J. OPERATOR THEORY
60:2(2008), 301–316

ON MAZUR'S PROPERTY AND PROPERTY (X)

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Communicated by Kenneth R. Davidson

ABSTRACT. We give a complete characterization of those von Neumann algebras whose preduals have Mazur's Property. We further show that for preduals of von Neumann algebras, Mazur's Property is actually equivalent to Property (X) which was first studied by Godefroy and Talagrand in [11]. Moreover, we introduce and study natural generalizations of the latter properties to the level of arbitrary cardinal numbers κ, as suggested in [9] for Property (X). In particular, using Edgar's partial ordering of Banach spaces [4], we prove that Property (X) of level κ only differs from the original one in the case where κ is a measurable cardinal number. Several applications of our results to some concrete spaces such as $L_1(\mathcal{G})$ for a locally compact group $\mathcal{G}$ and the space of trace class operators $\mathcal{T}(\mathcal{H})$ on a Hilbert space are also discussed.

KEYWORDS: *Mazur's Property, Property (X), predual of von Neumann algebra, measurable cardinal.*

MSC (2000): 03E55, 22D05, 46B10, 46B26, 46L10.

1. INTRODUCTION

Our principal aim is to characterize precisely those preduals of von Neumann algebras which enjoy Mazur's Property. This yields in particular a normality criterion which is very useful for operator algebraists (cf. Theorems 2.12 and 2.18 below): roughly speaking, a functional on a von Neumann algebra $\mathcal{M}$ is normal if and only if it is *sequentially* w^*-continuous, provided that $\mathcal{M}$ is not pathologically big. Moreover, we shall show that for preduals of von Neumann algebras, Mazur's Property and Property (X) as introduced by Godefroy and Talagrand, which is stronger in general, are in fact equivalent. Namely, we shall prove that the predual of a κ-decomposable von Neumann algebra has the latter properties if and only if κ is a non-measurable cardinal. Here, we define κ-decomposability as a natural generalization of the well-known concept of countable decomposability of von Neumann algebras.

This clarifies completely the situation of the above properties in the context of von Neumann algebras. To our knowledge, Mazur's Property and Property (X) have only been established for *separable* preduals of von Neumann algebras so far, where the first is obtained by a standard argument, the second quite easily. We have to stress here that in contrast to what is claimed at different places in the literature, it is *not* true that the predual of any von Neumann algebra has Mazur's Property, let alone Property (X). A counterexample has been pointed out by Edgar.

THEOREM 1.1. *For an abstract set Γ, the space $\ell_1(\Gamma)$ has Mazur's Property if and only if it has Property* (X), *which in turn happens if and only if $|\Gamma|$ is a non-measurable cardinal.*

Proof. For these two results, see Theorem 5.10 of [3], and Proposition 12 of [4]. ∎

Even assuming that measurable cardinals do not exist, no proof is known to us which would establish Property (X) or at least Mazur's Property for preduals of von Neumann algebras. We already remark at this point that, when investigating Property (X) in our general situation, we will also find another quite large class of Banach spaces having Property (X) (see Theorem 2.21 below).

We will further introduce and study natural generalizations of Mazur's Property and Property (X) for an arbitrary cardinal number κ. We remark that it has been proposed in the "Erratum" added to [9] to investigate such a variant of Property (X), but to our knowledge, nobody has taken this route so far. In doing so, we shall prove the surprising result that this new property only differs from the classical one in the case where the cardinal number involved is measurable. This will be obtained by using the ordering of Banach spaces introduced by Edgar in [4].

Our original interest was stemming from the question whether the space $L_1(\mathcal{G})$, where $\mathcal{G}$ denotes an arbitrary locally compact group, has Mazur's Property or even Property (X). By the above mentioned theorem, this is not true in general. So the question was to find out under which circumstances the answer is positive and how one could change the property in order to obtain a variant which in turn would hold for *any* locally compact group. We were also interested in having some normality criteria at hand in the case of the non-commutative counterpart of the above, namely the trace class operators $\mathcal{T}(\mathcal{H})$, where $\mathcal{H}$ denotes a Hilbert space (e.g., $\mathcal{H} = L_2(\mathcal{G})$). All these questions will turn out to be special cases of the theorems we shall prove, and the corresponding answers will thus serve as applications of the more general results.

The paper is organized as follows: we shall first establish the equivalence of Mazur's Property and Property (X) of the predual $\mathcal{M}_*$ as well as the κ-decomposability for non-measurable κ for an arbitrary von Neumann algebra $\mathcal{M}$; then we shall discuss the generalizations of both properties to the level of arbitrary cardinal numbers as briefly sketched above.

We would like to stress that the concepts and results contained in this paper have recently led to various important applications in Banach algebra theory and abstract harmonic analysis. Dales and Lau [2] as well as the author [30] have used results such as Theorem 2.26 and Theorem 3.4 below, respectively, in their study of the topological centres of the biduals of Beurling algebras. Moreover, the notion of decomposability introduced and studied in Section 2 below has been of fundamental importance in the recent article [18] by Hu and the author, and in Hu's subsequent papers [17] and [16], of which the latter completes the programme started in [15]. Also, the author has used Theorem 3.4 in his new approach to strong Arens irregularity of group algebras [28], as well as in his recent proof [27] of the Hofmeier–Wittstock conjecture [14] concerning the automatic boundedness of linear (left) $L_\infty(\mathcal{G})^*$-module maps on $L_\infty(\mathcal{G})$. He has further used Theorem 2.12 below in his recent proof [29] of the Ghahramani–Lau conjecture (see [22] and [8]) on the topological centre of the bidual of the measure algebra $M(\mathcal{G})$. Finally, Theorem 2.12 has also proved useful in his study [26] of amplifications of completely bounded operators on von Neumann algebras.

2. MAZUR'S PROPERTY AND PROPERTY (X) FOR PREDUALS OF VON NEUMANN ALGEBRAS

We shall begin by briefly recalling the definition of Mazur's Property as well as the (stronger) Property (X) of a Banach space Y. In both cases, the property enables one to characterize completely the normality of functionals on Y^* only by their *sequential* behaviour.

DEFINITION 2.1. Let Y be a Banach space. Then Y is said to have *Mazur's Property* (or to satisfy the *condition of Mazur*) if the following criterion for normality holds: a functional in Y^{**} is normal, i.e., defines an element of Y, if and only if it is w*-sequentially continuous on the unit ball of Y^*.

We remark that spaces with Mazur's Property sometimes are also called d-complete (cf. [21]) or μB spaces (cf. [38]).

Property (X) has been introduced in [9], Définition 5, in connection with the question of unicity of preduals (up to isometry) of dual Banach spaces (cf. also Définition 3 of [11]; p. 158, condition $(**)$ in Theorem V.3 of [10]; p. 147 of [13]). To formulate the latter property, we briefly recall the notion of weakly unconditionally Cauchy (wuC) series, also called weakly unconditionally convergent (wuc) series in the literature (cf. p. 157–158 of [10]). We follow the terminology of III.3.3, p. 127 in [13].

DEFINITION 2.2. Let X be a Banach space. A series $\sum f_n$ in X is called *weakly unconditionally Cauchy (wuC)* if for every functional $\Phi \in X^*$ we have $\sum\limits_{n=1}^{\infty} |\langle \Phi, f_n\rangle| < \infty$.

Now, still following III.5, p. 147 of [13], we come to

DEFINITION 2.3. A Banach space Y is said to have *Property (X)* if the following normality criterion is true: if $f \in Y^{**}$ is a functional such that for every wuC series $\sum y_n$ in Y^*, the equality $\langle f, \text{w}^*\text{-}\sum y_n \rangle = \sum \langle f, y_n \rangle$ holds, then we have $f \in Y$. (Here, the limit w*-$\sum y_n$ is taken in the $\sigma(Y^*, Y)$-topology.)

For later purposes, we collect some basic properties of the above properties.

REMARK 2.4. As is very easily seen, Mazur's Property as well as Property (X) are stable under isomorphism. They are also hereditary, i.e., they pass to (closed) subspaces. (In both cases, this can be obtained in essentially the same fashion as one proves that reflexivity is hereditary.)

For these statements (without proof), see Proposition 2.2 (1) and p. 625 of [21], or p. 51 of [23], concerning Mazur's Property, and Proposition 4 in [11] for Property (X).

REMARK 2.5. We briefly point out examples of spaces which have Mazur's Property but not Property (X). To this end, we recall that if a Banach space X has Property (X), then X is weakly sequentially complete (see Remark following Theorem V.3, p. 159 of [10]). Hence, the spaces c_0 and $\mathcal{K}(\mathcal{H})$, where $\mathcal{H}$ is an infinite-dimensional separable Hilbert space, do not have Property (X), but being separable, they have Mazur's Property.

In fact, if a Banach space X has Property (X), then X is the unique isometric predual of X^* (see Theorem V.3 (i) of [10]); clearly Mazur's Property does not imply the latter.

Our investigations start with a

DEFINITION 2.6. Let κ be a cardinal number. A von Neumann algebra $\mathcal{M}$ will be called *κ-decomposable* if any family of pairwise orthogonal non-zero projections in $\mathcal{M}$ has at most cardinality κ.

REMARK 2.7. Of course, in the case where $\kappa = \aleph_0$, this is the usual notion of countable decomposability (or σ-finiteness).

First, as in the classical setting, we will show the connection between decomposability and the existence of separating families for $\mathcal{M}$.

PROPOSITION 2.8. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $\kappa \geqslant \aleph_0$ a cardinal number. Then $\mathcal{M}$ is κ-decomposable if and only if there exists a family $(\zeta_i)_{i \in I} \subseteq \mathcal{H}$ with $|I| \leqslant \kappa$ which is separating for $\mathcal{M}$ (i.e., cyclic for $\mathcal{M}'$, the commutant of $\mathcal{M}$).*

Proof. Suppose first that $\mathcal{M}$ is κ-decomposable, and let $(\zeta_i)_{i \in I}$ be a maximal family of non-zero vectors in $\mathcal{H}$, such that the spaces $\overline{\mathcal{M}'\zeta_i}$ and $\overline{\mathcal{M}'\zeta_j}$ are orthogonal for $i, j \in I$ with $i \neq j$. Since the projections e_i from $\mathcal{H}$ onto $\overline{\mathcal{M}'\zeta_i}$ are elements of $\mathcal{M}$ and are pairwise orthogonal, we deduce from our assumption that $|I| \leqslant \kappa$.

Maximality of the family $(\zeta_i)_{i\in I}$ guarantees that $\bigoplus_{i\in I} \overline{\mathcal{M}'\zeta_i} = \mathcal{H}$, hence, $(\zeta_i)_{i\in I}$ is cyclic for $\mathcal{M}'$.

To prove the second implication, let $(\zeta_i)_{i\in I}$ be a family which is separating for $\mathcal{M}$, where $|I| \leqslant \kappa$. Let further $(p_l)_{l\in L} \subseteq \mathcal{M}$ be an arbitrary family of orthogonal non-zero projections. Since for all $i \in I$ we have $\sum_{l\in L} \|p_l\zeta_i\|^2 \leqslant \|\zeta_i\|^2$, we clearly have that for each $i \in I$, there is an index set $J_i \subseteq L$ which is at most countable and such that $p_l\zeta_i = 0$ for all $l \notin J_i$. Putting $J := \bigcup_{i\in I} J_i \subseteq L$, we thus get an index set of cardinality $|J| \leqslant \kappa$ such that $p_l\zeta_i = 0$ for all $i \in I$ and $l \notin J$. Finally, $(\zeta_i)_{i\in I}$ being separating for $\mathcal{M}$, we obtain $L = J$. ∎

COROLLARY 2.9. *Let κ be a cardinal number. The von Neumann algebra $\mathcal{B}(\mathcal{H})$ is κ-decomposable if and only if* $\dim(\mathcal{H}) \leqslant \kappa$, *where* dim *denotes the Hilbert space dimension.*

Proof. The assertion is clear in the case where κ is finite. If $\kappa \geqslant \aleph_0$, one has just to take into account the fact that $\mathcal{B}(\mathcal{H})' = \mathbb{C}\mathbf{1}$, and to apply Proposition 2.8. ∎

We now recall a

DEFINITION 2.10. A cardinal number κ is called *(real-valued) measurable* if for every abstract set Γ of cardinality κ there exists a probability measure on the power set $\mathfrak{P}(\Gamma)$ which is diffused, i.e., vanishes on singletons.

Obviously, measurability is a property of "large" cardinals. We remark that, if κ_1 and κ_2 are two cardinal numbers with $\kappa_1 \leqslant \kappa_2$, and if κ_2 is non-measurable, then of course κ_1 also is non-measurable.

REMARK 2.11. Let us stress that the existence of measurable cardinals cannot be proven in ZFC (= the axioms of Zermelo-Fraenkel and the axiom of choice); cf. Section 1, p. 106 and 108 of [20]. On the other hand, it is consistent with ZFC to assume that measurable cardinals do not exist ([7], Section 4, Theorem 4.14, p. 972).

As for the somewhat pathological nature of the assumption of the existence of measurable cardinals, we note that (cf. Section 4, Theorem 4.14, p. 972 of [7]) in ZFC the following statements are equiconsistent (i.e., if one is consistent with ZFC, so is the other one): "there exists a measurable cardinal" and "Lebesgue measure admits a σ-additive (not translation invariant) extension on the power set $\mathfrak{P}(\mathbb{R})$". We finish our brief discussion of measurable cardinals by remarking that, to give a concrete example, there do not exist measurable cardinals in ZFC + "$V = L$", where the latter denotes Gödel's constructibility axiom (p. 521 of [32]). The reader will find detailed information about measurable cardinals, e.g., in [6] and [34].

Our first main goal is the following result which is very handy for operator algebraists: it shows that a functional on a von Neumann algebra $\mathcal{M}$ is normal if and only if it is *sequentially* w*-continuous, whenever the cardinality of pairwise orthogonal (non-zero) projections in $\mathcal{M}$ is non-measurable. Note that a von Neumann algebra which does not satisfy the latter condition can never be shown to exist (in ZFC)!

THEOREM 2.12. *Let $\mathcal{M}$ be a von Neumann algebra. Then the following are equivalent:*

(i) *The predual $\mathcal{M}_*$ of $\mathcal{M}$ has Mazur's Property.*

(ii) *The von Neumann algebra $\mathcal{M}$ is κ-decomposable for some non-measurable cardinal number κ.*

REMARK 2.13. The implication "(ii) $\Rightarrow$(i)" will be strengthened in Theorem 2.18 below.

We shall now present some auxiliary results preparing the proof of Theorem 2.12, which will also be useful in our later study of Property (X).

LEMMA 2.14. *Let $\mathcal{M}$ be a von Neumann algebra. A linear form on $\mathcal{M}$ is normal if and only if its restriction to every abelian von Neumann subalgebra of $\mathcal{M}$ is normal.*

Proof. See Corollary 1 of [35]. ∎

For the following, we recall that a Banach space is said to be *weakly compactly generated (WCG)* if it has a weakly compact subset whose linear span is dense. A Banach space X is *weakly countably determined (WCD)* or a *Vašák space* if there is a sequence $(K_n)_{n\in\mathbb{N}} \subseteq X^{**}$ of w*-compact sets such that for $x \in X$ and $u \in X^{**} \setminus X$, there is $n \in \mathbb{N}$ such that $x \in K_n$ and $u \notin K_n$. Finally, a Banach space X is called *weakly Lindelöf determined (WLD)* if $\mathrm{Ball}(X^*)$ in its w*-topology is a Corson compact (i.e., homeomorphic to a subset of $\{x \in [-1,1]^\Gamma : |\{\gamma \in \Gamma : x_\gamma \neq 0\}| \leqslant \aleph_0\}$ for some set Γ). We stress that every WCG Banach space is WCD, and every WCD Banach space in turn is WLD, but none of the converse assertions holds (cf. Theorem 3.8 of [39]).

For a detailed discussion of these classes of Banach spaces, we refer to the excellent survey paper [39]. The author is grateful to Václav Zizler for discussions about several properties of WLD spaces and also for providing him with an early copy of [39].

LEMMA 2.15. *Every* WLD *Banach space has Mazur's Property.*

Proof. If X is a WLD space, by definition, $(\mathrm{Ball}(X^*), \mathrm{w}^*)$ is a Corson compact, hence an angelic space (see Remarks 6.3 (c), p. 1100 of [25]). Owing to Proposition 2.3 in [21], (cf. also p. 564, Figure 1 of [3]), this implies that X has Mazur's Property. ∎

EXAMPLE 2.16. Let (Ω, μ) be a σ-finite measure space. Then $L_1(\Omega, \mu)$ is weakly compactly generated and therefore enjoys Mazur's Property.

Proof. Following Corollary 2 of [24] (or Chapter 10, p. 217 Examples of [12]), the space $L_1(\Omega, \mu)$ is weakly compactly generated precisely in the case where (Ω, μ) is σ-finite. ∎

LEMMA 2.17. *Let I be an abstract set whose cardinality is non-measurable. Let further $(Y_i)_{i \in I}$ be a family of Banach spaces*

(i) *with Mazur's Property, or*

(ii) *with Property* (X).

Then the space $Y = \bigoplus_{i \in I}^{\ell_1} Y_i$ also has the respective property.

Proof. Assertion (i) follows from Theorem 4.2 in [23]; assertion (ii) is Proposition 15 in [4]. ∎

We are now ready to prove Theorem 2.12.

Proof. (i) $\Rightarrow$ (ii) Assuming the contrary, let $(P_\alpha)_{\alpha \in I}$ be a family of pairwise orthogonal, non-zero projections in $\mathcal{M}$, where $|I|$ is a measurable cardinal. Now let $\mathcal{R}$ be the von Neumann algebra in $\mathcal{M}$ generated by these projections. Then $\mathcal{R}$ is an abelian atomic von Neumann subalgebra of $\mathcal{M}$. Hence, by p. 23–24 of [36], there exists a normal projection of norm 1 from $\mathcal{M}$ onto $\mathcal{R}$. We thus see that the predual $\mathcal{R}_*$ is isometrically isomorphic to a (closed) subspace of the predual $\mathcal{M}_*$. Since the latter has Mazur's Property, which is both hereditary and stable under isomorphism (see Remark 2.4), we deduce that $\mathcal{R}_*$ also has Mazur's Property.

Now, by a standard argument (cf. p. 669 of [19]), we see that $\mathcal{R} = \left\{ \sum_{\alpha \in I} f_\alpha P_\alpha : (f_\alpha) \in \ell_\infty(I) \right\}$. (In fact, by minimality of P_α we have for arbitrary $A \in \mathcal{R}$ that $AP_\alpha = P_\alpha A P_\alpha = f_\alpha P_\alpha$ with $f_\alpha \in \mathbb{C}$, $|f_\alpha| \leqslant \|A\|$; hence, since $\mathbf{1}_{\mathcal{R}} = \sum P_\alpha$, we obtain $A = \sum AP_\alpha = \sum f_\alpha P_\alpha$.) Consider the mapping

$$\begin{aligned} \pi : \ell_\infty(I) &\longrightarrow \mathcal{R} = \left\{ \sum_{\alpha \in I} f_\alpha P_\alpha : (f_\alpha) \in \ell_\infty(I) \right\} \\ f &\longmapsto \sum_{\alpha \in I} f_\alpha P_\alpha. \end{aligned}$$

Obviously, π is a $*$-isomorphism from $\ell_\infty(I)$ onto $\mathcal{R}$, hence normal (and isometric). We conclude that $\mathcal{R}_*$ and $\ell_1(I)$ are isomorphic as Banach spaces. Since $\mathcal{R}_*$ has Mazur's Property, the same is true for $\ell_1(I)$. But this implies ([3], Theorem 5.10) that $|I|$ is a non-measurable cardinal, which is the desired contradiction.

(ii)$\Rightarrow$ (i) Let $\mathcal{M}$ be a κ-decomposable von Neumann algebra. Thanks to Lemma 2.14, it is easily seen that we can restrict ourselves to the case where $\mathcal{M}$ is an abelian von Neumann algebra.

We follow the discussion of abelian w*-algebras presented in 1.18 of [31], in particular Proposition 1.18.1 (with proof), as well as the terminology used there. For a finite measure space (Ω, μ), we denote by $L_\infty(\Omega, \mu)$ the abelian w*-algebra of essentially bounded μ-measurable functions on Ω. In the proof of Proposition 1.18.1 in [31], it is shown that for an abelian von Neumann algebra, and so for $\mathcal{M}$, we have $\mathcal{M} = \bigoplus_{\alpha\in I}^{\ell_\infty} L_\infty(\Omega_\alpha, \mu_\alpha)$ as von Neumann algebras, where $(\Omega_\alpha, \mu_\alpha)$ are finite measure spaces. We thus get an isometric isomorphism: $\mathcal{M}_* = \bigoplus_{\alpha\in I}^{\ell_1} L_1(\Omega_\alpha, \mu_\alpha)$. By Example 2.16, the spaces $L_1(\Omega_\alpha, \mu_\alpha)$ have Mazur's Property. Since obviously $|I| \leqslant \kappa$, and κ is non-measurable, we see that $|I|$ also is non-measurable. Hence, noting that Mazur's Property is stable under isomorphism (see Remark 2.4), we deduce from Lemma 2.17 (i) that $\mathcal{M}_*$ has Mazur's Property, as desired. ∎

We now state one of our main theorems.

THEOREM 2.18. *Let $\mathcal{M}$ be a von Neumann algebra. Then the following are equivalent:*

(i) *The predual $\mathcal{M}_*$ of $\mathcal{M}$ has Property* (X) *of Godefroy-Talagrand.*

(ii) *The predual $\mathcal{M}_*$ of $\mathcal{M}$ has Mazur's Property.*

(iii) *The von Neumann algebra $\mathcal{M}$ is κ-decomposable for some non-measurable cardinal number κ.*

REMARK 2.19. An intermediate property between Mazur's Property and Property (X) has been introduced as "Property $(*)$" by Godefroy (cf. p. 155 of [10]). A Banach space Y is said to have *Property* $(*)$ if the following holds: a functional $f \in Y^{**}$ belongs to Y if for every weakly Cauchy sequence $(y_n) \subseteq Y^*$, one has $\langle f, \text{w}^*\text{-}\lim y_n\rangle = \lim\langle f, y_n\rangle$. By our Theorem 2.18, Property $(*)$ is equivalent to Property (X) for preduals of von Neumann algebras.

Before presenting the preparations needed to prove Theorem 2.18, we give an important example.

COROLLARY 2.20. *Let $\mathcal{H}$ be a Hilbert space. Then the space $\mathcal{T}(\mathcal{H})$ of trace class operators on $\mathcal{H}$ has Mazur's Property or Property* (X)*, respectively, if and only if the Hilbert space dimension* $\dim(\mathcal{H})$ *is a non-measurable cardinal.*

Proof. This follows from Theorem 2.18 and Corollary 2.9. Note that the necessity of $\dim(\mathcal{H})$ being non-measurable can also be seen by using Theorem 1.1 and the fact that each of the properties is hereditary. ∎

We shall first derive the following general result.

THEOREM 2.21. *Every weakly Lindelöf determined, weakly sequentially complete direct factor of a Banach lattice has Property* (X).

COROLLARY 2.22. *If (Ω, μ) is a σ-finite measure space, then the space $L_1(\Omega, \mu)$ has Property* (X).

Proof. We know from Example 2.16 that $L_1(\Omega, \mu)$ is weakly compactly generated, hence weakly Lindelöf determined, and the remaining assertions are evident. ∎

REMARK 2.23. Our Theorem 2.21 generalizes a well-known result (cf. Example (2), p. 161 of [10]; Example (iii), p. 90 of [4]) from separable to weakly Lindelöf determined spaces. Note also that weak sequential completeness is a necessary condition to ensure Property (X), as pointed out in Theorem V.3 with Remark thereafter on p. 159 of [10].

As noted in Lemma 2.15, weakly Lindelöf determined Banach spaces always have Mazur's Property. The above quite sharp result (cf. Remark 2.23) shows how much stronger Property (X) is.

LEMMA 2.24. *Let Y be a* WLD *Banach space, and let $(y_n) \subseteq \mathrm{Ball}(Y^*)$ be a sequence. Then there exists a projection $P : Y \to Y$ of norm* 1 *such that $P(Y)$ is separable and $(y_n) \subseteq P^*(Y^*)$.*

Proof. This follows from Lemma 1 in [37]. ∎

Now let us turn to Theorem 2.21.

Proof. Let Y be a Banach space satisfying the conditions of the theorem. Fix $f \in Y^{**}$ such that for every wuC series $\sum x_n$ in Y^* the equality $\langle f, \mathrm{w}^*\text{-}\sum x_n \rangle = \sum \langle f, x_n \rangle$ holds. We wish to prove that $f \in Y$. Being weakly Lindelöf determined, Y has Mazur's Property by Lemma 2.15. Hence it suffices to show that for every sequence $(y_n) \subseteq \mathrm{Ball}(Y^*)$ converging w* to 0 we have $\lim_n \langle f, y_n \rangle = 0$. Fix such a sequence (y_n). Thanks to Lemma 2.24, there is a projection $P \in \mathcal{B}(Y)$ with $\|P\| = 1$, such that $P(Y)$ is separable and $(y_n) \subseteq P^*(Y^*)$. Of course, $P^{**}(f) \in P^{**}(Y^{**}) = P(Y)^{**}$. But $P(Y)$ is a *separable* weakly sequentially complete direct factor of a Banach lattice (note our assumptions about Y and the properties of P). Hence, applying the theorem stated in Remark 2.23, $P(Y)$ has Property (X). Now one easily deduces from the assumptions about f that $P^{**}(f) \in P(Y)$. Since $(y_n) \subseteq P^*(Y^*)$, we conclude that f satisfies $\lim_n \langle f, y_n \rangle = 0$. ∎

REMARK 2.25. By the same argument, using a similar result from [10] for the separable case ([10], p. 160–161), one can show the following variant of Theorem 2.21: Every weakly Lindelöf determined, weakly sequentially complete subspace of a Banach lattice with order continuous norm has Property (X).

Now Theorem 2.18 is only a few steps away.

Proof. (i) $\Rightarrow$ (ii) Evident.
(ii) $\Rightarrow$ (iii) See Theorem 2.12, "(i)$\Rightarrow$ (ii)".

(iii)⇒ (i) We proceed analogously to the proof of Theorem 2.12. Thanks to Lemma 2.14, to prove Property (X), we can assume that the von Neumann algebra $\mathcal{M}$ is abelian. As in the proof of Theorem 2.12, we obtain an isometric isomorphism: $\mathcal{M}_* = \bigoplus_{\alpha \in I}^{\ell_1} L_1(\Omega_\alpha, \mu_\alpha)$, where $|I| \leqslant \kappa$ is a non-measurable cardinal and $(\Omega_\alpha, \mu_\alpha)$ are finite measure spaces. By Corollary 2.22, the spaces $L_1(\Omega_\alpha, \mu_\alpha)$ enjoy Property (X). Since the latter is stable under isomorphism (cf. Remark 2.4), we conclude by applying Lemma 2.17 (ii). ∎

As an application, we state the following result which is useful in abstract harmonic analysis. For a locally compact group $\mathcal{G}$, we denote by $\mathfrak{k}(\mathcal{G})$ its compact covering number, i.e., the smallest cardinality of a compact covering of $\mathcal{G}$. Note that for every open σ-compact subgroup $\mathcal{H}$ of $\mathcal{G}$ we have $\mathfrak{k}(\mathcal{G}) \cdot \aleph_0 = |\mathcal{G}/\mathcal{H}| \cdot \aleph_0$, cf. Section 3, Lemma 3.11, p. 1167 of [1] (there, $\mathcal{H}$ is in addition assumed to be compactly generated; inspection of the proof shows that this assumption is unnecessary).

THEOREM 2.26. *Let $\mathcal{G}$ be a locally compact group such that $\mathfrak{k}(\mathcal{G})$ is non-measurable. Then $L_1(\mathcal{G})$ has Property* (X).

Proof. In view of Corollary 2.22, we can assume that $\mathcal{G}$ is not σ-compact. Let $\mathcal{H}$ be an open σ-compact subgroup of $\mathcal{G}$. Then, by the above remark, $|\mathcal{G}/\mathcal{H}| = \mathfrak{k}(\mathcal{G})$, hence $|\mathcal{G}/\mathcal{H}|$ is non-measurable. Writing $\mathcal{G} = \dot{\bigcup}_{x\mathcal{H} \in \mathcal{G}/\mathcal{H}} x\mathcal{H}$, we obtain an isometric isomorphism $L_1(\mathcal{G}) = \bigoplus_{x\mathcal{H} \in \mathcal{G}/\mathcal{H}}^{\ell_1} L_1(x\mathcal{H})$, where $x\mathcal{H}$ are σ-finite measure spaces (of course, the measure is the restriction of left Haar measure $\lambda_{\mathcal{G}}$ to $x\mathcal{H}$). Again by Corollary 2.22, we see that the spaces $L_1(x\mathcal{H})$, $x\mathcal{H} \in \mathcal{G}/\mathcal{H}$, all have Property (X). Now Lemma 2.17 finishes the proof, where we use the stability of Property (X) under isomorphism (cf. Remark 2.4). ∎

3. MAZUR'S PROPERTY AND PROPERTY (X) OF HIGHER CARDINAL LEVEL

We start again with a

DEFINITION 3.1. Let X be a Banach space and $\kappa \geqslant \aleph_0$ a cardinal number.
(i) A functional $f \in X^{**}$ will be called w*-*κ-continuous* if for all nets $(x_\alpha)_{\alpha \in I} \subseteq \mathrm{Ball}(X^*)$ of cardinality $\aleph_0 \leqslant |I| \leqslant \kappa$ with $x_\alpha \xrightarrow{w^*} 0$, we have $\langle f, x_\alpha \rangle \longrightarrow 0$.
(ii) We say that X has *Mazur's Property of level κ* if every w*-κ-continuous functional $f \in X^{**}$ is an element of X.

REMARK 3.2. Of course, the classical property of Mazur implies Mazur's Property of level $\aleph_0$. We further remark that obviously the property becomes weaker with increasing cardinality.

One shows the following elementary facts by slightly modifying the proofs from the classical situation.

REMARK 3.3. Let $\kappa \geqslant \aleph_0$ be a cardinal number. Mazur's Property of level κ is stable under isomorphism and passes to (closed) subspaces.

First we shall obtain the following

THEOREM 3.4. *Let $\mathcal{G}$ be a locally compact group with compact covering number $\mathfrak{k}(\mathcal{G})$. Then $L_1(\mathcal{G})$ has Mazur's Property of level $\mathfrak{k}(\mathcal{G}) \cdot \aleph_0$.*

REMARK 3.5. If the group $\mathcal{G}$ is σ-compact, the above statement follows from Example 2.16.

The theorem follows by an argument analogous to the proof of Theorem 2.26, by using Theorem 3.6 and Remark 3.3, once the following general result has been established.

THEOREM 3.6. *Let I be an abstract set with $|I| \geqslant \aleph_0$. Let further $(X_i)_{i\in I}$ be a family of Banach spaces each having Mazur's Property of level $|I|$. Then the space $\overset{\ell_1}{\bigoplus_{i\in I}} X_i$ also has this property.*

REMARK 3.7. Compare the above with Theorem 3.1 in [21] or Theorem 4.2 in [23], respectively (cf. Lemma 2.17 above).

Proof. We have an isometric isomorphism $\Big(\overset{\ell_1}{\bigoplus_{i\in I}} X_i\Big)^* = \overset{\ell_\infty}{\bigoplus_{i\in I}} X_i^*$. Put $\kappa := |I|$. Now let $\Phi \in \Big(\overset{\ell_1}{\bigoplus_{i\in I}} X_i\Big)^{**}$ be a w*-κ-continuous functional. For each $i \in I$ denote by x_i the restriction of Φ to X_i^*. By assumption, we see that $x_i \in X_i$ for all $i \in I$. We also have that $\sum_{i\in I} \|x_i\| \leqslant \|\Phi\|$. Hence, the family $x := (x_i)_{i\in I}$ defines an element in $\overset{\ell_1}{\bigoplus_{i\in I}} X_i$.

We have to show that $\Phi = x$. To this end, fix $\varphi := (\varphi_i)_{i\in I} \in \overset{\ell_\infty}{\bigoplus_{i\in I}} X_i^*$. We claim that $\Phi(\varphi) = \sum_{i\in I} \langle \varphi_i, x_i \rangle$. Denote by $\mathfrak{P}_{\text{fin}}(I)$ the set of all finite subsets of I. Since $|I| \geqslant \aleph_0$, it follows that $\kappa = |I| = |\mathfrak{P}_{\text{fin}}(I)|$. Now for each $F \in \mathfrak{P}_{\text{fin}}(I)$, define the functional $\varphi_F \in \Big(\overset{\ell_1}{\bigoplus_{i\in I}} X_i\Big)^*$ through:

$$\langle \varphi_F, z \rangle := \sum_{i\in F} \langle \varphi_i, z_i \rangle \quad \text{for } z = (z_i)_{i\in I} \in \overset{\ell_1}{\bigoplus_{i\in I}} X_i.$$

In other words: $\varphi_F := (\chi_F(i)\varphi_i)_{i\in I}$. Now, $(\varphi_F)_{F\in\mathfrak{P}_{\mathrm{fin}}(I)}$ is a bounded net (directed by inclusion) which converges w* to φ. For if $z \in \bigoplus_{i\in I}^{\ell_1} X_i$, we see that

$$\langle\varphi, z\rangle = \sum_{i\in I}\langle\varphi_i, z_i\rangle = \lim_F \sum_{i\in F}\langle\varphi_i, z_i\rangle = \lim_F\langle\varphi_F, z\rangle.$$

Hence, thanks to the w*-κ-continuity of Φ (noting that $\kappa = |\mathfrak{P}_{\mathrm{fin}}(I)|$), we obtain:

$$\langle\Phi, \varphi\rangle = \left\langle\Phi, \text{w}^*\text{-}\lim_F \varphi_F\right\rangle = \lim_F\langle\Phi, \varphi_F\rangle = \lim_F \sum_{i\in F}\langle x_i, \varphi_i\rangle = \sum_{i\in I}\langle\varphi_i, x_i\rangle,$$

which ends the proof. ∎

Now we come to a corresponding natural generalization of Godefroy-Talagrand's Property (X) which has already been suggested, but never carried out, in the "Erratum" of [9] that we mentioned before. Our aim is to show that this notion of "Property (X) of level κ", where now κ is an arbitrary cardinal number, only differs from the classical Property (X) if κ is measurable.

DEFINITION 3.8. Let X be a Banach space. A series $\sum_{\alpha\in I} f_\alpha$ in X will be called *weakly unconditionally Cauchy (wuC)* if for every functional $\Phi \in X^*$ one has $\sum_{\alpha\in I} |\langle\Phi, f_\alpha\rangle| < \infty$.

REMARK 3.9. One can show, cf. proof of Lemma 15.1 in [33], that a series $\sum f_\alpha$ in a dual Banach space X^* is wuC if and only if it is w*uC (i.e., for all $x \in X$ we have $\sum_{\alpha\in I} |\langle f_\alpha, x\rangle| < \infty$).

These preparations are sufficient for our next

DEFINITION 3.10. Let X be a Banach space, and $\kappa \geqslant \aleph_0$ a cardinal number. We say that X has *Property (X) of level* κ if the following normality criterion holds:

A functional $\Phi \in X^{**}$ belongs to X if for every wuC series $\sum_{\alpha\in I} f_\alpha$ in X^* of cardinality $|I| \leqslant \kappa$ one has $\left\langle\Phi, \text{w}^*\text{-}\sum_{\alpha\in I} f_\alpha\right\rangle = \sum_{\alpha\in I}\langle\Phi, f_\alpha\rangle$. (Here, the limit w*-$\sum_{\alpha\in I} f_\alpha$ is taken in the $\sigma(X^*, X)$-topology.)

REMARK 3.11. (i) The property clearly becomes weaker with increasing cardinality. For a very interesting, if not surprising!, connection between the classical Property (X) and Property (X) of higher cardinal level, see Corollary 3.15 below.

(ii) Property (X) of level κ obviously implies Mazur's Property of level κ.

The following statement can easily be shown by mimicking the proof of the classical case.

REMARK 3.12. Let $\kappa \geqslant \aleph_0$ be a cardinal number. Property (X) of level κ is hereditary and stable under isomorphism.

We now briefly recall the partial ordering of Banach spaces introduced by Edgar in [4]. If X and Y are Banach spaces, one defines $X \prec Y$ whenever each functional $\Phi \in X^{**}$ such that $T^{**}(\Phi) \in Y$ for all $T \in \mathcal{B}(X, Y)$, already belongs to X.

Edgar shows in Proposition 10 of [4], that a Banach space X enjoys Property (X) if and only if $X \prec \ell_1$. The argument can easily be generalized to yield the following

PROPOSITION 3.13. *Let X be a Banach space, and $\kappa \geqslant \aleph_0$ a cardinal number. Then X has Property* (X) *of level κ if and only if $X \prec \ell_1(I)$, where $|I| = \kappa$.*

REMARK 3.14. In Proposition 3 of [4], it is shown that a Banach space X has Mazur's Property if and only if $X \prec c_0$. We note that a characterization of Mazur's Property of level κ which would correspond to Proposition 3.13, involving the space $c_0(I)$, $|I| = \kappa$, does not hold. Comparing the statements (1) "X has Mazur's Property of level κ" and (2) "$X \prec c_0(I)$ with $|I| = \kappa$", we only have that (2) implies (1) in general.

To see this, recall that $c_0(\Gamma) \prec c_0$ for every abstract set Γ ([5], Proposition 1). Hence, if (2) holds, we already obtain that $X \prec c_0$ which by the above means that X has the classical property of Mazur, whence (1) follows. The converse does not hold in general since, if Γ is a discrete group of measurable cardinality, we deduce from Theorem 3.4 that $\ell_1(\Gamma)$ has Mazur's Property of level $|\Gamma|$; but (2) would imply that $\ell_1(\Gamma) \prec c_0$, which is a contradiction to $|\Gamma|$ being measurable.

We shall now deduce that Property (X) of level κ can only differ from the classical Property (X) in the case where κ is measurable.

COROLLARY 3.15. *Let $\kappa \geqslant \aleph_0$ be a non-measurable cardinal, X a Banach space with Property* (X) *of level κ. Then X has the classical Property* (X).

Proof. From Proposition 3.13 we know that $X \prec \ell_1(I)$, where $|I| = \kappa$. But since $|I| = \kappa$ is non-measurable, by Proposition 12 of [4], it follows that $\ell_1(I) \prec \ell_1$. Transitivity of $\prec$ (cf. p. 84 of [4]) gives $X \prec \ell_1$. Owing to Proposition 10 in [4], this finishes the proof. ∎

COROLLARY 3.16. *If $\kappa \leqslant \aleph_1$, then Property* (X) *of level κ is equivalent to the classical Property* (X).

By Proposition 3 in [5], for an arbitrary index set I with $|I| \geqslant \aleph_0$, and a family of Banach spaces $(X_\alpha)_{\alpha\in I}$ where $X_\alpha \prec \ell_1(I)$ $(\alpha \in I)$, one has $\bigoplus_{\alpha\in I}^{\ell_1} X_\alpha \prec \ell_1(I)$. Together with Proposition 3.13, this yields the following analogue of Theorem 3.6.

THEOREM 3.17. *Let I be an index set with $|I| \geqslant \aleph_0$, and let $(X_\alpha)_{\alpha\in I}$ be a family of Banach spaces each having Property* (X) *of level $|I|$. Then the latter is also shared by the space $\bigoplus_{\alpha\in I}^{\ell_1} X_\alpha$.*

We close by deriving the following sharpening of Theorem 3.4.

THEOREM 3.18. *Let $\mathcal{G}$ be a locally compact group with compact covering number $\mathfrak{k}(\mathcal{G})$. Then the space $L_1(\mathcal{G})$ enjoys Property* (X) *of level $\mathfrak{k}(\mathcal{G}) \cdot \aleph_0$.*

Proof. This follows in the same way as Theorem 3.4 by now using Theorem 3.17 and Remark 3.12. ∎

Acknowledgements. Some parts of this article are taken from the author's Ph.D. Dissertation. It is a pleasure to thank my supervisor Gerd Wittstock for many very fruitful discussions. I would also like to thank Gilles Godefroy for various helpful discussions which led to some of the results presented here concerning Property (X).

This work was partially supported by NSERC; the support of NSERC is gratefully acknowledged.

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Received April 8, 2006.