathrm{LUC}(\mathcal{G})$ under suitable conditions involving the above notion.

Lemma 3.2. *Let $\mathcal{G}$ be a topological group. Consider a family of functions $\{u_i \mid i \in I\} \subseteq \mathrm{LUC}(\mathcal{G})$ that is bounded and equi-*LUC. *Suppose further that the sets* $\mathrm{supp}(u_i)$ *are left uniformly separated. Then the (pointwise defined) function $u = \sum_{i \in I} u_i$ belongs to* $\mathrm{LUC}(\mathcal{G})$.

Proof. This is [25, Proposition 3.2.6]; see also [28]. □

We now come to the factorization theorems which constitute the main tool of the paper, and are of interest in their own right. They show that the factorization result [25, Satz 3.2.7] (cf. also [28])—which concerns families of functions of cardinality $\kappa(\mathcal{G})$ (the compact covering number of $\mathcal{G}$) on locally compact, non-compact groups—can be extended beyond the realm of local compactness under some weak assumptions on $\mathcal{G}$.

Theorem 3.3. *Let $\mathcal{G}$ be an unbounded, ω-bounded group. Then there exists a sequence $(\psi_n) \subseteq \mathcal{G}^{\mathrm{LUC}}$ such that for every equi-*LUC *sequence $(f_n) \subseteq \mathrm{Ball}(\mathrm{LUC}(\mathcal{G}))$ there is a single function $f \in \mathrm{Ball}(\mathrm{LUC}(\mathcal{G}))$ such that the factorization*

$$f_n = \psi_n \cdot f$$

holds for all $n \in \mathbb{N}$.

Proof.[1] By Remark 2.8(i) there exists a symmetric (open) neighbourhood V of e such that, for all $m \in \mathbb{N}$, the group $\mathcal{G}$ cannot be covered by finitely many sets of the form $sV^m t$ with $s, t \in \mathcal{G}$.

[1] Note added in proof: Theorem 3.3 can also be shown with the assumption 'unbounded' replaced by 'not precompact' by an argument similar to the one given for Theorem 3.4.

Since $\mathcal{G}$ is ω-bounded, there is a countable set $\{x_k \mid k \in \mathbb{N}\}$ in $\mathcal{G}$ such that the family $\{Vx_k \mid k \in \mathbb{N}\}$ is a covering of $\mathcal{G}$.

Let $W_{k,m} := V^m x_k$. Consider the family of all unions of finitely many sets $W_{k,m}$. This family still forms a countable covering of $\mathcal{G}$ which we denote by $\{K_n \mid n \in I\}$, with $|I| = \omega$.

We define a relation on I by

$$n \prec' l \quad \Leftrightarrow \quad VK_n \subsetneq K_l.$$

Obviously, $\prec'$ is transitive. Moreover, $\prec'$ directs I. In order to see this, let $n, l \in I$. Since $\mathcal{G}$ is unbounded, there exists $p \in I$ such that $K_p \nsubseteq VK_n \cup VK_l$. By construction, there exists $q \in I$ such that $K_q = VK_n \cup VK_l \cup K_p$. This implies that $n \prec' q$ and $l \prec' q$.

We consider the set $\tilde{I} := I \times J$, where $|J| = \omega$. For $\tilde{n} = (n, h) \in \tilde{I}$ we define $K_{\tilde{n}} := K_n$ and $A_{\tilde{n}} := V^2 K_{\tilde{n}}$.

We construct by induction a net $(y_{\tilde{n}})_{\tilde{n} \in \tilde{I}}$ with the property that

$$A_{\tilde{n}} y_{\tilde{n}}^{-1} \cap A_{\tilde{m}} y_{\tilde{m}}^{-1} = \emptyset \quad \text{if } \tilde{n} \neq \tilde{m}. \tag{3.1}$$

To this end we well-order $\tilde{I}$ by $\prec_w$ and impose that $y_{\tilde{n}}^{-1} \notin A_{\tilde{n}}^{-1} A_{\tilde{m}} y_{\tilde{m}}^{-1}$ for all $\tilde{m} \prec_w \tilde{n}$. This is possible because each set $A_{\tilde{n}}^{-1} A_{\tilde{m}} y_{\tilde{m}}^{-1}$ is a finite union of double translates of powers of V.

We define a relation on $\tilde{I}$ by

$$\tilde{n} \prec \tilde{m} \quad \Leftrightarrow \quad VK_{\tilde{n}} \subsetneq K_{\tilde{m}}.$$

Clearly, this relation is transitive and directs $\tilde{I}$. Note that, for all (n, h) and (m, k) in $\tilde{I}$, we have $(n, h) \prec (m, k)$ if and only if $n \prec' m$.

Let $\{U_i\}_{i=1}^{\infty}$ be a sequence of symmetric (open) neighbourhoods of e such that $U_1 = V$ and $U_{i+1}^2 \subseteq U_i$ for all $i \in \mathbb{N}$. By [16, Theorem 8.2], there exists a right invariant pseudo-metric d on $\mathcal{G}$ such that:

(i) d is uniformly continuous for the right uniform structure of $\mathcal{G} \times \mathcal{G}$;
(ii) $\frac{1}{2^{i-1}} \leqslant d(x, y)$ whenever $xy^{-1} \notin U_i$.

For $\tilde{n} \in \tilde{I}$, we define the functions

$$u_{\tilde{n}}(x) := 1 - \min\{1, d(x, K_{\tilde{n}})\} \quad (x \in \mathcal{G}).$$

Since d is right invariant and continuous, $\{u_{\tilde{n}} \mid \tilde{n} \in \tilde{I}\}$ is an equi-LUC family. Moreover, the functions $u_{\tilde{n}}$ have values in $[0, 1]$ and satisfy $u_{\tilde{n}}|_{K_{\tilde{n}}} \equiv 1$. Also,

$$\operatorname{supp}(u_{\tilde{n}}) \subseteq VK_{\tilde{n}}. \tag{3.2}$$

For if $x \notin VK_{\tilde{n}}$, then by property (ii) of d, we have $d(x, K_{\tilde{n}}) \geqslant 1$, whence $u_{\tilde{n}}(x) = 0$. We conclude that, for $\tilde{n}, \tilde{m} \in \tilde{I}$,

$$u_{\tilde{n}} u_{\tilde{m}} = u_{\tilde{n}} \quad \text{whenever } \tilde{n} \prec \tilde{m}. \tag{3.3}$$

By (3.1), for each $\tilde{n} \in \tilde{I}$, we have

$$V \operatorname{supp}(u_{\tilde{n}}) \subseteq V^2 K_{\tilde{n}} = A_{\tilde{n}}. \tag{3.4}$$

Consider the functions

$$v_{\tilde{n}} = v_{(n,h)} := r_{y_{(n,h)}}(u_{(n,h)} f_h) \in \mathrm{LUC}(\mathcal{G})$$

for $\tilde{n} = (n,h) \in \tilde{I}$.

For $\tilde{n} = (n,h)$, $\tilde{m} = (m,k) \in \tilde{I}$ with $\tilde{n} \neq \tilde{m}$ we have

$$\begin{aligned}
V\,\mathrm{supp}(v_{\tilde{n}}) \cap V\,\mathrm{supp}(v_{\tilde{m}}) &= V\,\mathrm{supp}(u_{(n,h)} f_h) y_{(n,h)}^{-1} \cap V\,\mathrm{supp}(u_{(m,k)} f_k) y_{(m,k)}^{-1} \\
&\subseteq V\,\mathrm{supp}(u_{(n,h)}) y_{(n,h)}^{-1} \cap V\,\mathrm{supp}(u_{(m,k)}) y_{(m,k)}^{-1} \\
&\subseteq A_{(n,h)} y_{(n,h)}^{-1} \cap A_{(m,k)} y_{(m,k)}^{-1} \quad \big(\text{by (3.4)}\big) \\
&= A_{\tilde{n}} y_{\tilde{n}}^{-1} \cap A_{\tilde{m}} y_{\tilde{m}}^{-1} \\
&= \emptyset \quad \big(\text{by (3.1)}\big).
\end{aligned}$$

Hence, the sets $\mathrm{supp}(v_{\tilde{n}})$, $\tilde{n} \in \tilde{I}$, are uniformly separated in the right uniformity of $\mathcal{G}$. Since the families $\{u_{(n,h)} \mid (n,h) \in \tilde{I}\}$ and $\{f_h\}_{h \in J}$ are equi-LUC and norm-bounded, the family $\{v_{\tilde{n}} \mid \tilde{n} \in \tilde{I}\}$ is equi-LUC. Moreover, the family $\{v_{\tilde{n}} \mid \tilde{n} \in \tilde{I}\}$ is norm-bounded. By Lemma 3.2, the last three facts imply that the function f defined pointwise by

$$f := \sum_{\tilde{n} \in \tilde{I}} v_{\tilde{n}} = \sum_{n \in I} \sum_{h \in J} r_{y_{(n,h)}}(u_{(n,h)} f_h)$$

belongs to $\mathrm{LUC}(\mathcal{G})$. Obviously, $\|f\|_\infty \leqslant 1$.

Let $\mathcal{F}$ be an ultrafilter on I which dominates the order filter. For $h \in J$, we define

$$\psi_h := w^*\text{-}\lim_{n \to \mathcal{F}} \delta_{y_{(n,h)}^{-1}} \in \mathcal{G}^{\mathrm{LUC}}.$$

We now show that the factorization

$$f_h = \psi_h \cdot f$$

holds for all $h \in J$. By (3.1) and (3.4) we know that for all $(n,h), (m,k) \in \tilde{I}$ with $(n,h) \neq (m,k)$:

$$\mathrm{supp}(r_{y_{(n,h)}} u_{(n,h)}) \cap \mathrm{supp}(r_{y_{(m,k)}} u_{(m,k)}) = \emptyset. \tag{3.5}$$

We note that for all $(n,h), (m,k), (l,c) \in \tilde{I}$ with $(l,c) \prec (m,k)$:

$$\begin{aligned}
u_{(l,c)}\big(r_{y_{(m,k)}^{-1}} r_{y_{(n,h)}}(u_{(n,h)} f_h)\big) &= u_{(l,c)}\big(u_{(m,k)} r_{y_{(m,k)}^{-1}} r_{y_{(n,h)}}(u_{(n,h)} f_h)\big) \quad \big(\text{by (3.3)}\big) \\
&= u_{(l,c)}\big(r_{y_{(m,k)}^{-1}}\big(r_{y_{(m,k)}}(u_{(m,k)}) r_{y_{(n,h)}}(u_{(n,h)} f_h)\big)\big) \\
&= \delta_{(n,h),(m,k)} u_{(l,c)} f_k \quad \big(\text{by (3.3) and (3.5)}\big).
\end{aligned}$$

Fix $x \in \mathcal{G}$. By the above we obtain for all $k \in J$ and $(l,c) \in \tilde{I}$:

$$\begin{aligned} u_{(l,c)}(x)\langle\psi_k, \ell_x f\rangle &= \lim_{m\to\mathcal{F}} u_{(l,c)}(x)\big(r_{y^{-1}_{(m,k)}}(f)\big)(x) \\ &= \lim_{m\to\mathcal{F}} \sum_{n\in I}\sum_{h\in J} u_{(l,c)}(x)\big(r_{y^{-1}_{(m,k)}} r_{y_{(n,h)}}(u_{(n,h)} f_h)\big)(x) \\ &= u_{(l,c)}(x) f_k(x). \end{aligned}$$

Since $u_{(l,c)} \to \mathbf{1}$ pointwise, we have $\psi_k \cdot f = f_k$ for all $k \in J$, as desired. □

If the group $\mathcal{G}$ is not just assumed to be ω-bounded but separable, we can considerably weaken the condition of unboundedness, namely to non-precompactness (recall that for precompact groups, the topological centre problem is trivial).

Theorem 3.4. *Let $\mathcal{G}$ be a separable, not precompact group. Then there exists a sequence $(\psi_n) \subseteq \mathcal{G}^{\mathrm{LUC}}$ such that for every equi-*LUC *sequence $(f_n) \subseteq \mathrm{Ball}(\mathrm{LUC}(\mathcal{G}))$ there is a single function $f \in \mathrm{Ball}(\mathrm{LUC}(\mathcal{G}))$ such that the factorization*

$$f_n = \psi_n \cdot f$$

holds for all $n \in \mathbb{N}$.

Proof. The argument is similar to the one given for Theorem 3.3 but quicker. Since $\mathcal{G}$ is not precompact, in view of Remark 2.8(ii), there exists a symmetric neighbourhood V of e such that $\mathcal{G}$ cannot be covered by finitely many sets of the form sV^4t with $s, t \in \mathcal{G}$. Let $\{x_k \mid k \in \mathbb{N}\}$ be a countable dense set in $\mathcal{G}$, and set $K_n := \{x_1, \dots, x_n\}$ for all $n \in \mathbb{N}$.

For $\tilde{n} = (n, h) \in \mathbb{N} \times \mathbb{N}$ we define $K_{\tilde{n}} := K_n$ and $A_{\tilde{n}} := V^2 K_{\tilde{n}}$.

We construct by induction a net $(y_{\tilde{n}})_{\tilde{n}\in\mathbb{N}\times\mathbb{N}}$ satisfying

$$A_{\tilde{n}} y_{\tilde{n}}^{-1} \cap A_{\tilde{m}} y_{\tilde{m}}^{-1} = \emptyset \quad \text{if } \tilde{n} \neq \tilde{m}.$$

To this end we well-order $\mathbb{N} \times \mathbb{N}$ by $\prec_w$ and impose that $y_{\tilde{n}}^{-1} \notin A_{\tilde{n}}^{-1} A_{\tilde{m}} y_{\tilde{m}}^{-1}$ for all $\tilde{m} \prec_w \tilde{n}$. This is possible because each set $A_{\tilde{n}}^{-1} A_{\tilde{m}} y_{\tilde{m}}^{-1}$ is of the form $F_1 V^4 F_2$ with finite sets $F_1, F_2 \subseteq \mathcal{G}$.

Let the sequence of neighbourhoods $\{U_i\}_{i=1}^{\infty}$, the pseudo-metric d and the functions $u_{\tilde{n}}$ ($\tilde{n} \in \mathbb{N} \times \mathbb{N}$) be as in the proof of Theorem 3.3. (Note that instead of (3.3) we will only use $u_{\tilde{n}}|_{K_{\tilde{n}}} \equiv 1$.) Also, define the LUC-functions $v_{\tilde{n}}$ ($\tilde{n} \in \mathbb{N} \times \mathbb{N}$) and f as before. Consider $\mathbb{N}$ with its natural order, and let $\mathcal{F}$ be an ultrafilter on $\mathbb{N}$ which dominates the order filter. For $h \in \mathbb{N}$, define

$$\psi_h := w^*\text{-}\lim_{n\to\mathcal{F}} \delta_{y^{-1}_{(n,h)}} \in \mathcal{G}^{\mathrm{LUC}}.$$

To prove the factorization, note that, as before, for all $(n, h), (m, k) \in \mathbb{N} \times \mathbb{N}$ with $(n, h) \neq (m, k)$:

$$\mathrm{supp}(r_{y_{(n,h)}} u_{(n,h)}) \cap \mathrm{supp}(r_{y_{(m,k)}} u_{(m,k)}) = \emptyset.$$

Using this and the fact that $u_{\tilde{n}}|_{K_{\tilde{n}}} \equiv 1$ ($\tilde{n} \in \mathbb{N} \times \mathbb{N}$), we have for all $(n, h), (m, k) \in \mathbb{N} \times \mathbb{N}$, $j \in \mathbb{N}$ with $j \leqslant m$:

$$\begin{aligned}
\big(r_{y^{-1}_{(m,k)}} r_{y_{(n,h)}}(u_{(n,h)} f_h)\big)(x_j) &= \big(u_{(m,k)} r_{y^{-1}_{(m,k)}} r_{y_{(n,h)}}(u_{(n,h)} f_h)\big)(x_j) \\
&= \big(r_{y^{-1}_{(m,k)}}\big(r_{y_{(m,k)}}(u_{(m,k)}) r_{y_{(n,h)}}(u_{(n,h)} f_h)\big)\big)(x_j) \\
&= \delta_{(n,h),(m,k)} f_k(x_j).
\end{aligned}$$

Since for any $k \in \mathbb{N}$, both functions $\psi_k \cdot f$ and f_k are continuous, we only need to check that they coincide on the set $\{x_j \mid j \in \mathbb{N}\}$. For fixed $j \in \mathbb{N}$, using the above, we obtain for all $k \in \mathbb{N}$:

$$\begin{aligned}
(\psi_k \cdot f)(x_j) = \langle \psi_k, \ell_{x_j} f\rangle &= \lim_{m \to \mathcal{F}} \big(r_{y^{-1}_{(m,k)}}(f)\big)(x_j) \\
&= \lim_{m \to \mathcal{F}} \sum_{n \in \mathbb{N}} \sum_{h \in \mathbb{N}} \big(r_{y^{-1}_{(m,k)}} r_{y_{(n,h)}}(u_{(n,h)} f_h)\big)(x_j) \\
&= f_k(x_j).
\end{aligned}$$

We thus obtain our factorization. □

4. The topological centres of LUC($\mathcal{G}$)* and $\mathcal{G}^{\mathrm{LUC}}$

This section contains the main results of the paper. In particular, we shall employ Theorem 3.4 to completely determine the topological centres of $\mathrm{LUC}(\mathcal{G})^*$ and $\mathcal{G}^{\mathrm{LUC}}$ for all second countable groups $\mathcal{G}$.

We shall start with relating the algebra $\mathrm{Leb}(\mathcal{G})$ of uniform measures (as defined in Section 2) both with the measure algebra $\mathrm{M}(\mathcal{G})$ and the topological centre $Z_t(\mathrm{LUC}(\mathcal{G})^*)$ for an *arbitrary* topological group $\mathcal{G}$. Let us briefly recall the definition and basic properties of $\mathrm{M}(\mathcal{G})$; in our discussion we follow [7].

Definition 4.1. Let $\mathcal{G}$ be a topological group.

(1) A positive measure μ is called *K-regular* if

$$\mu(E) = \sup_{E \supset K \in \mathfrak{K}} \mu(K),$$

where $\mathfrak{K}$ denotes the class of all compact subsets of $\mathcal{G}$.

(2) A finite, complex Borel measure μ on $\mathcal{G}$ is called *K-regular* if $|\mu|$ is K-regular.

(3) By $\mathrm{M}(\mathcal{G})$ we denote the *measure algebra* of $\mathcal{G}$, i.e., the collection of all finite, complex, K-regular Borel measures on $\mathcal{G}$.

Regarding (3) in the definition above we recall that the convolution of finite, complex, K-regular measures on $\mathcal{G}$ may be defined in the usual way to give another K-regular measure; in fact, $\mathrm{M}(\mathcal{G})$ with convolution is a Banach algebra. Moreover, $\mathrm{M}(\mathcal{G})$ is isometrically embedded in $\mathrm{LUC}(\mathcal{G})^*$ via integration.

For any topological group $\mathcal{G}$, we shall denote by $\widehat{G}$ the completion of $\mathcal{G}$ with respect to its right uniform structure. As is well known, $\widehat{G}$ is a subsemigroup of $\mathcal{G}^{\mathrm{LUC}}$ with jointly continuous multiplication. Analogously to the above definition, one considers the *measure algebra* $\mathrm{M}(\widehat{G})$, i.e., the space of finite, complex, K-regular Borel measures on $\widehat{G}$, which is a closed subalgebra of $\mathrm{LUC}(\widehat{\mathcal{G}})^* = \mathrm{LUC}(\mathcal{G})^*$. Indeed, the topological semigroup $\widehat{\mathcal{G}}$ is *C-distinguished*, i.e., the real-valued functions in $C_b(\widehat{\mathcal{G}})$ separate the points of $\widehat{\mathcal{G}}$ (for this, use, e.g., [17, Exercise 21.5.3]); hence, by [10, Theorem 3.7], $\mathrm{M}(\widehat{\mathcal{G}})$ with convolution product and the usual total variation norm, is a Banach algebra.

Proposition 4.2. *Let $\mathcal{G}$ be any topological group. Then we have*

$$\mathrm{M}(\widehat{\mathcal{G}}) \subseteq \mathrm{Leb}(\widehat{\mathcal{G}}) = \mathrm{Leb}(\mathcal{G}) \subseteq Z_t\big(\mathrm{LUC}(\mathcal{G})^*\big).$$

Proof. A bounded, equi-LUC net of functions on $\widehat{\mathcal{G}}$ that converges pointwise to 0, converges to 0 uniformly on compact subsets of $\widehat{\mathcal{G}}$. Combining this observation with the K-regularity of a measure $\mu \in \mathrm{M}(\widehat{\mathcal{G}})$, one easily deduces the first inclusion.

For the equality $\mathrm{Leb}(\widehat{\mathcal{G}}) = \mathrm{Leb}(\mathcal{G})$, note that $\mathrm{LUC}(\widehat{\mathcal{G}})$ can be identified with $\mathrm{LUC}(\mathcal{G})$, and that the corresponding bounded equi-LUC sets coincide, as well as the topologies of pointwise convergence on such sets.

To prove the last inclusion, let $n \in \mathrm{Leb}(\mathcal{G})$, and let $(f_\alpha) \subseteq \mathrm{Ball}(\mathrm{LUC}(\mathcal{G})^*)$ be a net which converges w^* to 0. We need to show that, for every $g \in \mathrm{LUC}(\mathcal{G})$, we have $\langle n \cdot f_\alpha, g\rangle \to 0$. By definition, $\langle n \cdot f_\alpha, g\rangle = \langle n, f_\alpha \cdot g\rangle$ and, since for every $x \in \mathcal{G}$ we have that $(f_\alpha \cdot g)(x) = \langle f_\alpha, \ell_x g\rangle \to 0$, $(f_\alpha \cdot g)$ is a bounded net of LUC functions which converges pointwise to 0. Moreover, the net $(f_\alpha \cdot g)$ is equi-LUC since, for all $x \in \mathcal{G}$, we have that

$$\big\|\ell_x(f_\alpha \cdot g) - f_\alpha \cdot g\big\|_\infty = \big\|f_\alpha \cdot (\ell_x g) - f_\alpha \cdot g\big\|_\infty \leqslant \|\ell_x f - f\|_\infty.$$

Hence, we get the desired convergence $\langle n \cdot f_\alpha, g\rangle = \langle n, f_\alpha \cdot g\rangle \to 0$ since n is ρ-continuous. $\square$

Remark 4.3. The inclusion $\mathrm{M}(\widehat{\mathcal{G}}) \subseteq Z_t(\mathrm{LUC}(\mathcal{G})^*)$ was shown for locally compact groups in [37, Lemma 3.1]. An alternative proof for arbitrary topological groups is as follows. Fix $\mu \in \mathrm{M}(\widehat{\mathcal{G}})$. Let $(n_\alpha) \subseteq \mathrm{Ball}(\mathrm{LUC}(\widehat{\mathcal{G}})^*) = \mathrm{Ball}(\mathrm{LUC}(\mathcal{G})^*)$ be a net that converges to 0 in the w^*-topology. We have to show that $\mu \cdot n_\alpha \to 0$ (w^*). Fix $f \in \mathrm{LUC}(\widehat{\mathcal{G}}) = \mathrm{LUC}(\mathcal{G})$. Then we have:

$$\langle \mu \cdot n_\alpha, f\rangle = \langle \mu, n_\alpha \cdot f\rangle = \int_{\widehat{\mathcal{G}}} (n_\alpha \cdot f)(x)\, d\mu(x) = \int_{\widehat{\mathcal{G}}} \langle n_\alpha, \ell_x f\rangle\, d\mu(x),$$

where the net of functions $n_\alpha \cdot f$ in $\mathrm{LUC}(\widehat{\mathcal{G}})$ converges to 0 pointwise, is bounded, and—as an easy calculation shows—is equi-LUC. Since $\mu \in \mathrm{M}(\widehat{\mathcal{G}})$ is K-regular, we conclude that $\langle \mu \cdot n_\alpha, f\rangle \to 0$.

We shall now give the proof of the difficult inclusion for the topological centre of $\mathrm{LUC}(\mathcal{G})^*$ which is the main application of our factorization results (Theorems 3.3 and 3.4). Our argument follows the lines of [25, Satz 3.5.1] (see also [28]), which is concerned with locally compact groups.

Theorem 4.4. *Let $\mathcal{G}$ be a topological group.*

(i) *If $\mathcal{G}$ is separable and not precompact, then $Z_t(\mathrm{LUC}(\mathcal{G})^*) \subseteq \mathrm{Leb}^s(\mathcal{G}) = \mathrm{Leb}(\mathcal{G})$.*

(ii) *If $\mathcal{G}$ is ω-bounded and unbounded, then* $Z_t(\mathrm{LUC}(\mathcal{G})^*) \subseteq \mathrm{Leb}^s(\mathcal{G})$.

Proof. Let $m \in Z_t(\mathrm{LUC}(\mathcal{G})^*)$. Consider a bounded, equi-LUC sequence of functions (f_n) in $\mathrm{LUC}(\mathcal{G})$ such that $f_n \to 0$ pointwise. In either case (i) or (ii), we only have to show that $\langle m, f_n\rangle \to 0$. It is enough to prove that every convergent subsequence of $(\langle m, f_n\rangle)_n$ converges to 0. Let $(\langle m, f_{n_k}\rangle)_k$ be such a sequence. Now we factorize the sequence $(f_n)_{n\in\mathbb{N}}$. According to Theorem 3.4 for case (i), and Theorem 3.3 for case (ii), there exist a sequence $(\psi_n)_{n\in\mathbb{N}} \subseteq \mathcal{G}^{\mathrm{LUC}}$ and a single function $f \in \mathrm{LUC}(\mathcal{G})$ such that

$$f_n = \psi_n \cdot f$$

for all $n \in \mathbb{N}$. Consider the subsequence $(\psi_{n_k})_k$ of $(\psi_n)_n$. There exists a w^*-convergent subnet whose limit we denote by E:

$$E := w^*\text{-}\lim_{\gamma} \psi_{n_{k_\gamma}} \in \mathcal{G}^{\mathrm{LUC}} \subseteq \mathrm{Ball}\big(\mathrm{LUC}(\mathcal{G})^*\big).$$

Now, on the one hand, we have, using that $m \in Z_t(\mathrm{LUC}(\mathcal{G})^*)$:

$$\begin{aligned}
\langle m, E\cdot f\rangle &= \langle m\cdot E, f\rangle \\
&= \lim_{\gamma}\langle m\cdot \psi_{n_{k_\gamma}}, f\rangle \\
&= \lim_{\gamma}\langle m, \psi_{n_{k_\gamma}}\cdot f\rangle \\
&= \lim_{\gamma}\langle m, f_{n_{k_\gamma}}\rangle \\
&= \lim_{k}\langle m, f_{n_k}\rangle,
\end{aligned}$$

since the latter limit exists. On the other hand, we obtain for arbitrary $x \in \mathcal{G}$:

$$\begin{aligned}
(E\cdot f)(x) &= \langle E, \ell_x f\rangle \\
&= \lim_{\gamma}\langle \psi_{n_{k_\gamma}}, \ell_x f\rangle \\
&= \lim_{\gamma}(\psi_{n_{k_\gamma}}\cdot f)(x) \\
&= \lim_{\gamma} f_{n_{k_\gamma}}(x) \\
&= \lim_{n} f_n(x) = 0.
\end{aligned}$$

Thus we have $E\cdot f = 0$, whence

$$\lim_{k}\langle m, f_{n_k}\rangle = \langle m, E\cdot f\rangle = 0,$$

which proves our claim. □

Remark 4.5. The above proof shows that the points in $\mathcal{G}^{\mathrm{LUC}}$ are enough to determine the topological centre of $\mathrm{LUC}(\mathcal{G})^*$ in the sense that w^*-continuity of left multiplication by $m \in \mathrm{LUC}(\mathcal{G})^*$ on $\mathcal{G}^{\mathrm{LUC}}$ only already implies that $m \in \mathrm{Leb}^s(\mathcal{G})$. This phenomenon has first been observed, in the locally compact case, in [25, Bemerkung 3.5.2]; see also [26]. In the intriguing recent memoir [9], H.G. Dales, A.T.-M. Lau and D. Strauss introduce and study the concept of a dtc set (i.e., a set which is determining for the topological centre), a notion that captures abstractly the situation described above. In fact, it is shown in [9] that for every locally compact, non-compact group $\mathcal{G}$, there exists a 2-point dtc set in $\mathcal{G}^{\mathrm{LUC}}$; see [13] and [18] for related results in the locally compact setting. Since the present paper characterizes completely the topological centre of $\mathrm{LUC}(\mathcal{G})^*$ for all separable groups $\mathcal{G}$, one might now ask if two testing points are also sufficient in the non-locally compact, non-precompact case.

Corollary 4.6. *Let $\mathcal{G}$ be any separable, or any precompact group. Then the topological centre of* $\mathrm{LUC}(\mathcal{G})^*$ *equals precisely the algebra of uniform measures, i.e.,* $Z_t(\mathrm{LUC}(\mathcal{G})^*) = \mathrm{Leb}(\mathcal{G})$.

Proof. If $\mathcal{G}$ is separable and not precompact, this follows from Theorem 4.4(i) and Proposition 4.2. If $\mathcal{G}$ is precompact, the assertion is clear by Remarks 2.2 and 2.5(iv). □

To characterize the topological centre in terms of standard measure theory for a very large class of groups, we shall invoke the following result.

Proposition 4.7. *Let X be a complete, metric uniform space. Then the space* $\mathrm{M}(X)$ *of all finite, complex, K-regular Borel measures on X coincides with the space* $\mathrm{Leb}(X)$ *of uniform measures on X (defined analogously to* $\mathrm{Leb}(\mathcal{G})$*).*

Proof. This is [30, Theorem 2.1.3]; for proofs, see [24, Note, p. 18], [5], or [11]. □

We can now state one of the main results of this paper which gives a simple, complete description of the topological centre of $\mathrm{LUC}(\mathcal{G})^*$ for a class of groups comprising *every* second countable group.

Corollary 4.8. *Let $\mathcal{G}$ be a topological group.*

(i) *If $\mathcal{G}$ is precompact, then* $Z_t(\mathrm{LUC}(\mathcal{G})^*) = \mathrm{LUC}(\mathcal{G})^*$.
(ii) *If $\mathcal{G}$ is second countable, then* $Z_t(\mathrm{LUC}(\mathcal{G})^*) = \mathrm{M}(\widehat{\mathcal{G}})$.

Proof. For (i), see Remark 2.2. To show (ii), first note that since $\mathcal{G}$ is separable, our Corollary 4.6 implies that $Z_t(\mathrm{LUC}(\mathcal{G})^*) = \mathrm{Leb}(\mathcal{G}) = \mathrm{Leb}(\widehat{\mathcal{G}})$. But $\mathcal{G}$ is metric, whence Proposition 4.7 applied to $X := \widehat{\mathcal{G}}$ yields $\mathrm{Leb}(\widehat{\mathcal{G}}) = \mathrm{M}(\widehat{\mathcal{G}})$, as desired. □

Let us point out that in the locally compact case, our results hold without any further condition on the group:

Theorem 4.9. *Let $\mathcal{G}$ be a locally compact group. Then we have*

$$Z_t\big(\mathrm{LUC}(\mathcal{G})^*\big) = \mathrm{M}(\mathcal{G}) = \mathrm{Leb}(\mathcal{G}).$$

Proof. The first equality is the main result of [20] (see [26] for another proof). The second equality follows from [25, Korollar 3.1.5] (see also [29]). □

We shall now give yet another important application of our Theorem 4.4. In [8, Remark (ii), p. 33], I. Csiszár formulates the following conjecture: if $L \in Z_t(\mathrm{LUC}(\mathcal{G})^*)$ then we have $L_\alpha \cdot M_\alpha \xrightarrow{w^*} L \cdot M$ whenever $L_\alpha \xrightarrow{w^*} L$ and $M_\alpha \xrightarrow{w^*} M$, where $L_\alpha, M_\alpha, M \in \mathrm{LUC}(\mathcal{G})^*$. The question is motivated by Theorem 1 in [8] which says that the above is in fact true if L is assumed to be ρ-continuous, L_α, L are positive functionals, and the net M_α is bounded (note that the latter condition is missing in the statement of [8, Theorem 1] but needed in the proof given there; we shall therefore consider Csiszár's conjecture only for *bounded* nets M_α). As Csiszár points out (cf. [8, Remark (i), p. 33]), dropping the positivity conditions on L_α and L, while "interesting," is of course "irrelevant" from his main point of view, which is the one of probability theory. Hence, his question is mainly concerned with the condition of ρ-continuity. Now, our Theorem 4.4(i) shows that it is indeed possible to drop the latter condition—since it is automatically satisfied—whenever $\mathcal{G}$ is separable and not precompact, thus answering partially Csiszár's conjecture:

Corollary 4.10. *Let $\mathcal{G}$ be a separable, not precompact group, and let $L \in Z_t(\mathrm{LUC}(\mathcal{G})^*)$. Then we have $L_\alpha \cdot M_\alpha \xrightarrow{w^*} L \cdot M$ whenever $L_\alpha \xrightarrow{w^*} L$ and $M_\alpha \xrightarrow{w^*} M$, where $L_\alpha, M_\alpha, M \in \mathrm{LUC}(\mathcal{G})^*$ with L_α, L positive and the net M_α bounded.*

Finally, we shall turn to the description of the topological centre $\Lambda(\mathcal{G}^{\mathrm{LUC}})$ of the semigroup $\mathcal{G}^{\mathrm{LUC}}$. To this end, let us consider the analogue of uniform measures in the context of the LUC-compactification of $\mathcal{G}$:

$$\mathrm{DL}(\mathcal{G}) := \mathcal{G}^{\mathrm{LUC}} \cap \mathrm{Leb}(\mathcal{G}) \quad \text{and} \quad \mathrm{DL}^s(\mathcal{G}) := \mathcal{G}^{\mathrm{LUC}} \cap \mathrm{Leb}^s(\mathcal{G}).$$

Clearly, $\mathrm{DL}(\mathcal{G})$ and $\mathrm{DL}^s(\mathcal{G})$ are subsemigroups of $\mathcal{G}^{\mathrm{LUC}}$. They may be characterized by the following double limit criterion: $m \in \mathrm{DL}(\mathcal{G})$ (respectively, $m \in \mathrm{DL}^s(\mathcal{G})$) if and only if m can be approximated in the w^*-topology by a net (x_α) in $\mathcal{G}$ such that, given any bounded, pointwise convergent, equi-LUC net (respectively, sequence) (f_β), we have:

$$\lim_\beta \lim_\alpha f_\beta(x_\alpha) = \lim_\alpha \lim_\beta f_\beta(x_\alpha).$$

We shall derive descriptions of $\Lambda(\mathcal{G}^{\mathrm{LUC}})$ analogous to the ones given above for $Z_t(\mathrm{LUC}(\mathcal{G})^*)$. We start with an immediate consequence of our Proposition 4.2.

Proposition 4.11. *Let $\mathcal{G}$ be any topological group. Then we have*

$$\widehat{\mathcal{G}} \subseteq \mathrm{DL}(\widehat{\mathcal{G}}) = \mathrm{DL}(\mathcal{G}) \subseteq \Lambda\big(\mathcal{G}^{\mathrm{LUC}}\big).$$

Remark 4.12. The inclusion $\widehat{\mathcal{G}} \subseteq \Lambda(\mathcal{G}^{\mathrm{LUC}})$ also follows from [12, Theorem 3.4], where a different method is used.

Theorem 4.13. *Let $\mathcal{G}$ be a topological group.*

(i) *If $\mathcal{G}$ is separable and not precompact, then $\Lambda(\mathcal{G}^{\mathrm{LUC}}) \subseteq \mathrm{DL}^s(\mathcal{G}) = \mathrm{DL}(\mathcal{G})$.*

(ii) *If $\mathcal{G}$ is ω-bounded and unbounded, then $\Lambda(\mathcal{G}^{\mathrm{LUC}}) \subseteq \mathrm{DL}^s(\mathcal{G})$.*

Proof. This can be shown in the same way as Theorem 4.4 (cf. Remark 4.5). □

Corollary 4.14. *Let $\mathcal{G}$ be any separable, or any precompact group. Then $\Lambda(\mathcal{G}^{\mathrm{LUC}}) = \mathrm{DL}(\mathcal{G})$.*

Proof. If $\mathcal{G}$ is separable and not precompact, use Theorem 4.13(i) and Proposition 4.11. If $\mathcal{G}$ is precompact, the equality is obvious in view of Remark 2.2, which yields $\Lambda(\mathcal{G}^{\mathrm{LUC}}) = \mathcal{G}^{\mathrm{LUC}}$, and Remark 2.5(iv). □

In order to derive a simple description of $\Lambda(\mathcal{G}^{\mathrm{LUC}})$ for a very large class of second countable groups, we need the following lemma. We recall that for a topological group $\mathcal{G}$, the topological semigroup $\widehat{\mathcal{G}}$ (i.e., its completion with respect to the right uniform structure) is a topological group if and only if $\mathcal{G}$ admits a group completion; cf. [36, Theorem 10.15]. This in turn is the case, for instance, if $\mathcal{G}$ is an almost SIN (= ASIN) group; cf. [36, Proposition 10.16]. A topological group $\mathcal{G}$ is called ASIN if there is a neighbourhood U of e in $\mathcal{G}$ on which the left and right uniform structures coincide. The class of ASIN groups comprises, for example, all topological groups containing an open SIN group, and all locally precompact groups; see [36, Remark–Definition 9.27].

Lemma 4.15. *Let $\mathcal{G}$ be any topological group that admits a group completion* (*e.g., any ASIN group*)*. Then we have*

$$\mathrm{M}(\widehat{\mathcal{G}}) \cap \mathcal{G}^{\mathrm{LUC}} = \widehat{\mathcal{G}}.$$

Proof. Note in the following that due to our assumption, $\widehat{\mathcal{G}}$ is a topological group. Let $m \in \mathrm{M}(\widehat{\mathcal{G}}) \cap \mathcal{G}^{\mathrm{LUC}}$. Then $m \in \mathrm{M}(\widehat{\mathcal{G}})^+$, hence there is a compact set $K \subseteq \widehat{\mathcal{G}}$ such that for all $f \in \mathrm{LUC}(\widehat{\mathcal{G}}) = \mathrm{LUC}(\mathcal{G})$ with $0 \leqslant f \leqslant 1$ and $f|_K \equiv 0$ we have $m(f) < \frac{1}{2}$. Since $m \in \mathcal{G}^{\mathrm{LUC}}$, there is a net $(x_\alpha)_{\alpha \in I} \subseteq \mathcal{G}$ such that $m = w^*\text{-}\lim_\alpha x_\alpha$.

Now, assume towards a contradiction that there is a symmetric (open) neighbourhood $V \subseteq \widehat{\mathcal{G}}$ of e such that for all $\alpha \in I$ we have $x_\alpha \in \widehat{\mathcal{G}} \setminus VK$. For the topological group $\widehat{\mathcal{G}}$, consider the pseudo-metric d used in the proof of Theorem 3.3. Then $d \in \mathrm{LUC}(\widehat{\mathcal{G}} \times \widehat{\mathcal{G}})$, and we have $d(x, K) \geqslant 1$ for all $x \in \widehat{\mathcal{G}} \setminus VK$. Define $f(x) := \min\{1, d(x, K)\}$ $(x \in \widehat{\mathcal{G}})$. Then $f \in \mathrm{LUC}(\widehat{\mathcal{G}})$, $0 \leqslant f \leqslant 1$, and $f|_K \equiv 0$; thus, $m(f) < \frac{1}{2}$. But $x_\alpha \notin VK$ for all $\alpha \in I$ implies that $m(f) = \lim_\alpha f(x_\alpha) = 1$, a contradiction. Hence, for any symmetric (open) neighbourhood $V \subseteq \widehat{\mathcal{G}}$ of e there is $\alpha_V \in I$ such that $x_{\alpha_V} \in VK$. It now follows easily from the compactness of K that, if V runs through a descending symmetric (open) neighbourhood base of e in $\widehat{\mathcal{G}}$, some subnet of (x_{α_V}) converges to some $y \in K$, whence $m = y \in \widehat{\mathcal{G}}$. □

We are now ready to present another main result of this paper.

Corollary 4.16. *Let $\mathcal{G}$ be a topological group.*

(i) *If $\mathcal{G}$ is precompact, then $\Lambda(\mathcal{G}^{\mathrm{LUC}}) = \mathcal{G}^{\mathrm{LUC}}$.*
(ii) *If $\mathcal{G}$ admits a group completion* (*e.g., $\mathcal{G}$ is ASIN*) *and is second countable, then $\Lambda(\mathcal{G}^{\mathrm{LUC}}) = \widehat{\mathcal{G}}$.*

Proof. (i) follows from Remark 2.2. Regarding (ii), $\mathcal{G}$ being separable, our Corollary 4.14 gives

$$\Lambda\big(\mathcal{G}^{\mathrm{LUC}}\big) = \mathrm{DL}(\mathcal{G}) = \mathrm{DL}(\widehat{\mathcal{G}}) = \mathcal{G}^{\mathrm{LUC}} \cap \mathrm{Leb}(\widehat{\mathcal{G}}).$$

Since $\mathcal{G}$ is metric, Proposition 4.7 implies $\mathrm{Leb}(\widehat{\mathcal{G}}) = \mathrm{M}(\widehat{\mathcal{G}})$, and we conclude by using Lemma 4.15. □

Again, let us briefly recall the situation in the locally compact case.

Theorem 4.17. *For a locally compact group $\mathcal{G}$, we have*

$$\Lambda\big(\mathcal{G}^{\mathrm{LUC}}\big) = \widehat{\mathcal{G}} = \mathcal{G} = \mathrm{DL}(\mathcal{G}).$$

Proof. Since $\mathcal{G}$ is locally compact, we have $\widehat{\mathcal{G}} = \mathcal{G}$. The equality $\Lambda(\mathcal{G}^{\mathrm{LUC}}) = \mathcal{G}$ is the main result of [21]; see also [22,27,35]. The last equality is obvious in view of Theorem 4.9 and the simple fact that $\mathrm{M}(\mathcal{G}) \cap \mathcal{G}^{\mathrm{LUC}} = \mathcal{G}$. □

We close by giving an application of our factorization theorems to automatic continuity of module maps on $\mathrm{LUC}(\mathcal{G})$. (See [25, Satz 2.4.1] and [28] for general locally compact groups.)

Theorem 4.18. *Let $\mathcal{G}$ be a topological group. Assume that $\mathcal{G}$ is separable and not precompact, or $\mathcal{G}$ is ω-bounded and unbounded. Then every linear left* $\mathrm{LUC}(\mathcal{G})^*$*-module map on* $\mathrm{LUC}(\mathcal{G})$ *is automatically continuous.*

Proof. Assume towards a contradiction that $\Phi : \mathrm{LUC}(\mathcal{G}) \to \mathrm{LUC}(\mathcal{G})$ is an unbounded linear left $\mathrm{LUC}(\mathcal{G})^*$-module map. Then there is a sequence $(g_n)_{n\in\mathbb{N}} \subseteq \mathrm{LUC}(\mathcal{G})$ with $\|g_n\|_\infty = 1$ and $\|\Phi(g_n)\|_\infty \geqslant n^2$ for all $n \in \mathbb{N}$. Then $(f_n)_{n\in\mathbb{N}}$, where $f_n := \frac{1}{n} g_n$, $n \in \mathbb{N}$, is a sequence of functions in $\mathrm{LUC}(\mathcal{G})$ such that $\|f_n\|_\infty \to 0$ and $\|\Phi(f_n)\|_\infty \geqslant n$ $(n \in \mathbb{N})$.

Obviously, the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded and equi-LUC. Our Theorem 3.3 (respectively, Theorem 3.4, depending on the assumption on $\mathcal{G}$) yields

$$f_n = \psi_n \cdot f \quad (n \in \mathbb{N})$$

with functionals $\psi_n \in \mathcal{G}^{\mathrm{LUC}} \subseteq \mathrm{Ball}(\mathrm{LUC}(\mathcal{G})^*)$ and a single function $f \in \mathrm{LUC}(\mathcal{G})$. So we obtain for all $n \in \mathbb{N}$:

$$n \leqslant \big\|\Phi(f_n)\big\|_\infty = \big\|\Phi(\psi_n \cdot f)\big\|_\infty = \big\|\psi_n \cdot \Phi(f)\big\|_\infty \leqslant \big\|\Phi(f)\big\|_\infty < \infty,$$

a contradiction. Hence Φ is continuous. □

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