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On a conjecture by Ghahramani–Lau and related problems concerning topological centres

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Abstract

Let $\mathscr{A}$ be a Banach algebra, and consider $\mathscr{A}^{**}$ equipped with the first Arens product. We establish a general criterion which ensures that $\mathscr{A}$ is left strongly Arens irregular, i.e., the first topological centre of $\mathscr{A}^{**}$ is reduced to $\mathscr{A}$ itself. Using this criterion, we prove that for a very large class of locally compact groups, Ghahramani–Lau's conjecture (cf. [Lau 94] and [Gha-Lau 95]) stating the left strong Arens irregularity of the measure algebra $\mathrm{M}(\mathscr{G})$, holds true. (Our methods obviously yield as well the right strong Arens irregularity in the situation considered.)

Furthermore, the same condition used above implies that every linear left $\mathscr{A}^{**}$-module homomorphism on $\mathscr{A}^*$ is automatically bounded and w^*-continuous. We finally show that our criterion also yields a partial answer to a question raised by Lau-Ülger (Trans. Amer. Math. Soc. 348 (3) (1996) 1191) on the topological centre of the algebra $(\mathscr{A}^* \odot \mathscr{A})^*$, for $\mathscr{A}$ having a right approximate identity bounded by 1.

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1. Introduction

In 1951, Arens showed that there are two canonical ways of extending the product on a Banach algebra $\mathcal{A}$ to the level of its bidual $\mathcal{A}^{**}$ (see [Are 51]); these are called the *Arens products*. Comparing the two products, there are two extreme cases which are naturally of major interest. In the extreme case where the two products coincide, the algebra $\mathcal{A}$ is called *Arens regular*. Examples are all C^*-algebras, and also the algebra $(\ell_1, \cdot)$ with pointwise multiplication. On the other hand, for an infinite locally compact group $\mathcal{G}$, the algebra $L_1(\mathcal{G})$ with convolution is not Arens regular. Hence, the question arises how one can measure the Arens (ir)regularity of a Banach algebra $\mathcal{A}$.

A natural procedure is to consider the so-called *topological centres*:

$$\begin{aligned} Z_t^1(\mathcal{A}^{**}) &= \{F \in \mathcal{A}^{**} \mid F \odot G = F.G \ \forall G \in \mathcal{A}^{**}\} \\ &= \{F \in \mathcal{A}^{**} \mid \mathcal{A}^{**} \ni G \mapsto F \odot G \ \ w^* - w^*\text{-continuous}\} \end{aligned}$$

and

$$\begin{aligned} Z_t^2(\mathcal{A}^{**}) &= \{F \in \mathcal{A}^{**} \mid G \odot F = G.F \ \forall G \in \mathcal{A}^{**}\} \\ &= \{F \in \mathcal{A}^{**} \mid \mathcal{A}^{**} \ni G \mapsto G.F \ \ w^* - w^*\text{-continuous}\}. \end{aligned}$$

Here we denote by "$\odot$" the first, by "." the second Arens product. In the following, we shall restrict ourselves to the first Arens product and the first topological centre—our methods and results obviously admit analogues for the second topological centre. (We use the notation $\odot$ for the first Arens product and the canonical module operations of $\mathcal{A}^{**}$ and $\mathcal{A}$ on $\mathcal{A}^*$. For a detailed account of Arens products and topological centres, we refer the reader to [Dal 00, Lau-Ülg 96, Pal 94].) We have $Z_t^1(\mathcal{A}^{**}) = \mathcal{A}^{**}$ if and only if $\mathcal{A}$ is Arens regular. The other extreme situation is $Z_t^1(\mathcal{A}^{**}) = \mathcal{A}$, in which case $\mathcal{A}$ is called *left strongly Arens irregular*; cf. [Dal-Lau 04].

It is well-known that for all locally compact groups, $L_1(\mathcal{G})$ is left (and right) strongly Arens irregular—a result which has a history of about 15 years and was obtained in full generality by Lau and Losert (cf. [Lau-Los 88]). In [Lau,Gha-Lau 95], Ghahramani and Lau conjectured that the measure algebra $\mathrm{M}(\mathcal{G})$ also shares this property; this is Problem 11 in [Lau 94, p. 89], and Problem 1 in [Gha-Lau 95, p. 184]. Our aim is to prove that for a very large class of locally compact groups, this conjecture holds.

At this point we would like to mention the preprint [Ess 04] by Esslamzadeh which presents an attempt to prove the Ghahramani–Lau conjecture for every locally compact group, by an approach completely different from ours. The proof, however, contains a gap, and, to our knowledge, it has until now been impossible to fix it.

We shall obtain Ghahramani–Lau's conjecture as a corollary of a general Banach algebraic principle which is a powerful criterion for strong Arens irregularity. At the same time, we shall prove that this principle also implies the automatic boundedness and w^*-continuity of all linear $\mathcal{A}^{**}$-module maps on $\mathcal{A}^*$; it establishes the abstract framework for the methods first developed in [Neu 04a]. The crucial idea is to combine the concept of the Mazur property of higher cardinal level for $\mathcal{A}$, as introduced in [Neu 04b], with a certain factorization property for bounded families in $\mathcal{A}^*$ of the same cardinality. Finally, we shall derive a "dual" variant of our criterion which in turn completely describes the structure of the topological centre for algebras of the form $(\mathcal{A}^* \odot \mathcal{A})^*$, where $\mathcal{A}$ has a right approximate identity bounded by 1. This result provides a partial answer to a question raised by Lau-Ülger [laul, Section 6, Question (f)]. We recall that in the above situation, $(\mathcal{A}^* \odot \mathcal{A})^*$ is a Banach algebra as a quotient of $\mathcal{A}^{**}$ (endowed with the first Arens product), and naturally its topological centre is defined to be

$$Z_t((\mathcal{A}^* \odot \mathcal{A})^*) = \{m \in (\mathcal{A}^* \odot \mathcal{A})^* \mid (\mathcal{A}^* \odot \mathcal{A})^* \ni n \mapsto mn \ w^* - w^*\text{-continuous}\}.$$

For a Banach algebra $\mathcal{A}$, we denote by $\mathcal{L}_{\mathcal{A}^{**}}(\mathcal{A}^*)$ the space of linear left $\mathcal{A}^{**}$-module maps on $\mathcal{A}^*$; the subspaces of bounded respectively w^*-w^*-continuous module maps are denoted by $\mathcal{B}_{\mathcal{A}^{**}}(\mathcal{A}^*)$ and $\mathcal{B}^{\sigma}_{\mathcal{A}^{**}}(\mathcal{A}^*)$, respectively.

For any locally compact group $\mathcal{G}$, we denote by $\kappa(\mathcal{G})$ the *compact covering number*, i.e., the least cardinality of a covering of $\mathcal{G}$ by compact subsets. We write $b(\mathcal{G})$ for the *local weight* of $\mathcal{G}$, i.e., the least cardinality of an open basis at the neutral element of $\mathcal{G}$. It is a classical result that these two cardinals are "dual" to each other in the following sense: if $\mathcal{G}$ is abelian with dual group $\widehat{\mathcal{G}}$, then the equality $\kappa(\mathcal{G}) = b(\widehat{\mathcal{G}})$ holds.

We recall that a cardinal number κ is called (*real-valued-*)*measurable* if for any set Γ with cardinality $|\Gamma| = \kappa$, there exists a diffused probability measure on the power set $\mathfrak{P}(\Gamma)$. One class of groups we shall consider in the sequel, are groups with non-measurable cardinality, which is a natural assumption. We list below a few properties of measurable cardinals several of which show their somewhat pathological nature:

- It cannot be proven in ZFC that measurable cardinals exists at all.
- It is consistent with ZFC to assume that measurable cardinals do not exist.
- The cardinals $\aleph_0$ (trivially) and $\aleph_1$ are non-measurable.
- Martin's Axiom implies that $\mathfrak{c}$—and hence every cardinal below $\mathfrak{c}$—is non-measurable. (So, in particular, assuming the Continuum Hypothesis implies that $\mathfrak{c}$ is non-measurable.)
- In ZFC, the statements "a measurable cardinal exists" and "Lebesgue measure can be extended to a measure defined on the power set of $\mathbf{R}$" are equiconsistent.

The above results can be found in [Jec 97, Part III, Chapter 5, §27], [Gar-Pfe 84, §4, p. 972–973]. For more information on measurable cardinals, we refer the reader to [Sol, 71, Fre 93]; especially the latter text shows "how enormously complicated real-valued-measurable cardinals have to be" (ibid., p. 159)—if one assumes their existence.

2. The topological centre of $\mathcal{A}^{**}$ and automatic continuity of module homomorphisms on $\mathcal{A}^*$

We introduce the following crucial concept which is a general property for Banach algebras.

Definition 2.1. Let $\mathcal{A}$ be a Banach algebra, and let κ be a cardinal number. We say that $\mathcal{A}^*$ has

(i) the left $\mathcal{A}^{**}$ factorization property of level κ [property (F_κ), for short] if for any family of functionals $(h_\alpha)_{\alpha\in I} \subseteq \mathrm{Ball}(\mathcal{A}^*)$ with $|I| = \kappa$, there exist a family $(\psi_\alpha)_{\alpha\in I} \subseteq \mathrm{Ball}(\mathcal{A}^{**})$ and one single functional $h \in \mathcal{A}^*$ such that the factorization formula

$$h_\alpha = \psi_\alpha \odot h$$

holds for all $\alpha \in I$;

(ii) the left uniform $\mathcal{A}^{**}$ factorization property of level κ [property (UF_κ), for short] if there is a family $(\psi_\alpha)_{\alpha\in I} \subseteq \mathrm{Ball}(\mathcal{A}^{**})$ with $|I| = \kappa$, such that for any family of functionals $(h_\alpha)_{\alpha\in I} \subseteq \mathrm{Ball}(\mathcal{A}^*)$, there is one single functional $h \in \mathcal{A}^*$ such that the factorization formula

$$h_\alpha = \psi_\alpha \odot h$$

holds for all $\alpha \in I$.

We recall from [Neu 04b] the following:

Definition 2.2. Let X be a Banach space and $\kappa \geqslant \aleph_0$ a cardinal number.

(i) A functional $f \in X^{**}$ is called w^*-κ-continuous if for all nets $(x_\alpha)_{\alpha\in I} \subseteq \mathrm{Ball}(X^*)$ of cardinality $\aleph_0 \leqslant |I| \leqslant \kappa$ with $x_\alpha \xrightarrow{w^*} 0$, we have: $\langle f, x_\alpha \rangle \longrightarrow 0$.

(ii) We say that X has the Mazur property of level κ [property (M_κ), for short] if every w^*-κ-continuous functional $f \in X^{**}$ actually is an element of X.

As is well-known a Banach space X is said to have the (classical) Mazur property if every w^* sequentially continuous functional $f \in X^{**}$ belongs to X.

We now come to our general criterion for both strong Arens irregularity and the automatic boundedness and w^*-continuity of module homomorphisms. The idea is to combine the factorization property with the Mazur property of the same cardinal level.

Theorem 2.3. *Let $\mathcal{A}$ be a Banach algebra satisfying (M_κ) and whose dual $\mathcal{A}^*$ has the property (F_κ), for some $\kappa \geqslant \aleph_0$. Then the following two statements hold*:

(i) *The Banach algebra $\mathcal{A}$ is left strongly Arens irregular*; *i.e.*,

$$Z_t^1(\mathcal{A}^{**}) = \mathcal{A}.$$

(ii) *Every linear left $\mathcal{A}^{**}$-module homomorphism on $\mathcal{A}^*$ is automatically bounded and w^*-w^*-continuous*; *i.e.*,

$$\mathcal{L}_{\mathcal{A}^{**}}(\mathcal{A}^*) = \mathcal{B}^\sigma_{\mathcal{A}^{**}}(\mathcal{A}^*).$$

Remark 2.4. Obviously, assuming the right version of the factorization property (F_κ), one can deduce the right strong Arens irregularity of $\mathcal{A}$ and the continuity of right $\mathcal{A}^{**}$-module maps on $\mathcal{A}^*$.

Proof of Theorem 2.3. (i) Let $m \in Z_t^1(\mathcal{A}^{**})$. Consider a net $(h_\alpha)_{\alpha\in I} \subseteq \mathrm{Ball}(\mathcal{A}^*)$, where $|I| \leqslant \kappa$, which converges w^* to 0. By property (M_κ), we only have to show that $\langle m, h_\alpha\rangle$ converges to 0. It suffices to prove that every convergent subnet of $(\langle m, h_\alpha\rangle)_\alpha$ converges to 0. Fix such a convergent subnet $(\langle m, h_{\alpha_\beta}\rangle)_\beta$. By property (F_κ), for all $\alpha \in I$, we have the factorization $h_\alpha = \psi_\alpha \odot h$, where $(\psi_\alpha)_{\alpha\in I} \subseteq \mathrm{Ball}(\mathcal{A}^{**})$ and $h \in \mathcal{A}^*$. Since the net $(\psi_{\alpha_\beta})_\beta \subseteq \mathrm{Ball}(\mathcal{A}^{**})$ is bounded, there exists a w^*-convergent subnet $(\psi_{\alpha_{\beta_\gamma}})_\gamma$; let $E \bullet w^* - \lim_\gamma \psi_{\alpha_{\beta_\gamma}} \in \mathrm{Ball}(\mathcal{A}^{**})$. We obtain $E \odot h = 0$ since for all $a \in \mathcal{A}$:

$$\langle E \odot h, a\rangle = \langle E, h \odot a\rangle = \lim_\gamma \langle \psi_{\alpha_{\beta_\gamma}}, h \odot a\rangle = \lim_\gamma \langle \psi_{\alpha_{\beta_\gamma}} \odot h, a\rangle = \lim_\gamma \langle h_{\alpha_{\beta_\gamma}}, a\rangle = 0.$$

Hence, we finally deduce, using that $m \in Z_t^1(\mathcal{A}^{**})$:

$$\begin{aligned}\lim_\beta \langle m, h_{\alpha_\beta}\rangle &= \lim_\gamma \langle m, h_{\alpha_{\beta_\gamma}}\rangle = \lim_\gamma \langle m, \psi_{\alpha_{\beta_\gamma}} \odot h\rangle = \lim_\gamma \langle m \odot \psi_{\alpha_{\beta_\gamma}}, h\rangle = \langle m \odot E, h\rangle \\ &= \langle m, E \odot h\rangle = 0,\end{aligned}$$

which yields the desired convergence.

(ii) Our procedure is similar to the proofs given for Theorems 3.1 and 3.2 in [Neu 04a]. We shall outline the argument for the convenience of the reader. Let $\Phi \in \mathcal{L}_{\mathcal{A}^{**}}(\mathcal{A}^*)$. We shall first prove the statement concerning the automatic boundedness. To this end, assume that Φ is unbounded. Thus, there is a sequence $(h_n)_{n\in\mathbb{N}} \subseteq \mathrm{Ball}(\mathcal{A}^*)$ such that

$$\|\Phi(h_n)\| \geqslant n$$

holds for all $n \in \mathbb{N}$. Using the factorization

$$h_n = \psi_n \odot h \quad (n \in \mathbb{N}),$$

where $\psi_n \in \mathrm{Ball}(\mathcal{A}^{**})$ and $h \in \mathcal{A}^*$, we obtain that for all $n \in \mathbb{N}$:

$$n \leqslant \|\Phi(h_n)\| = \|\Phi(\psi_n \odot h)\| = \|\psi_n \odot \Phi(h)\| \leqslant \|\Phi(h)\|,$$

a contradiction.

Let us now prove that a mapping $\Phi \in \mathcal{B}_{\mathcal{A}^{**}}(\mathcal{A}^*)$ is automatically w^*-w^*-continuous. Due to property (M_κ), we only need to show that for any net $(h_\alpha)_{\alpha\in I} \subseteq \mathrm{Ball}(\mathcal{A}^*)$, $\aleph_0 \leqslant |I| \leqslant \kappa$, such that $h_\alpha \to 0$ (w^*), we have $\Phi(h_\alpha) \to 0$ (w^*). Fix $a \in \mathcal{A}$. Obviously, it is enough to show that any convergent subnet $\langle \Phi(h_{\alpha_\beta}), a\rangle$ converges to 0. Property

(F_κ) entails the factorization $h_\alpha = \psi_\alpha \odot h$ with $(\psi_\alpha)_{\alpha\in I} \subseteq \mathrm{Ball}(\mathcal{A}^{**})$ and $h \in \mathcal{A}^*$. Let $\psi := w^* - \lim_\gamma \psi_{\alpha_{\beta_\gamma}} \in \mathrm{Ball}(\mathcal{A}^{**})$ be a w^*-cluster point. We have $\psi \odot h = 0$, since for all $b \in \mathcal{A}$:

$$\langle \psi \odot h, b\rangle = \lim_\gamma \langle \psi_{\alpha_{\beta_\gamma}} \odot h, b\rangle = \lim_\gamma \langle h_{\alpha_{\beta_\gamma}}, b\rangle = 0.$$

Hence we obtain

$$\begin{aligned}\lim_\beta \langle \Phi(h_{\alpha_\beta}), a\rangle &= \lim_\beta \langle \Phi(\psi_{\alpha_\beta} \odot h), a\rangle = \lim_\gamma \langle \psi_{\alpha_{\beta_\gamma}} \odot \Phi(h), a\rangle = \langle \psi \odot \Phi(h), a\rangle \\ &= \langle \Phi(\psi \odot h), a\rangle = 0,\end{aligned}$$

which finishes the proof. □

Remark 2.5. Consider the special case $\mathcal{A} = L_1(\mathcal{G})$. We have shown in [Neu 04b] that $L_1(\mathcal{G})$ has the property $(M_{\kappa(\mathcal{G})\cdot\aleph_0})$, and in [Neu 04a] that, for all locally compact non-compact groups, $L_\infty(\mathcal{G})$ has the property $(UF_{\kappa(\mathcal{G})\cdot\aleph_0})$. Hence, it follows from part (i) of the above theorem that $L_1(\mathcal{G})$ is left strongly Arens irregular for all non-compact groups $\mathcal{G}$ (in the compact case, a very quick proof of this fact has been given by Lau and Losert [Lau-Los 88]); this is the main result of [Lau-Los 88]. See [Neu 04].

Moreover, part (ii) of Theorem 2.3, applied to the algebra $\mathcal{A} = L_1(\mathcal{G})$, gives an affirmative answer to a conjecture formulated by Hofmeier and Wittstock [Hof-Wit 97] concerning the automatic boundedness of left $L_\infty(\mathcal{G})^*$-module maps on $L_\infty(\mathcal{G})$ and even derives their w^*-continuity (cf. [Gha-McC 92] for the latter). See [Neu 04a].

The above result establishes a common abstract Banach algebraic setting for the strong Arens irregularity of a Banach algebra $\mathcal{A}$ and the automatic w^*-continuity of $\mathcal{A}^{**}$-module maps on $\mathcal{A}^*$. Let us briefly note that the first property always implies the second, without any assumption on $\mathcal{A}$. Indeed, the following gives an alternative proof of part of the assertion (ii) in Theorem 2.3 above, by using (i).

Proposition 2.6. *Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{A}$ is left strongly Arens irregular, then every bounded left $\mathcal{A}^{**}$-module map on $\mathcal{A}^*$ is automatically w^*-continuous, and hence the adjoint of a left multiplier of $\mathcal{A}$ (i.e., a right $\mathcal{A}$-module map on $\mathcal{A}$). In other words: If $Z_t^1(\mathcal{A}^{**}) = \mathcal{A}$, then $\mathcal{B}_{\mathcal{A}^{**}}(\mathcal{A}^*) = \mathcal{B}^\sigma_{\mathcal{A}^{**}}(\mathcal{A}^*)$.*

Proof. The argument for showing the automatic w^*-continuity is implicitly contained in the proof of [Gha-McC 92, Theorem 1.8], where the case $\mathcal{A} = L_1(\mathcal{G})$ is considered; we shall give a variant of the short proof here for the convenience of the reader. Let $\Phi \in \mathcal{B}_{\mathcal{A}^{**}}(\mathcal{A}^*)$. We have to show that $\Phi^*(\mathcal{A}) \subseteq \mathcal{A}$, which by our assumption is equivalent to $\Phi^*(\mathcal{A}) \subseteq Z_t^1(\mathcal{A}^{**})$. Fix $a \in \mathcal{A}$. Then, indeed, $\Phi^*(a) \in Z_t^1(\mathcal{A}^{**})$ for if $(n_\alpha) \subseteq \mathcal{A}^{**}$ is a bounded net converging w^* to 0, then we have, for all $h \in \mathcal{A}^*$:

$$\langle \Phi^*(a) \odot n_\alpha, h\rangle = \langle a, \Phi(n_\alpha \odot h)\rangle = \langle a, n_\alpha \odot \Phi(h)\rangle = \langle n_\alpha, \Phi(h) \odot a\rangle \longrightarrow 0.$$

The fact that Φ is a left multiplier of $\mathcal{A}$ is seen by a quick calculation (cf. the proof of [Neu 04a, Corollary 3.4]). $\square$

3. Application to the conjecture by Ghahramani–Lau

In the following, we shall establish the left uniform $\mathrm{M}(\mathcal{G})^{**}$ factorization property of level $\kappa(\mathcal{G})$ for $\mathrm{M}(\mathcal{G})^*$, where $\mathcal{G}$ is any locally compact non-compact group. Our procedure is a more complicated version of the technique applied in [Neu 04a] to prove a corresponding factorization result for $L_\infty(\mathcal{G})$. There, we used translation of functions by group elements in order to "move" projections in $L_\infty(\mathcal{G})$. In the case of $\mathrm{M}(\mathcal{G})^*$ whose elements are of course not functions in general, our substitute for translation is the canonical module action of point masses of group elements (viewed as belonging to $\mathrm{M}(\mathcal{G})^{**}$) on $\mathrm{M}(\mathcal{G})^*$.

We begin by collecting a few facts that we will need later, concerning the product in the von Neumann algebra $\mathrm{M}(\mathcal{G})^* = \mathrm{C}_0(\mathcal{G})^{**}$ as well as the module operation just mentioned. We omit the proofs, the arguments being standard. We denote by ι the canonical embedding of $\mathrm{M}(\mathcal{G})$ in its second dual. If $x \in \mathcal{G}$, we write r_x for the right translation by x, i.e., $(r_x f)(y) = f(yx)$ for any function f on $\mathcal{G}$ and $y \in \mathcal{G}$.

Lemma 3.1. *Let $\mathcal{G}$ be any locally compact group. If $K \subseteq \mathcal{G}$ is a compact subset, we regard the characteristic function χ_K as an element of $\mathrm{M}(\mathcal{G})^*$, via integration: $\langle \chi_K, \mu\rangle = \mu(K)$ for all $\mu \in \mathrm{M}(\mathcal{G})$. Then the following hold:*

(i) *Let K and K' be compact subsets of $\mathcal{G}$. Then we have $\chi_K \chi_{K'} = \chi_{K\cap K'}$ in $\mathrm{M}(\mathcal{G})^*$.*
(ii) *If $K \subseteq \mathcal{G}$ is compact and $x \in \mathcal{G}$, we have: $\iota(\delta_x) \odot \chi_K = r_x \chi_K = \chi_{Kx^{-1}}$.*
(iii) *Let $h, f \in \mathrm{M}(\mathcal{G})^*$ and $x \in \mathcal{G}$. Then $\iota(\delta_x) \odot (hf) = (\iota(\delta_x) \odot h)(\iota(\delta_x) \odot f)$.*

We are now prepared to prove our factorization result.

Theorem 3.2. *Let $\mathcal{G}$ be a locally compact non-compact group with compact covering number $\kappa(\mathcal{G})$. Then $\mathrm{M}(\mathcal{G})^*$ has the property $(UF_{\kappa(\mathcal{G})})$; here, the factorizing functionals ψ_j even belong to $\overline{\iota(\delta_{\mathcal{G}})}^{w^*} \subseteq \mathrm{Ball}(\mathrm{M}(\mathcal{G})^{**})$.*

Remark 3.3. A straightforward modification of the proof presented below shows that $\mathrm{M}(\mathcal{G})^*$ also enjoys the right (instead of left) version of the factorization property $(UF_{\kappa(\mathcal{G})})$.

Proof of Theorem 3.2. We can cover $\mathcal{G}$ by $\kappa(\mathcal{G})$ many open sets with compact closure, and we may assume the covering being closed under finite unions. We shall denote by $(K_\alpha)_{\alpha\in I}$ this family of compacta. Set $\widetilde{I} := I \times I$. For $\widetilde{\alpha} = (\alpha, i) \in \widetilde{I}$, define $K_{\widetilde{\alpha}} = K_{(\alpha,i)} := K_\alpha$. Then $(K_{\widetilde{\alpha}})_{\widetilde{\alpha}\in\widetilde{I}}$ is a covering of $\mathcal{G}$ with the same properties as the original one. By Lemma 3 in [Lau-Los 88], there exists a family $(y_{\widetilde{\alpha}})_{\widetilde{\alpha}\in\widetilde{I}} \subseteq \mathcal{G}$ such that

$$K_{\widetilde{\alpha}} y_{\widetilde{\alpha}} \cap K_{\widetilde{\beta}} y_{\widetilde{\beta}} = \varnothing \quad \forall\ \widetilde{\alpha},\ \widetilde{\beta} \in \widetilde{I}, \widetilde{\alpha} \neq \widetilde{\beta}. \tag{1}$$

Define the following natural partial orderings on $\widetilde{I}$ and I by setting, for $(\alpha, i), (\beta, j) \in \widetilde{I}$:

$$(\alpha, i) \preceq (\beta, j) :\Longleftrightarrow K_{(\alpha,i)} \subseteq K_{(\beta,j)} \Longleftrightarrow K_\alpha \subseteq K_\beta \Longleftrightarrow: \alpha \preceq' \beta. \tag{2}$$

Let $\mathfrak{F}$ be an ultrafilter on I which dominates the order filter. Define, for $j \in I$,

$$\psi_j := w^* - \lim_{\beta \to \mathfrak{F}} \imath(\delta_{y_{(\beta,j)}}) \in \overline{\imath(\delta_{\mathcal{G}})}^{w^*} \subseteq \mathrm{Ball}\left(\mathrm{M}(\mathcal{G})^{**}\right).$$

Consider $(\alpha, i) \in \widetilde{I}$. By Lemma 3.1(ii), (iii) we have:

$$\begin{aligned}
\imath(\delta_{y^{-1}_{(\alpha,i)}}) \odot (\chi_{K_{(\alpha,i)}} h_i) &= (\imath(\delta_{y^{-1}_{(\alpha,i)}}) \odot \chi_{K_{(\alpha,i)}})(\imath(\delta_{y^{-1}_{(\alpha,i)}}) \odot h_i) \\
&= \chi_{K_{(\alpha,i)} y_{(\alpha,i)}} (\imath(\delta_{y^{-1}_{(\alpha,i)}}) \odot h_i).
\end{aligned}$$

Thanks to (1), we see by Lemma 3.1(i) that the projections $\chi_{K_{(\alpha,i)} y_{(\alpha,i)}}$ are pairwise orthogonal. Hence, noting that $\mathrm{M}(\mathcal{G})^*$ is a commutative von Neumann algebra and that the family $(h_i)_{i \in I}$ is bounded, we obtain

$$h := \sum_{\alpha \in I} \sum_{i \in I} \imath(\delta_{y^{-1}_{(\alpha,i)}}) \odot \left(\chi_{K_{(\alpha,i)}} h_i\right) \in \mathrm{M}(\mathcal{G})^* \quad (w^*\text{-limits}).$$

By (1), using Lemma 3.1 in a crucial fashion, we obtain for all $(\alpha, i), (\beta, j), (\gamma, k) \in \widetilde{I}$ with $(\gamma, k) \preceq (\beta, j)$:

$$\begin{aligned}
&\chi_{K_{(\gamma,k)}} \left[\imath(\delta_{y_{(\beta,j)}}) \odot \imath(\delta_{y^{-1}_{(\alpha,i)}}) \odot \left(\chi_{K_{(\alpha,i)}} h_i\right)\right] \\
&= \chi_{K_{(\gamma,k)}} \chi_{K_{(\beta,j)}} \left[\imath(\delta_{y_{(\beta,j)}}) \odot \imath(\delta_{y^{-1}_{(\alpha,i)}}) \odot \left(\chi_{K_{(\alpha,i)}} h_i\right)\right] \\
&= \chi_{K_{(\gamma,k)}} \left[\imath(\delta_{y_{(\beta,j)}}) \odot \left(\left(\imath(\delta_{y^{-1}_{(\beta,j)}}) \odot \chi_{K_{(\beta,j)}}\right) \imath(\delta_{y^{-1}_{(\alpha,i)}}) \odot \left(\chi_{K_{(\alpha,i)}} h_i\right)\right)\right] \\
&= \delta_{(\alpha,i),(\beta,j)} \chi_{K_{(\gamma,k)}} h_j.
\end{aligned}$$

By (2), we finally get for all $j \in I$ and $(\gamma, k) \in \widetilde{I}$:

$$\begin{aligned}
\chi_{K_{(\gamma,k)}} \left(\psi_j \odot h\right) &= w^* - \lim_{\beta \to \mathfrak{F}} \sum_{\alpha \in I} \sum_{i \in I} \underbrace{\chi_{K_{(\gamma,k)}} \imath(\delta_{y_{(\beta,j)}}) \odot \imath(\delta_{y^{-1}_{(\alpha,i)}}) \odot \left(\chi_{K_{(\alpha,i)}} h_i\right)}_{=\delta_{(\alpha,i),(\beta,j)} \chi_{K_{(\gamma,k)}} h_j} \\
&= \chi_{K_{(\gamma,k)}} h_j.
\end{aligned}$$

Taking w^*-limits yields the factorization formula that we have claimed. $\square$

In order to apply Theorem 2.3 to the algebra $\mathcal{A} = \mathrm{M}(\mathcal{G})$, we need to consider the Mazur property of a certain level, as established in [Hu-Neu 04, Corollary 5.6]. We shall recall this result here, with a very brief indication of the procedure followed in its proof.

Proposition 3.4. *Let $\mathcal{G}$ be a locally compact group.*

(i) *If the cardinality $|\mathcal{G}|$ is non-measurable, then $\mathrm{M}(\mathcal{G})$ has the classical Mazur property.*
(ii) *The space $\mathrm{M}(\mathcal{G})$ always has the Mazur property of level $|\mathcal{G}| \cdot \aleph_0$.*

Proof. (i) In [Neu 04b, Theorem 3.16], it is shown that the predual $\mathcal{M}_*$ of a von Neumann algebra $\mathcal{M}$ (in our case $\mathrm{M}(\mathcal{G})^*$) has the Mazur property if and only if the decomposability number $\mathrm{dec}(\mathcal{M})$ (i.e., the largest cardinality of a family of pairwise orthogonal non-zero projections in $\mathcal{M}$) is non-measurable. The assertion then follows from the equality $\mathrm{dec}(\mathrm{M}(\mathcal{G})^*) = |\mathcal{G}|$ which has been established in [Hu-Neu 04, Theorem 5.5(ii)].

(ii) As shown in [Hu-Neu 04, Theorem 2.2], the predual $\mathcal{M}_*$ of a von Neumann algebra $\mathcal{M}$ always has the Mazur property of level $\mathrm{dec}(\mathcal{M}) \cdot \aleph_0$, whence $\mathrm{M}(\mathcal{G})$ has the Mazur property of level $|\mathcal{G}| \cdot \aleph_0$, due to the equality given at the end of the above proof of part (i). $\square$

Our factorization (for non-compact $\mathcal{G}$) is at level $\kappa(\mathcal{G}) \leqslant |\mathcal{G}|$. But it is well-known that for every infinite locally compact group $\mathcal{G}$, we have precisely $|\mathcal{G}| = \kappa(\mathcal{G}) \cdot 2^{b(\mathcal{G})}$ (cf. [Com 84, Theorem 3.12(iii)]). Therefore, whenever $\mathcal{G}$ is an infinite locally compact group with $\kappa(\mathcal{G}) \geqslant 2^{b(\mathcal{G})}$, we have $\kappa(\mathcal{G}) = |\mathcal{G}|$, and hence, by part (ii) of the above Proposition, the space $\mathrm{M}(\mathcal{G})$ has the Mazur property of level $\kappa(\mathcal{G})$.

So, combining Theorem 3.2 with Proposition 3.4 entails that for all non-compact groups $\mathcal{G}$ with non-measurable cardinality, $\mathrm{M}(\mathcal{G})^*$ has $(UF_{\aleph_0})$ and $\mathrm{M}(\mathcal{G})$ has $(M_{\aleph_0})$. Furthermore, whenever $\mathcal{G}$ is an infinite locally compact group such that $\kappa(\mathcal{G}) \geqslant 2^{b(\mathcal{G})}$, then $\mathrm{M}(\mathcal{G})^*$ has $(UF_{\kappa(\mathcal{G})})$ and $\mathrm{M}(\mathcal{G})$ has $(M_{\kappa(\mathcal{G})})$. We wish to stress here that the latter class of groups in particular includes every group $\mathcal{G}$ which is the product of an *arbitrary* first countable locally compact group with any discrete group of cardinality at least $\mathfrak{c}$. Also, this class of groups of course does not involve any consideration of large—such as measurable—cardinals.

Thus, Theorem 2.3(i) in particular yields the answer to Ghahramani–Lau's conjecture for both the above classes of groups. We state the "left" version below; in view of Remarks 2.4 and 3.3, it is easy to see that the corresponding "right" versions of the following assertions (i) and (ii) hold as well.

Theorem 3.5. *Let $\mathcal{G}$ be either*

- *an infinite locally compact group such that $\kappa(\mathcal{G}) \geqslant 2^{b(\mathcal{G})}$—for example, $\mathcal{G} = \mathcal{H} \times \mathcal{D}$ where $\mathcal{H}$ is* any *first countable locally compact group, and $\mathcal{D}$ is any discrete group of cardinality $\mathfrak{c}$ or higher;*

- *or a locally compact non-compact group with non-measurable cardinality*.

Then in both cases, the following two statements hold.

(i) *The Banach algebra* $\mathrm{M}(\mathcal{G})$ *is left strongly Arens irregular*; *i.e.*,

$$Z_t^1(\mathrm{M}(\mathcal{G})^{**}) = \mathrm{M}(\mathcal{G}).$$

(ii) *Every linear left* $\mathrm{M}(\mathcal{G})^{**}$*-module homomorphism on* $\mathrm{M}(\mathcal{G})^*$ *is automatically bounded and* w^**-*w^**-continuous*; *i.e.*,

$$\mathcal{L}_{\mathrm{M}(\mathcal{G})^{**}}(\mathrm{M}(\mathcal{G})^*) = \mathcal{B}^\sigma_{\mathrm{M}(\mathcal{G})^{**}}(\mathrm{M}(\mathcal{G})^*).$$

Remark 3.6. We would like to point out an intriguing connection of a special case of the above Theorem to the main result of [Lau-Los 04], recently obtained by Lau and Losert (see [Lau-Los 04, Theorem 4.2] and also [Lau-Los 93, p. 22, Remark]). The authors are concerned with the topological centre problem for the Fourier algebra $A(\mathcal{G})$. Of course, since the latter is commutative, both topological centres of $A(\mathcal{G})^{**}$ coincide with the (algebraic) centre of $A(\mathcal{G})^{**}$. It was shown in [Lau-Los 93] that the centre equals $A(\mathcal{G})$ for a large class of amenable locally compact groups including the Heisenberg group, the "$ax+b$"-group and the motion group (cf. also [Hu-Neu 04, §8] for further results). However, in general, the determination of the centre is still an open problem even for compact or discrete groups $\mathcal{G}$.

In [Lau-Los 04, Theorem 4.2] it is shown that the centre of $A(\mathcal{G})^{**}$ is $A(\mathcal{G})$ for groups $\mathcal{G}$ of the form $\mathcal{G} = \mathcal{G}_0 \times \prod_{i=1}^\infty \mathcal{G}_i$, where each $\mathcal{G}_i$, $i \geqslant 0$, is a second countable locally compact group, $\mathcal{G}_0$ is amenable, and each $\mathcal{G}_i$, $i \geqslant 1$, is compact and non-trivial. Of course, the group algebra $L_1(\mathcal{G})$ and the Fourier algebra $A(\mathcal{G})$, on the one hand, as well as the measure algebra $\mathrm{M}(\mathcal{G})$ and the Fourier–Stieltjes algebra $B(\mathcal{G})$, on the other hand, form pairs of objects "dual" to each other; this is true for general locally compact groups in the framework of the duality theory for Kac algebras, and for abelian groups follows from classical Pontryagin duality. Hence, the role that discrete groups play in results on $L_1(\mathcal{G})$ or $\mathrm{M}(\mathcal{G})$ is the same as compact groups play for $A(\mathcal{G})$ or $B(\mathcal{G})$.

From this point of view, the special case of groups $\mathcal{G} = \mathcal{H} \times \mathcal{D}$ given as an example in our Theorem 3.5 above is very similar to the class of groups $\mathcal{G} = \mathcal{G}_0 \times \prod_{i=1}^\infty \mathcal{G}_i$ considered in [Lau-Los 04]: instead of a countably infinite product of compact, non-trivial groups, we have a discrete group of cardinality (at least) $\mathfrak{c}$. Moreover, this analogy is reflected in a somewhat similar pattern of the approaches to the two results: in [Lau-Los 04], the second group factor guarantees "enough" characters; in our case it provides "enough" group elements to translate compacta in $\mathcal{G}$ (so that they become pairwise disjoint), cf. Theorem 3.2.

Finally, it may be possible to combine our technique with the method used in [Lau-Los 04] towards a solution of the (topological) centre problem for the Fourier–Stieltjes algebra $B(\mathcal{G})$, for large classes of groups. This is of course a very hard question; it was raised by Lau, see [Lau 94, Problem 10, p. 89].

We shall finish this section by presenting an important consequence of Theorem 3.5, namely a partial solution to Problem 2 in [Gha-Lau 95, p. 184]. The latter asks whether an isometric (Banach algebra) isomorphism between $\mathrm{M}(\mathcal{G}_1)^{**}$ and $\mathrm{M}(\mathcal{G}_2)^{**}$, for two locally compact groups $\mathcal{G}_1$ and $\mathcal{G}_2$, forces the groups to be topologically isomorphic. Ghahramani and Lau note, after stating Problem 2, that for locally compact abelian groups $\mathcal{G}_1$ and $\mathcal{G}_2$, the answer is of course affirmative provided that for both groups the topological centre conjecture holds. Thus we have the following.

Corollary 3.7. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two abelian groups both satisfying either one of the conditions stated at the beginning of Theorem* 3.5. *If $\mathrm{M}(\mathcal{G}_1)^{**}$ and $\mathrm{M}(\mathcal{G}_2)^{**}$ are isometrically isomorphic, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are topologically isomorphic*.

4. The topological centre of $(\mathcal{A}^* \odot \mathcal{A})^*$

In the sequel, we shall denote by $\mathcal{B}_{\mathcal{A},r}(\mathcal{A}^*)$ the algebra of bounded linear right $\mathcal{A}$-module maps on $\mathcal{A}^*$. We begin with a general observation.

Proposition 4.1. *Let $\mathcal{A}$ be a Banach algebra with a right approximate identity bounded by* 1. *Then the canonical map*

$$\rho : (\mathcal{A}^* \odot \mathcal{A})^* \longrightarrow \mathcal{B}_{\mathcal{A},r}(\mathcal{A}^*),$$

where $\langle \rho(m)(h), a\rangle = \langle m, h \odot a\rangle$ for $m \in (\mathcal{A}^* \odot \mathcal{A})^*$, $h \in \mathcal{A}^*$ and $a \in \mathcal{A}$, is a linear isometric isomorphism. Its inverse τ is given by

$$\langle \tau(\Phi), f\rangle = \langle E, \Phi(f)\rangle \quad (\Phi \in \mathcal{B}_{\mathcal{A},r}(\mathcal{A}^*),\ f \in \mathcal{A}^* \odot \mathcal{A}),$$

where E is any w^*-cluster point in $\mathcal{A}^{**}$ of the bounded right approximate identity.

Proof. The first assertion follows from [Lau 87, Corollary 5.2] (inspection of the proof shows that the assumption about $\mathcal{A}$ being an F-algebra made there is actually not needed). It is easy to check that τ is well-defined, and that ρ and τ are inverse to each other. $\square$

The next result determines the topological centre of the algebra $(\mathcal{A}^* \odot \mathcal{A})^*$ for Banach algebras $\mathcal{A}$ with (M_κ) such that $\mathcal{A}^*$ has (F_κ). It thus provides a partial answer to a question posed by Lau-Ülger [laul, Section 6, Question (f)].

Theorem 4.2. *Let $\mathcal{A}$ be a Banach algebra with a right approximate identity bounded by* 1, *which has (M_κ) and whose dual has (F_κ), for some $\kappa \geqslant \aleph_0$. Then the topological centre of $(\mathcal{A}^* \odot \mathcal{A})^*$ can be identified* (*up to isometric algebra isomorphism*) *with the algebra* $\mathrm{RM}(\mathcal{A})$ *of right multipliers of* $\mathcal{A}$; *i.e.*,

$$Z_t((\mathcal{A}^* \odot \mathcal{A})^*) \cong \mathrm{RM}(\mathcal{A}).$$

Remark 4.3. The above theorem provides an abstract Banach algebraic version of the approach presented in [Neu 04] to the topological centre problem for $\mathrm{LUC}(\mathcal{G})^*$ (cf. [Lau 86]). Indeed, if $\mathcal{A} = L_1(\mathcal{G})$, then we have $\mathcal{A}^* \odot \mathcal{A} = \mathrm{LUC}(\mathcal{G})$, the space of bounded left uniformly continuous functions on $\mathcal{G}$, and $\mathrm{RM}(\mathcal{A}) = \mathrm{M}(\mathcal{G})$, by Wendel's theorem. Also, as already mentioned in Remark 2.5, $L_1(\mathcal{G})$ has the property $(M_{\kappa(\mathcal{G})\cdot\aleph_0})$, and, for any locally compact non-compact group, $L_\infty(\mathcal{G})$ has the property $(UF_{\kappa(\mathcal{G})\cdot\aleph_0})$. Thus, Theorem 4.2 yields, for the special case $\mathcal{A} = L_1(\mathcal{G})$, the main result of [Lau 86] in full generality (note that the assertion is simple for compact groups $\mathcal{G}$), and puts the latter into a general Banach algebraic framework.

Proof of Theorem 4.2. First we note that taking adjoints yields a canonical identification of $\mathrm{RM}(\mathcal{A})$ with the subalgebra of w^*-w^*-continuous elements in $\mathcal{B}_{\mathcal{A},r}(\mathcal{A}^*)$, which we shall denote by $\mathcal{B}^\sigma_{\mathcal{A},r}(\mathcal{A}^*)$. We claim that $Z_t((\mathcal{A}^* \odot \mathcal{A})^*) \cong \mathcal{B}^\sigma_{\mathcal{A},r}(\mathcal{A}^*)$, the identification being given by the map ρ presented in Proposition 4.1. Indeed, an argument analogous to the one given in the proof of Theorem 2.3(i) shows that if $m \in Z_t((\mathcal{A}^* \odot \mathcal{A})^*)$, then $\rho(m)$ is a w^*-w^*-continuous map on $\mathcal{A}^*$. On the other hand, fix $\Phi \in \mathcal{B}^\sigma_{\mathcal{A},r}(\mathcal{A}^*)$. In order to show that $\tau(\Phi) \in Z_t((\mathcal{A}^* \odot \mathcal{A})^*)$, consider a bounded net (n_α) in $(\mathcal{A}^* \odot \mathcal{A})^*$ converging w^* to 0, and $f = h \odot a \in \mathcal{A}^* \odot \mathcal{A}$. Recall the canonical action of $n_\alpha \in (\mathcal{A}^* \odot \mathcal{A}^*)$ on $h \in \mathcal{A}^*$ via $\langle n_\alpha h, b\rangle = \langle n_\alpha, h \odot b\rangle$, for all $b \in \mathcal{A}$. We obtain:

$$\begin{aligned}\langle \tau(\Phi) n_\alpha, f\rangle &= \langle E, \Phi((n_\alpha h) \odot a)\rangle = \langle E, \Phi(n_\alpha h) \odot a\rangle = \langle \Phi(n_\alpha h), a\rangle \\ &= \langle n_\alpha, h \odot \Phi_*(a)\rangle \xrightarrow{\alpha} 0,\end{aligned}$$

which finishes the proof. $\square$

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