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A unified approach to the topological centre problem for certain Banach algebras arising in abstract harmonic analysis

By

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Abstract. Let $\mathcal{G}$ be a locally compact group. Consider the Banach algebra $L_1(\mathcal{G})^{**}$, equipped with the first Arens multiplication, as well as the algebra $\mathrm{LUC}(\mathcal{G})^*$, the dual of the space of bounded left uniformly continuous functions on $\mathcal{G}$, whose product extends the convolution in the measure algebra $\mathrm{M}(\mathcal{G})$. We present (for the most interesting case of a non-compact group) completely different - in particular, *direct* - proofs and even obtain *sharpened* versions of the results, first proved by Lau-Losert in [9] and Lau in [8], that the topological centres of the latter algebras precisely are $L_1(\mathcal{G})$ and $\mathrm{M}(\mathcal{G})$, respectively. The special interest of our new approach lies in the fact that it shows a fairly general pattern of solving the topological centre problem for various kinds of Banach algebras; in particular, it avoids the use of any measure theoretical techniques. At the same time, deriving both results in perfect parallelity, our method reveals the nature of their close relation.

1. Introduction. In this note, we wish to present a new and, for the first time, *unified* approach to two theorems which may be considered as the fundamental results, known by now, concerning the topological centre problem for concrete Banach algebras studied in abstract harmonic analysis. Namely, for a period of nearly 15 years (beginning in the seventies with [17]), research in the topological centre question was centred around the Banach algebras $L_1(\mathcal{G})^{**}$, endowed with the first Arens product, and its quotient algebra $\mathrm{LUC}(\mathcal{G})^*$, where $\mathcal{G}$ denotes a locally compact group. Here, we write $\mathrm{LUC}(\mathcal{G})$ for the space of complex-valued bounded left uniformly continuous functions on $\mathcal{G}$; the definition of the product on the dual $\mathrm{LUC}(\mathcal{G})^*$ will be briefly recalled below. The questions were eventually answered in full generality by the decisive work of Lau [8] and Lau-Losert [9]. In the present note, we will derive both these results (see Theorems 1.1 and 1.2 below) by organizing the arguments in a parallel fashion, which at the same time not only will yield direct proofs, but even sharpenings of the statements.

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Theorem 1.1. *The topological centre $Z_t(L_1(\mathcal{G})^{**})$ of the Banach algebra $L_1(\mathcal{G})^{**}$ is precisely $L_1(\mathcal{G})$.*

Proof. This is the main result, Thm. 1, in [9]. □

We shall restrict ourselves to the most interesting situation where $\mathcal{G}$ is non-compact. In case $\mathcal{G}$ is compact, the above assertion may be obtained by a very short argument ([9, Thm. 1]; also cf. [6, Thm. 3.3 (vi)]).

The second theorem even is only of interest in the non-compact case – for in case $\mathcal{G}$ is compact, $\mathrm{LUC}(\mathcal{G})^*$ equals the measure algebra $\mathrm{M}(\mathcal{G})$, and the assertion is immediately verified.

Theorem 1.2. *The topological centre $Z_t(\mathrm{LUC}(\mathcal{G})^*)$ of the Banach algebra $\mathrm{LUC}(\mathcal{G})^*$ is* $\mathrm{M}(\mathcal{G})$.

Proof. This is the main result, Thm. 1, in [8]. □

In the sequel, we will have to distinguish between the two Arens products on the bidual $L_1(\mathcal{G})^{**}$. We denote by $\odot$ the first (= left) and by . the second (= right) Arens product, and use these symbols as well for the various module operations linking $L_1(\mathcal{G})$, its dual and bidual, as follows. Let $m, n \in L_1(\mathcal{G})^{**}$, $h \in L_1(\mathcal{G})^*$, $f, g \in L_1(\mathcal{G})$. Denoting by $*$ the convolution product of functions (whenever it is defined), we write:

$$\langle h \odot f, g\rangle := \langle h, f * g\rangle$$
$$\langle n \odot h, f\rangle := \langle n, h \odot f\rangle$$
$$\langle m \odot n, h\rangle := \langle m, n \odot h\rangle$$

and, following a completely symmetric pattern:

$$\langle f.h, g\rangle := \langle h, g * f\rangle$$
$$\langle h.m, f\rangle := \langle m, f.h\rangle$$
$$\langle m.n, h\rangle := \langle n, h.m\rangle.$$

We note that, in particular, $h \odot f = \widetilde{f} * h \in \mathrm{LUC}(\mathcal{G})$; here, as usual, we write $\widetilde{f} := \frac{1}{\Delta}\check{f}$, where Δ denotes the modular function of $\mathcal{G}$ and $\check{f}(x) := f(x^{-1})$ for all $x \in \mathcal{G}$. We refer to [15, §1.4], for a discussion of basic properties of Arens multiplication in the framework of general Banach algebras.

Whenever we consider $L_1(\mathcal{G})^{**}$ as a Banach algebra, we regard it as equipped with the first Arens product. We recall that the topological centre of $L_1(\mathcal{G})^{**}$ is defined to be the set of functionals $m \in L_1(\mathcal{G})^{**}$ which satisfy $m \odot n = m.n$ for all $n \in L_1(\mathcal{G})^{**}$. Equivalently, the topological centre consists of all the functionals $m \in L_1(\mathcal{G})^{**}$ such that left multiplication by m is w^*-w^*-continuous on $L_1(\mathcal{G})^{**}$. A detailed analysis of topological centres in the general context of biduals of Banach algebras can be found, e.g., in [11].

There is an analogous notion of topological centre for the Banach algebra $\mathrm{LUC}(\mathcal{G})^*$. First, let us recall the natural construction of the product in the latter space. If $n \in \mathrm{LUC}(\mathcal{G})^*$ and $f \in \mathrm{LUC}(\mathcal{G})$, then it is classical that the function $n \cdot f$, defined through

$$(n \cdot f)(x) := \langle n, l_x f \rangle \quad (x \in \mathcal{G})$$

still belongs to $\mathrm{LUC}(\mathcal{G})$; i.e., $\mathrm{LUC}(\mathcal{G})$ is left introverted. This operation gives rise to the product on the space $\mathrm{LUC}(\mathcal{G})^*$:

$$\langle m \cdot n, f \rangle := \langle m, n \cdot f \rangle \quad (m, n \in \mathrm{LUC}(\mathcal{G})^*,\ f \in \mathrm{LUC}(G))$$

under which the latter indeed becomes a Banach algebra, and $\mathrm{LUC}(\mathcal{G})$ becomes a left $\mathrm{LUC}(\mathcal{G})^*$-module with the action introduced above. In analogy to the case of $L_1(\mathcal{G})^{**}$, one defines the topological centre of $\mathrm{LUC}(\mathcal{G})^*$ to be the set of elements $m \in \mathrm{LUC}(\mathcal{G})^*$ such that left multiplication by m is w^*-w^*-continuous on $\mathrm{LUC}(\mathcal{G})^*$.

We shall also consider the natural (left) module operation of $\mathrm{LUC}(\mathcal{G})^*$ on $L_\infty(\mathcal{G})$ given by $\langle m \lozenge h, g \rangle := \langle m, h \odot g \rangle$, where $m \in \mathrm{LUC}(\mathcal{G})^*$, $h \in L_\infty(\mathcal{G})$, $g \in L_1(\mathcal{G})$. At this point, we note that, as is easily seen, one has $m \lozenge h = \widetilde{m} \odot h$, where $\widetilde{m}$ is an arbitrary Hahn-Banach extension of the functional m to $L_\infty(\mathcal{G})^*$.

The main interest of the approach presented here consists in the following:

- We obtain a sharpening of the non-trivial inclusion in the statements of both Theorem 1.1 and 1.2. Namely, as for Theorem 1.1, we shall see that for an element $m \in L_\infty(\mathcal{G})^*$ in order to belong to $L_1(\mathcal{G})$, it is sufficient to have $m \odot n = m.n$ only for all $n \in L_\infty(\mathcal{G})^*$ that are Hahn-Banach extensions of functionals in $\overline{\delta_\mathcal{G}}^{w^*} \subsetneqq \mathrm{Ball}(\mathrm{LUC}(\mathcal{G})^*)$ – instead of requiring the latter equality for all $n \in L_\infty(\mathcal{G})^*$, as does the definition of the topological centre. Here, we denote by $\delta_\mathcal{G}$ the set of all point evaluations $\delta_x (x \in \mathcal{G})$. Analogously, in the situation of Theorem 1.2, we will prove that an element $m \in \mathrm{LUC}(\mathcal{G})^*$ already belongs to $\mathrm{M}(\mathcal{G})$ if left multiplication by m is only w^*-w^*-continuous on $\overline{\delta_\mathcal{G}}^{w^*} \subsetneqq \mathrm{Ball}(\mathrm{LUC}(\mathcal{G})^*)$, instead of demanding this continuity on the whole unit ball of $\mathrm{LUC}(\mathcal{G})^*$.
- The only proof known so far of the topological centre theorem for $\mathrm{LUC}(\mathcal{G})^*$ is indirect (see Thm. 1 in [8]), and all proofs given for the corresponding theorem for $L_1(\mathcal{G})^{**}$ either heavily rely on the latter ([11], [2]) or are also indirect (see [9, Thm. 1]). Our proofs of the two results are independent, and in both cases *direct*. We remark *en passant* that the proofs of Thm. 5.4 and Cor. 5.5 in [11] are correct only under the additional set-theoretic assumption that the compact covering number $\mathfrak{k}(\mathcal{G})$ of the group $\mathcal{G}$ is a non-measurable cardinal, since the argument is based on *ibid.*, Lemma 5.3, which in turn has to be read with a similar set-theoretic assumption; we refer to [13] and [5] for a detailed discussion of these and related problems. The tool which enables us to overcome precisely those set-theoretic difficulties is provided by Lemma 2.1 below.
- The procedure we present shows a perfect analogy between the two topological centre theorems. Besides one additional structural result for $\mathrm{LUC}(\mathcal{G})^*$, the prerequisites are the same, and the proofs themselves follow completely parallel lines.
- The method of proof follows a purely Banach algebraic procedure and does not, in particular, rely on any measure theoretic argument, so that it might be applied equally well

in other situations. At this point, we only mention that in [5], we obtained an analogue of Lemma 2.1 below in the "dual" setting, i.e., with $L_\infty(\mathcal{G})$ replaced by the group von Neumann algebra $VN(\mathcal{G})$, and with $\mathfrak{k}(\mathcal{G})$ replaced by $b(\mathcal{G})$, the smallest cardinality of an open basis at the neutral element of $\mathcal{G}$. For Proposition 2.3 as well, a dual version is known at least in the case of amenable groups $\mathcal{G}$ – here, the algebras $\mathrm{UCB}(\widehat{\mathcal{G}})^*$ and $B_\rho(\mathcal{G})$ take over the place of $\mathrm{LUC}(\mathcal{G})^*$ and $\mathrm{M}(\mathcal{G})$, respectively. Thus, if we are able to prove a factorization result, for bounded families in $VN(\mathcal{G})$ of cardinality at most $b(\mathcal{G}) \cdot \aleph_0$, corresponding to Lemma 2.2, then our method of proof would immediately yield an affirmative answer (for the most interesting case of non-discrete groups $\mathcal{G}$) of two long-standing conjectures at once – namely, the topological centre of the bidual of the Fourier algebra $A(\mathcal{G})$ being just $A(\mathcal{G})$, i.e., $Z_t(A(\mathcal{G})^{**}) = A(\mathcal{G})$, and, for amenable groups $\mathcal{G}$, $Z_t(\mathrm{UCB}(\widehat{\mathcal{G}})^*) = B_\rho(\mathcal{G})$.

2. Preliminaries. For both proofs we will use the following two lemmata, which are of interest in their own right.

Lemma 2.1. *For an arbitrary locally compact group $\mathcal{G}$, the space $L_1(\mathcal{G})$ enjoys Mazur's property of level $\mathfrak{k}(\mathcal{G}) \cdot \aleph_0$, where $\mathfrak{k}(\mathcal{G})$ denotes the compact covering number of $\mathcal{G}$ (i.e., the least cardinality of a compact covering of $\mathcal{G}$). This means that a functional $m \in L_1(\mathcal{G})^{**}$ actually belongs to $L_1(\mathcal{G})$ if it carries bounded w^*-converging nets of cardinality at most $\mathfrak{k}(\mathcal{G}) \cdot \aleph_0$ into converging nets.*

Proof. This is Thm. 4.4 in [13]. □

Next we present our crucial tool, which is a general factorization theorem for bounded families in $L_\infty(\mathcal{G})$. It has already been used (cf. [14])

• to answer (in the affirmative) a question raised by Hofmeier-Wittstock in [4] concerning the automatic boundedness of left $L_\infty(\mathcal{G})^*$-module homomorphisms on $L_\infty(\mathcal{G})$;

• to give an alternative approach to the result on automatic w^*-w^*-continuity of the latter mappings, as first shown by Ghahramani-McClure in [2].

Lemma 2.2. *Let $\mathcal{G}$ be a locally compact non-compact group with compact covering number $\mathfrak{k}(\mathcal{G})$. Let further $(h_\alpha)_{\alpha\in I} \subseteqq L_\infty(\mathcal{G})$ be a bounded family of functions where $|I| \leqq \mathfrak{k}(\mathcal{G})$. Then there exist a family $(\psi_\alpha)_{\alpha\in I}$ of functionals in $\overline{\delta_{\mathcal{G}}}^{w^*} \subseteqq \mathrm{Ball}(\mathrm{LUC}(\mathcal{G})^*)$ and a function $h \in L_\infty(\mathcal{G})$ such that the factorization formula*

$$h_\alpha = \psi_\alpha \Diamond h$$

holds for all $\alpha \in I$. (Moreover, the functionals ψ_α, $\alpha \in I$, do not depend, except for the index set, on the given family $(h_\alpha)_{\alpha\in I}$; they are universal in the sense that they are obtained intrinsically from the group $\mathcal{G}$.)

Proof. See [12, Satz 3.6.2] (or [14]). □

In order to arrange the proof of Thm. 1.2 in a completely parallel manner to the one of Thm. 1.1, we only require the following proposition, whose first part is a classical structural result about the algebra $\mathrm{LUC}(\mathcal{G})^*$ going back to the pioneering work of Curtis-Figà-Talamanca ([1, Thm. 3.3]). We write $\mathcal{B}^\sigma(L_\infty(\mathcal{G}))$ for the space of normal (i.e., w^*-w^*-continuous) operators on $L_\infty(\mathcal{G})$.

Proposition 2.3. *The mapping*

$$\phi : \mathrm{LUC}(\mathcal{G})^* \longrightarrow \mathcal{B}(L_\infty(\mathcal{G}))$$

defined through

$$\phi(m)(h) := m \lozenge h \quad (m \in \mathrm{LUC}(\mathcal{G})^*,\ \ h \in L_\infty(\mathcal{G}))$$

is an isometric representation of $\mathrm{LUC}(\mathcal{G})^*$ *in* $\mathcal{B}(L_\infty(\mathcal{G}))$ *such that*

$$\phi(\mathrm{M}(\mathcal{G})) = \phi(\mathrm{LUC}(\mathcal{G})^*) \cap \mathcal{B}^\sigma(L_\infty(\mathcal{G})). \tag{1}$$

Proof. The first assertions are well-known (see, e.g., [7, Thm. 1], together with Lemma 1 and Remark 3; or [1, Thm. 3.3], for a proof in the unimodular case). The relation (1) is seen as follows (cf. [12, Prop. 3.1.1]). In order to prove the inclusion "$\subseteqq$", let $\mu \in \mathrm{M}(\mathcal{G})$. We have to show that $\phi(\mu) \in \mathcal{B}(L_\infty(\mathcal{G}))$ is normal. Consider a net $(h_\alpha)_\alpha \subseteqq \mathrm{Ball}(L_\infty(\mathcal{G}))$ such that $h_\alpha \stackrel{w^*}{\longrightarrow} 0$. Fix $g \in L_1(\mathcal{G})$. We claim that

$$\langle \phi(\mu)(h_\alpha), g\rangle \longrightarrow 0.$$

But we have:

$$\langle \phi(\mu)(h_\alpha), g\rangle = \langle \mu \lozenge h_\alpha, g\rangle = \langle \mu, \widetilde{g} * h_\alpha\rangle = \int_{\mathcal{G}} (\widetilde{g} * h_\alpha)(t)\, d\mu(t),$$

where (as is readily checked) the net $(\widetilde{g} * h_\alpha)_\alpha \subseteqq \mathrm{LUC}(\mathcal{G})$ is equicontinuous, bounded and converges pointwise to 0. It thus converges uniformly on compact subsets of $\mathcal{G}$, whence we conclude that the above integrals converge to 0, as desired.

Turning to the inclusion "$\supseteqq$", let us consider an arbitrary element $\phi(m)$ of the set on the right side, where $m \in \mathrm{LUC}(\mathcal{G})^*$. We denote by $\mathrm{C}_0(\mathcal{G})$ the space of all complex-valued continuous functions on $\mathcal{G}$ vanishing at infinity, and by $\mathrm{C}_0(\mathcal{G})^\perp$ its annihilator in $\mathrm{LUC}(\mathcal{G})^*$. Then we obviously have $\mathrm{LUC}(\mathcal{G})^* = \mathrm{M}(\mathcal{G}) \oplus \mathrm{C}_0(\mathcal{G})^\perp$ (the latter is actually an ℓ_1-direct sum – cf. Lemma 1.1 in [3] –, but we will not need this fact). Write $m = \mu + n$ with $\mu \in \mathrm{M}(\mathcal{G})$ and $n \in \mathrm{C}_0(\mathcal{G})^\perp$ according to this decomposition. It suffices to show that $\phi(n) = 0$.

First, using the inclusion proved above, we see that $\phi(n) = \phi(m) - \phi(\mu)$ is normal. Fix $h \in L_\infty(\mathcal{G})$. We have $h = \sigma(L_\infty(\mathcal{G}), L_1(\mathcal{G})) - \lim_\alpha h_\alpha$ for an appropriate net $(h_\alpha)_\alpha \subseteqq \mathrm{C}_0(\mathcal{G})$. Hence we obtain:

$$\phi(n)(h) = \sigma(L_\infty(\mathcal{G}), L_1(\mathcal{G})) - \lim_\alpha \phi(n)(h_\alpha) = 0,$$

which finishes the proof. For the last equality, note that for all α and arbitrary $g \in L_1(\mathcal{G})$:

$$\langle \phi(n)(h_\alpha), g\rangle = \langle n, \widetilde{g} * h_\alpha\rangle = 0,$$

since $\widetilde{g} * h_\alpha \in L_1(\mathcal{G}) * \mathrm{C}_0(\mathcal{G}) = \mathrm{C}_0(\mathcal{G})$. $\square$

R e m a r k 2.4. We only stated the specific properties of ϕ for the sake of completeness. In the sequel, we shall just use the fact that ϕ is bounded and injective, as well as the relation (1).

3. Application to the topological centre problem. We first present the proof of Theorem 1.1, for non-compact groups $\mathcal{G}$. To establish the non-trivial inclusion, let $m \in Z_t(L_1(\mathcal{G})^{**})$. The group $\mathcal{G}$ being non-compact, we infer from Lemma 2.1 that $L_1(\mathcal{G})$ has Mazur's property of level $\mathfrak{k}(\mathcal{G})$. So to prove $m \in L_1(\mathcal{G})$, let $(h_\alpha)_{\alpha\in I} \subsetneqq L_\infty(\mathcal{G})$ be a bounded net converging w^* to 0, where $|I| \leqq \mathfrak{k}(\mathcal{G})$. Thanks to Lemma 2.2, we have the factorization

$$h_\alpha = \psi_\alpha \lozenge h = \widetilde{\psi_\alpha} \odot h \quad (\alpha \in I)$$

with $\psi_\alpha \in \overline{\delta_{\mathcal{G}}}^{w^*} \subsetneqq \mathrm{Ball}(\mathrm{LUC}(\mathcal{G})^*)$ and $h \in L_\infty(\mathcal{G})$. Here, $\widetilde{\psi_\alpha}$ denotes some arbitrarily chosen Hahn-Banach extension of ψ_α to $L_\infty(\mathcal{G})^*$. We have to show that $a_\alpha := \langle m, h_\alpha\rangle \xrightarrow{\alpha} 0$. Due to the boundedness of $(h_\alpha)_\alpha$, it suffices to prove that every convergent subnet of $(a_\alpha)_\alpha$ tends to 0. Let $(\langle m, h_{\alpha_\beta}\rangle)_\beta$ be such a convergent subnet. Furthermore, let

$$E := (w^* - \lim_\gamma \psi_{\alpha_{\beta_\gamma}})\tilde{\ } \in \mathrm{Ball}(L_\infty(\mathcal{G})^*)$$

be an arbitrarily chosen Hahn-Banach extension, denoted by $(\ldots)\tilde{\ }$, of some w^*-cluster point of the net $(\psi_{\alpha_\beta})_\beta$ in $\mathrm{Ball}(\mathrm{LUC}(\mathcal{G})^*)$.

We first note that $E \odot h = 0$, since for arbitrary $g \in L_1(\mathcal{G})$ we obtain:

$$\begin{aligned}\langle E \odot h, g\rangle &= \langle E, h \odot g\rangle = \lim_\gamma \langle \psi_{\alpha_{\beta_\gamma}}, h \odot g\rangle \\ &= \lim_\gamma \langle \psi_{\alpha_{\beta_\gamma}} \lozenge h, g\rangle = \lim_\gamma \langle h_{\alpha_{\beta_\gamma}}, g\rangle = 0.\end{aligned}$$

Now we conclude that (using twice the assumption $m \in Z_t(L_1(\mathcal{G})^{**})$ and in particular the fact that $h.m \in L_1(\mathcal{G})^* \odot L_1(\mathcal{G}) = \mathrm{LUC}(\mathcal{G})$, cf. Lemma 3.1 a) in [11])

$$\begin{aligned}\lim_\beta \langle m, h_{\alpha_\beta}\rangle &= \lim_\beta \langle m, \widetilde{\psi_{\alpha_\beta}} \odot h\rangle = \lim_\beta \langle m \odot \widetilde{\psi_{\alpha_\beta}}, h\rangle \\ &= \lim_\beta \langle m.\widetilde{\psi_{\alpha_\beta}}, h\rangle = \lim_\beta \langle \widetilde{\psi_{\alpha_\beta}}, h.m\rangle \\ &= \lim_\beta \langle \psi_{\alpha_\beta}, h.m\rangle = \lim_\gamma \langle \psi_{\alpha_{\beta_\gamma}}, h.m\rangle \\ &= \langle w^* - \lim_\gamma \psi_{\alpha_{\beta_\gamma}}, h.m\rangle = \langle E, h.m\rangle \\ &= \langle m.E, h\rangle = \langle m \odot E, h\rangle = \langle m, E \odot h\rangle = 0,\end{aligned}$$

which gives the desired convergence.

We now turn to the proof of Theorem 1.2. The inclusion $\mathrm{M}(\mathcal{G}) \subsetneqq Z_t(\mathrm{LUC}(\mathcal{G})^*)$ being immediate and classical (cf. [16, Lemma 3.1]), we restrict our attention to the reverse inclusion, where in order to avoid trivialities, we assume the group $\mathcal{G}$ to be non-compact. Fix $m \in Z_t(\mathrm{LUC}(\mathcal{G})^*)$. Then, according to Proposition 2.3, we only have to prove that $\phi(m) \in \mathcal{B}(L_\infty(\mathcal{G}))$ is w^*-w^*-continuous. To this end, consider a bounded net $(h_\alpha)_{\alpha\in I} \subsetneqq L_\infty(\mathcal{G})$, where $|I| \leqq \mathfrak{k}(\mathcal{G})$, which converges w^* to 0. Lemma 2.2 yields the factorization

$$h_\alpha = \psi_\alpha \lozenge h \quad (\alpha \in I)$$

with $\psi_\alpha \in \overline{\delta_{\mathcal{G}}}^{w^*} \subseteqq \mathrm{Ball}(\mathrm{LUC}(\mathcal{G})^*)$ and $h \in L_\infty(\mathcal{G})$. By Lemma 2.1, we only have to show that $\phi(m)(h_\alpha)$ tends w^* to 0. The latter net being bounded, it suffices to prove that every convergent subnet tends to 0. Let $(\phi(m)(h_{\alpha_\beta}))_\beta$ be such a convergent subnet. Furthermore, let

$$E := w^* - \lim_\gamma \psi_{\alpha_{\beta_\gamma}} \in \overline{\delta_{\mathcal{G}}}^{w^*} \subseteqq \mathrm{Ball}(\mathrm{LUC}(\mathcal{G})^*)$$

be a w^*-cluster point of the net $(\psi_{\alpha_\beta})_\beta$ in $\mathrm{Ball}(\mathrm{LUC}(\mathcal{G})^*)$.

We remark that $E \Diamond h = 0$, since for arbitrary $g \in L_1(\mathcal{G})$ we obtain:

$$\begin{aligned}\langle E \Diamond h, g\rangle &= \langle E, h \odot g\rangle = \lim_\gamma \langle \psi_{\alpha_{\beta_\gamma}}, h \odot g\rangle \\ &= \lim_\gamma \langle \psi_{\alpha_{\beta_\gamma}} \Diamond h, g\rangle = \lim_\gamma \langle h_{\alpha_{\beta_\gamma}}, g\rangle = 0.\end{aligned}$$

In order to conclude, we will only require the following fact concerning the compatibility of our various module operations, which is easy to verify (cf. §2 in [10]): For $\psi \in \mathrm{LUC}(\mathcal{G})^*$, $h \in L_\infty(\mathcal{G})$ and $g \in L_1(\mathcal{G})$, we have

$$(\psi \Diamond h) \odot g = \psi \cdot (h \odot g).$$

Now we obtain:

$$\begin{aligned}\lim_\beta \langle \phi(m)(h_{\alpha_\beta}), g\rangle &= \lim_\beta \langle m \Diamond h_{\alpha_\beta}, g\rangle = \lim_\beta \langle m, h_{\alpha_\beta} \odot g\rangle \\ &= \lim_\beta \langle m, (\psi_{\alpha_\beta} \Diamond h) \odot g\rangle = \lim_\beta \langle m, \psi_{\alpha_\beta} \cdot (h \odot g)\rangle \\ &= \lim_\beta \langle m \cdot \psi_{\alpha_\beta}, h \odot g\rangle = \lim_\gamma \langle m \cdot \psi_{\alpha_{\beta_\gamma}}, h \odot g\rangle \\ &= \langle m \cdot E, h \odot g\rangle \quad \text{since } m \in Z_t(\mathrm{LUC}(\mathcal{G})^*) \\ &= \langle m, E \cdot (h \odot g)\rangle = \langle m, (E \Diamond h) \odot g\rangle = 0,\end{aligned}$$

which finishes the proof.

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