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On topological centre problems and SIN quantum groups ☆

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Abstract

Let A be a Banach algebra with a faithful multiplication and $\langle A^*A\rangle^*$ be the quotient Banach algebra of A^{**} with the left Arens product. We introduce a natural Banach algebra, which is a closed subspace of $\langle A^*A\rangle^*$ but equipped with a distinct multiplication. With the help of this Banach algebra, new characterizations of the topological centre $\mathfrak{Z}_t(\langle A^*A\rangle^*)$ of $\langle A^*A\rangle^*$ are obtained, and a characterization of $\mathfrak{Z}_t(\langle A^*A\rangle^*)$ by Lau and Ülger for A having a bounded approximate identity is extended to all Banach algebras. The study of this Banach algebra motivates us to introduce the notion of SIN locally compact quantum groups and the concept of quotient strong Arens irregularity. We give characterizations of co-amenable SIN quantum groups, which are even new for locally compact groups. Our study shows that the SIN property is intrinsically related to topological centre problems. We also give characterizations of quotient strong Arens irregularity for all quantum group algebras. Within the class of Banach algebras introduced recently by the authors, we characterize the unital ones, generalizing the corresponding result of Lau and Ülger. We study the interrelationships between strong Arens irregularity and quotient strong Arens irregularity, revealing the complex nature of topological centre problems. Some open questions by Lau and Ülger on $\mathfrak{Z}_t(\langle A^*A\rangle^*)$ are also answered.

Keywords: Banach algebras; Topological centres; Locally compact groups and quantum groups

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1. Introduction

Let A be a Banach algebra. As is well known, on the bidual A^{**} of A, there are two Banach algebra multiplications, called the left and the right Arens products, respectively, each extending the multiplication on A (cf. Arens [1]). By definition, the left Arens product $\Box$ is induced by the left A-module structure on A. That is, for $m, n \in A^{**}$, $f \in A^*$, and $a, b \in A$, we have

$$\langle f \cdot a, b\rangle = \langle f, ab\rangle, \qquad \langle n \Box f, a\rangle = \langle n, f \cdot a\rangle, \quad \text{and} \quad \langle m \Box n, f\rangle = \langle m, n \Box f\rangle.$$

The right Arens product $\Diamond$ is defined by considering A as a right A-module. It is known that

$$m \Box n = \mathrm{w}^*\text{-}\lim_{\alpha} \lim_{\beta} a_\alpha b_\beta \quad \text{and} \quad m \Diamond n = \mathrm{w}^*\text{-}\lim_{\beta} \lim_{\alpha} a_\alpha b_\beta$$

whenever (a_α) and (b_β) are nets in A converging to m and n, respectively, in $\sigma(A^{**}, A^*)$. The algebra A is said to be Arens regular if $\Box$ and $\Diamond$ coincide on A^{**}. Every operator algebra, in particular, every C^*-algebra, is Arens regular. However, Banach algebras studied in abstract harmonic analysis are typically far from being Arens regular. For example, for a locally compact group G, the group algebra $L_1(G)$ is Arens regular if and only if G is finite (cf. Young [47]).

In the study of Arens irregularity, the left and the right topological centres $\mathfrak{Z}_t(A^{**}, \Box)$ and $\mathfrak{Z}_t(A^{**}, \Diamond)$ of A^{**} are considered (see Section 2 for the definition). In general, $\mathfrak{Z}_t(A^{**}, \Box)$ and $\mathfrak{Z}_t(A^{**}, \Diamond)$ are norm closed subalgebras of A^{**} when A^{**} is equipped with either Arens product. It is obvious that A is Arens regular if and only if $\mathfrak{Z}_t(A^{**}, \Box) = \mathfrak{Z}_t(A^{**}, \Diamond) = A^{**}$. The left Arens product on A^{**} induces naturally a Banach algebra multiplication on $\langle A^*A\rangle^*$, also denoted by $\Box$, such that $(\langle A^*A\rangle^*, \Box) \cong (A^{**}, \Box)/\langle A^*A\rangle^\perp$. The topological centre $\mathfrak{Z}_t(\langle A^*A\rangle^*)$ of $(\langle A^*A\rangle^*, \Box)$ is defined in a similar fashion as $\mathfrak{Z}_t(A^{**}, \Box)$. As observed by several authors, there exists a close connection between the topological centres $\mathfrak{Z}_t(A^{**}, \Box)$ and $\mathfrak{Z}_t(\langle A^*A\rangle^*)$.

Under the canonical embedding $A \hookrightarrow A^{**}$, we have $A \subseteq \mathfrak{Z}_t(A^{**}, \Box) \subseteq A^{**}$ and $A \subseteq \mathfrak{Z}_t(A^{**}, \Diamond) \subseteq A^{**}$. In [6], Dales and Lau introduced the following concepts of Arens irregularity: A is said to be left strongly Arens irregular if $\mathfrak{Z}_t(A^{**}, \Box) = A$, right strongly Arens irregular if $\mathfrak{Z}_t(A^{**}, \Diamond) = A$, and strongly Arens irregular if A is both left and right strongly Arens irregular. When A has a bounded right approximate identity, one has $RM(A) \subseteq \mathfrak{Z}_t(\langle A^*A\rangle^*) \subseteq \langle A^*A\rangle^*$, where $RM(A)$ is the opposite right multiplier algebra of A. In general, $RM(A)$ and $\langle A^*A\rangle^*$ can be compared via their canonical images in $B_A(A^*)$, the Banach algebra of bounded right A-module homomorphisms on A^* (see Fact 2 in Section 4). In the spirit of the terminology of Dales and Lau, we say that A is *left quotient strongly Arens irregular* if $\mathfrak{Z}_t(\langle A^*A\rangle^*) \subseteq RM(A)$. Then, when A has a bounded right approximate identity, A is left quotient strongly Arens irregular if and only if $\mathfrak{Z}_t(\langle A^*A\rangle^*) = RM(A)$. The right quotient strong Arens irregularity is defined similarly by comparing the topological centre $\mathfrak{Z}_t(\langle AA^*\rangle^*)$ of $\langle AA^*\rangle^*$ with the left multiplier algebra of A, where $\langle AA^*\rangle^*$ is the Banach algebra with the multiplication induced by $\Diamond$. There have been a number of extensive studies of strong Arens irregularity and quotient strong Arens irregularity over the last three decades. The reader is referred to Dales [5], Dales and Lau [6], Dales, Lau and Strauss [7], Lau [25], Lau and Losert [26,27], Lau and Ülger [28], and Palmer [35] for more information on Arens products, topological centres, and related topics.

The present work is mainly motivated by the intriguing interrelationships between strong Arens irregularity and quotient strong Arens irregularity of general Banach algebras. The paper is organized as follows. In Section 2, we start with notation conventions, definitions, and preliminary results on Arens products and topological centres. We then introduce the Banach algebra $\langle A^*A\rangle_R^*$, which is a norm closed subspace of the quotient algebra $(\langle A^*A\rangle^*, \Box)$ on the one hand, but on the other hand its multiplication is induced by the right Arens product $\Diamond$ on A^{**}. The topological centre $\mathfrak{Z}_t(\langle A^*A\rangle_R^*)$ of $\langle A^*A\rangle^*$ is thereby defined, since $\langle A^*A\rangle_R^*$ is a left topological semigroup under the relative weak*-topology. It will be seen that the Banach algebras $\langle A^*A\rangle_R^*$ and $\mathfrak{Z}_t(\langle A^*A\rangle_R^*)$ help reveal the intrinsic structure of $\langle A^*A\rangle$, $\langle A^*A\rangle^*$, and A^{**}.

In Section 3, we present a new characterization of the topological centre $\mathfrak{Z}_t(\langle A^*A\rangle^*)$ (Theorem 2). In the case where A is the group algebra $L_1(G)$ of a locally compact group G, we use $LUC(G)_R^*$ to replace the Banach algebra $(Z_U(G), *)$, which was defined via the group action of G on $LUC(G)$ and was used by Lau [25, Lemma 2] to characterize $\mathfrak{Z}_t(LUC(G)^*)$ (see Section 3 for details, and see Dales and Lau [6, Proposition 11.6] for the extension of [25, Lemma 2] to convolution Beurling algebras). Therefore, Theorem 2 is a Banach algebraic version of [25, Lemma 2]. Our investigation of the SIN property and other problems shows that $LUC(G)_R^*$ has its advantages over $Z_U(G)$ in certain aspects. As a consequence of Theorem 2, we extend to all Banach algebras a characterization of $\mathfrak{Z}_t(\langle A^*A\rangle^*)$ by Lau and Ülger [28, Lemma 3.1c)], where they assumed that A has a bounded approximate identity (Corollary 3). The exploration of a Banach algebraic version of another characterization of $\mathfrak{Z}_t(LUC(G)^*)$ given by Neufang [34] leads to the concept of a strong identity of $\langle A^*A\rangle^*$, that is proven to be important for studying locally compact quantum groups.

Section 4 has initially been inspirited by the natural question of when $\langle A^*A\rangle^*$ has a strong identity. Generalizing the concept of SIN locally compact groups, we introduce the notion of SIN locally compact quantum groups. We characterize co-amenable SIN quantum groups $\mathbb{G}$ in terms of the Banach algebras $RM(L_1(\mathbb{G}))$, $\mathfrak{Z}_t(LUC(\mathbb{G})_R^*)$, and $LUC(\mathbb{G})_R^*$, and the existence of a strong identity of $LUC(\mathbb{G})^*$, respectively (Theorem 18), where we note that $RM(L_1(\mathbb{G})) = RM_{cb}(L_1(\mathbb{G}))$ (the completely bounded opposite right multiplier algebra of $L_1(\mathbb{G})$), since $\mathbb{G}$ is co-amenable. In particular, we prove that a quantum group $\mathbb{G}$ is co-amenable and SIN if and only if the Banach algebra $LUC(\mathbb{G})_R^*$ is unital. It is interesting to compare this result with [3, Theorem 3.1], where Bédos and Tuset showed that a quantum group $\mathbb{G}$ is co-amenable if and only if the algebra $C_0(\mathbb{G})^*$ is unital, which is also shown to be equivalent to $LUC(\mathbb{G})^*$ being unital (Theorem 15). The characterizations of SIN quantum groups obtained in this section are original even for locally compact groups. In the course of this investigation, we also obtain some other new characterizations of SIN-groups (Theorem 19). Results in this section illustrate that the SIN property is intrinsically related to topological centre problems.

Section 5 is devoted to the study of interrelationships between strong Arens irregularity and quotient strong Arens irregularity. First, using the connection between the Banach algebras $RM(A)$ and $\mathfrak{Z}_t(\langle A^*A\rangle^*)$, we extend a characterization of unital Banach algebras by Lau and Ülger [28] to the larger class of Banach algebras introduced recently by the authors [16] (Theorem 22). We prove a characterization of left quotient strong Arens irregularity in terms of $\mathfrak{Z}_t(A^{**}, \Box)$ (Theorem 23). Then, for the class of Banach algebras studied in [16], we give sufficient conditions to ensure the equivalence between (left, respectively, right) strong Arens irregularity and (left, respectively, right) quotient strong Arens irregularity. For involutive Banach algebras A from this class, we obtain some criteria for determining when the quotient strong Arens irregularity of A implies the strong Arens irregularity of A (Theorem 29). We prove that if the left quotient strong Arens irregularity of A is strengthened by replacing $RM(A)$ with the multiplier algebra $M(A)$, then the left strong Arens irregularity of A can be determined by testing elements of $\mathfrak{Z}_t(A^{**}, \Box)$ against one particular element of A^{**} (Theorem 30). In this situation, we have one test point in A^{**} to characterize A inside $\mathfrak{Z}_t(A^{**}, \Box)$. In this context, note that in [7, Definition 12.3], Dales, Lau, and Strauss introduced the concept of a dtc set in A^{**} (standing for "determining the topological centre") to characterize A inside A^{**} through a topological centre condition. We close Section 5 with characterizations of quotient strong Arens irregularity for all quantum group algebras as well as a characterization of quantum groups $\mathbb{G}$ satisfying $\mathfrak{Z}_t(LUC(\mathbb{G})^*_R) = RM(L_1(\mathbb{G}))$ (Theorems 32 and 33). The study in this section shows the complex nature of topological centre problems in certain aspects, noticing that Banach algebras like $RM(A), LM(A), \mathfrak{Z}_t(\langle A^*A\rangle^*), \mathfrak{Z}_t(\langle AA^*\rangle^*), \mathfrak{Z}_t(A^{**}, \Box)$, and $\mathfrak{Z}_t(A^{**}, \Diamond)$ all play a rôle even when only a one-sided topological centre is considered.

The paper concludes in Section 6 with some examples of Arens irregular Banach algebras, complementing some assertions in Section 5. Some open questions in Lau and Ülger [28] are answered there in the negative.

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2. Preliminaries

Let $\mathcal{B}$ be an algebra equipped with a topology σ such that $\mathcal{B}$ is a topological linear space. Assume that $\mathcal{B}$ is also a right topological semigroup under the multiplication. That is, for any fixed $y \in \mathcal{B}$, the map $x \longmapsto xy$ is continuous on $\mathcal{B}$ (cf. Berglund, Junghenn and Milnes [4]). The topological centre $\mathfrak{Z}_t(\mathcal{B})$ of $\mathcal{B}$ is defined to be the set of $y \in \mathcal{B}$ such that the map $x \longmapsto yx$ is continuous on $\mathcal{B}$. If $\mathcal{B}$ is a left topological semigroup under the multiplication, the topological centre $\mathfrak{Z}_t(\mathcal{B})$ of $\mathcal{B}$ is defined analogously. In the rest of the paper, if $\mathcal{B}$ is a subspace of a given dual Banach space, the topology σ on $\mathcal{B}$ is taken to be the relative weak*-topology.

Throughout this paper, A denotes a Banach algebra with a faithful multiplication; that is, for any $a \in A$, we have $a = 0$ if $aA = \{0\}$ or $Aa = \{0\}$. We use $\langle A^*A\rangle$ and $\langle AA^*\rangle$ to denote the closed linear spans of the module products A^*A and AA^*, respectively. By Cohen's factorization theorem, $\langle A^*A\rangle = A^*A$ if A has a bounded right approximate identity (BRAI), and $\langle AA^*\rangle = AA^*$ if A has a bounded left approximate identity (BLAI).

For any fixed $m \in A^{**}$, the maps $n \longmapsto n \Box m$ and $n \longmapsto m \Diamond n$ are weak*–weak* continuous on A^{**}. Then with the weak*-topology, $(A^{**}, \Box)$ is a right topological semigroup and $(A^{**}, \Diamond)$ is a left topological semigroup. Their topological centres

$$\mathfrak{Z}_t(A^{**}, \Box) = \{m \in A^{**}\colon \text{ the map } n \longmapsto m \Box n \text{ is weak*–weak* continuous on } A^{**}\}$$

and

$$\mathfrak{Z}_t(A^{**}, \Diamond) = \{m \in A^{**}\colon \text{ the map } n \longmapsto n \Diamond m \text{ is weak*–weak* continuous on } A^{**}\}$$

are called the left and the right topological centres of A^{**}, respectively. It is seen that

$$\mathfrak{Z}_t(A^{**}, \Box) = \{m \in A^{**}\colon m \Box n = m \Diamond n \text{ for all } n \in A^{**}\},$$

and

$$\mathfrak{Z}_t(A^{**}, \Diamond) = \{m \in A^{**}\colon n \Diamond m = n \Box m \text{ for all } n \in A^{**}\}.$$

Therefore, A is Arens regular if and only if $\mathfrak{Z}_t(A^{**}, \Box) = \mathfrak{Z}_t(A^{**}, \Diamond) = A^{**}$.

Clearly, $A \subseteq \mathfrak{Z}_t(A^{**}, \Box) \cap \mathfrak{Z}_t(A^{**}, \Diamond)$. The algebra A is left strongly Arens irregular (LSAI) if $\mathfrak{Z}_t(A^{**}, \Box) = A$, right strongly Arens irregular (RSAI) if $\mathfrak{Z}_t(A^{**}, \Diamond) = A$, and strongly Arens irregular if A is both LSAI and RSAI (cf. Dales and Lau [6]). In contrast to the situation for Arens regularity, there are LSAI Banach algebras which are not RSAI (cf. Section 6).

The Banach space $\langle A^*A\rangle$ is a closed A-submodule of A^*. It is also left introverted in A^*; that is, $m \Box x \in \langle A^*A\rangle$ for all $x \in \langle A^*A\rangle$ and $m \in \langle A^*A\rangle^*$, where $m \Box x = \widetilde{m} \Box x$ for any extension $\widetilde{m} \in A^{**}$ of m. This is equivalent to that $\langle A^*A\rangle$ is a left $(A^{**}, \Box)$-submodule of A^* (cf. Dales and Lau [6, Proposition 5.2]). Then $\langle A^*A\rangle^*$ is a Banach algebra under the multiplication defined by

$$\langle m \Box n, x\rangle = \langle m, n \Box x\rangle \big(x \in \langle A^*A\rangle,\ m, n \in \langle A^*A\rangle^*\big).$$

The multiplication $\Box$ on $\langle A^*A\rangle^*$ is induced by the left Arens product on A^{**}. That is, if m, $n \in \langle A^*A\rangle^*$ and $\widetilde{m}, \widetilde{n} \in A^{**}$ are extensions of m, n to A^*, respectively, then $\widetilde{m} \Box \widetilde{n}$ is an extension of $m \Box n$ to A^*. In fact, for $x \in A^*$, $n \in \langle A^*A\rangle^*$, and $w \in A^{**}$, $n \Box x \in A^*$ and $w \Box n \in A^{**}$ can be defined analogously. Then $\widetilde{m} \Box n = \widetilde{m} \Box \widetilde{n}$ for all $m, n \in \langle A^*A\rangle^*$. The canonical quotient map $\pi : A^{**} \longrightarrow \langle A^*A\rangle^*$ gives the isometric algebra isomorphism

$$\big(\langle A^*A\rangle^*, \Box\big) \cong (A^{**}, \Box)/\langle A^*A\rangle^{\perp},$$

where $\langle A^*A\rangle^{\perp} = \{m \in A^{**}\colon\ m|_{\langle A^*A\rangle} = 0\}$ is a closed ideal in $(A^{**}, \Box)$. For any fixed $m \in \langle A^*A\rangle^*$, the map $n \longmapsto n \Box m$ is weak*–weak* continuous on $\langle A^*A\rangle^*$. Hence, $(\langle A^*A\rangle^*, \Box)$ with the weak*-topology is a right topological semigroup, and its topological centre is given by

$$\mathfrak{Z}_t\big(\langle A^*A\rangle^*\big) = \big\{m \in \langle A^*A\rangle^*\colon\ n \longmapsto m \Box n \text{ is weak*–weak* continuous on } \langle A^*A\rangle^*\big\}.$$

We say that A is *left quotient Arens regular* if $\mathfrak{Z}_t(\langle A^*A\rangle^*) = \langle A^*A\rangle^*$.

Since $\langle A^*A\rangle$ is a left A-module, for $x \in \langle A^*A\rangle$ and $m \in \langle A^*A\rangle^*$, $x \Diamond \widetilde{m} \in A^*$ is independent of the choice of an extension $\widetilde{m} \in A^*$ of m. We denote $x \Diamond \widetilde{m}$ by $x \Diamond m$. However, when $\langle A^*A\rangle$ is not right introverted in A^*, $x \Diamond m$ may not be in $\langle A^*A\rangle$. Let

$$\langle A^*A\rangle^*_R = \big\{m \in \langle A^*A\rangle^*\colon\ \langle A^*A\rangle \Diamond m \subseteq \langle A^*A\rangle\big\}.$$

Then $\langle A^*A\rangle^*_R$ is a norm closed subspace of $\langle A^*A\rangle^*$. For $m \in \langle A^*A\rangle^*_R$ and $n \in \langle A^*A\rangle^*$, we define $m \Diamond n \in \langle A^*A\rangle^*$ by

$$\langle x, m \Diamond n\rangle = \langle x \Diamond m, n\rangle \big(x \in \langle A^*A\rangle\big),$$

and we see that $\widetilde{m} \Diamond \widetilde{n}$ is an extension of $m \Diamond n$ to A^*. It is also easy to see that for all m, $n \in \langle A^*A\rangle^*_R$, and $p \in \langle A^*A\rangle^*$, we have $m \Diamond n \in \langle A^*A\rangle^*_R$ as well as

$$\|m \Diamond n\| \leqslant \|m\|\|n\| \quad \text{and} \quad m \Diamond (n \Diamond p) = (m \Diamond n) \Diamond p.$$

In particular, $(\langle A^*A\rangle^*_R, \Diamond)$ is a Banach algebra.

It is evident that $\langle A^*A\rangle^{\perp}$ is a right ideal in $(A^{**}, \Diamond)$. It is seen that $\langle A^*A\rangle^{\perp}$ is a two-sided ideal in $(A^{**}, \Diamond)$ if and only if $\langle A^*A\rangle$ is two-sided introverted in A^* (cf. Dales and Lau [6, Proposition 5.4] for the "only if" part). Let

$$A^{**}_R = \pi^{-1}\big(\langle A^*A\rangle^*_R\big) = \big\{m \in A^{**}\colon\ \langle A^*A\rangle \Diamond m \subseteq \langle A^*A\rangle\big\}.$$

Then A_R^{**} is a closed subalgebra of $(A^{**},\Diamond)$, $\langle A^*A\rangle^\perp$ is a closed two-sided ideal in $(A_R^{**},\Diamond)$, and $\pi|_{A_R^{**}}$ induces the isometric algebra isomorphism

$$\big(\langle A^*A\rangle_R^*,\Diamond\big)\cong\big(A_R^{**},\Diamond\big)/\langle A^*A\rangle^\perp.$$

Moreover, A_R^{**} is the largest closed subalgebra B of $(A^{**},\Diamond)$ such that $\langle A^*A\rangle$ is a right B-submodule of A^* and $\langle A^*A\rangle^\perp$ is a closed ideal in B.

With the relative weak*-topology, $(\langle A^*A\rangle_R^*,\Diamond)$ is a left topological semigroup. By definition, $\mathfrak{Z}_t(\langle A^*A\rangle_R^*)$ is the set of $m\in\langle A^*A\rangle_R^*$ such that the map $n\longmapsto n\Diamond m$ is continuous on $\langle A^*A\rangle_R^*$ with respect to the relative $\sigma(\langle A^*A\rangle^*,\langle A^*A\rangle)$-topology. Since the multiplication in A is faithful, the map $A\longrightarrow\langle A^*A\rangle_R^*$, $a\longmapsto\dot a=a|_{\langle A^*A\rangle}$ is injective. Note that $\langle A^*A\rangle^*$ is also an A-module. It is easy to see that for all $a\in A$ and $n\in\langle A^*A\rangle^*$, $a\cdot n=\dot a\,\Box\,n=\dot a\Diamond n$. In the sequel, $a\cdot n$ will denote both $\dot a\,\Box\,n\in\langle A^*A\rangle^*$ and $a\,\Box\,n\in A^{**}$. It is obvious that $A\subseteq\mathfrak{Z}_t(\langle A^*A\rangle^*)\cap\mathfrak{Z}_t(\langle A^*A\rangle_R^*)$. Since $A\subseteq\langle A^*A\rangle^*$ is weak*-dense, we have

$$\mathfrak{Z}_t\big(\langle A^*A\rangle_R^*\big)=\big\{m\in\langle A^*A\rangle_R^*\colon\ n\Diamond m=n\,\Box\,m\text{ for all }n\in\langle A^*A\rangle_R^*\big\},$$

and $\mathfrak{Z}_t(\langle A^*A\rangle_R^*)$ is a norm closed subalgebra of $(\langle A^*A\rangle_R^*,\Diamond)$ and of $(\langle A^*A\rangle^*,\Box)$.

Note that A_R^{**} and $\langle A^*A\rangle_R^*$ both contain A. Therefore, A_R^{**} (respectively, $\langle A^*A\rangle_R^*$) is weak*-closed in A^{**} (respectively, in $\langle A^*A\rangle^*$) if and only if $A_R^{**}=A^{**}$ (respectively, $\langle A^*A\rangle_R^*=\langle A^*A\rangle^*$). On the other hand, it is clear that $A^{**}=A_R^{**}$ if and only if $\langle A^*A\rangle^*=\langle A^*A\rangle_R^*$ if and only if $\langle A^*A\rangle$ is introverted in A^*. In this situation, there are two Arens products on $\langle A^*A\rangle^*$ so that $\langle A^*A\rangle^*$ has two topological centres: the usual topological centre is now given by

$$\mathfrak{Z}_t\big(\langle A^*A\rangle^*\big)=\big\{m\in\langle A^*A\rangle^*\colon\ m\,\Box\,n=m\Diamond n\text{ for all }n\in\langle A^*A\rangle^*\big\},$$

and the other topological centre with respect to $\Diamond$ is just $\mathfrak{Z}_t(\langle A^*A\rangle_R^*)$.

Analogously, $\langle AA^*\rangle$ is an A-module and is right introverted in A^*. As in the $\langle A^*A\rangle$ case, one can define the Banach algebras $(\langle AA^*\rangle^*,\Diamond)$, $({}_L\langle AA^*\rangle^*,\Box)$, and $({}_LA^{**},\Box)$, and consider the topological centres $\mathfrak{Z}_t(\langle AA^*\rangle^*)$ and $\mathfrak{Z}_t({}_L\langle AA^*\rangle^*)$. The right quotient Arens regularity can be defined similarly.

We point out that for a norm closed left (respectively, right) introverted A-submodule X of A^*, the Banach algebras $(X^*,\Box)$ and $(X_R^*,\Diamond)$ (respectively, $(X^*,\Diamond)$ and $({}_LX^*,\Box)$), and their topological centres can also be defined. In the present paper, however, we will focus on the case where X is either $\langle A^*A\rangle$ or $\langle AA^*\rangle$. See Dales and Lau [6] for more information on topological centres $\mathfrak{Z}_t(X^*,\Box)$ and $\mathfrak{Z}_t(X^*,\Diamond)$. The reader is also referred to Grosser [12] for a systematic study of left (respectively, right) Banach modules of the form $\langle V^*A\rangle$ (respectively, $\langle AV^*\rangle$), where V is a left (respectively, right) Banach A-module.

It is well known that if A is the group algebra $L_1(G)$ of a locally compact group G, then $\langle A^*A\rangle=LUC(G)$ (respectively, $\langle AA^*\rangle=RUC(G)$), the C^*-algebra of bounded left (respectively, right) uniformly continuous functions on G. The space $LUC(G)\cap RUC(G)$ is denoted by $UC(G)$, which is the C^*-algebra of bounded uniformly continuous functions on G.

Let $LM(A)$ and $RM(A)$ be the left and the opposite right multiplier algebras of A, respectively. That is,

$$LM(A)=\big\{T\in B(A)\colon\ T(ab)=T(a)b\text{ for all }a,b\in A\big\},$$

and

$$RM(A) = \{T \in B(A)^{op}\colon\ T(ab) = aT(b) \text{ for all } a, b \in A\},$$

where $B(A)$ is the Banach algebra of bounded linear operators on A. As norm closed subalgebras of $B(A)$ and $B(A)^{op}$, respectively, $LM(A)$ and $RM(A)$ are Banach algebras. Let $M(A)$ denote the multiplier algebra of A, consisting of $(\mu_l, \mu_r) \in LM(A) \times RM(A)$ satisfying $a\mu_l(b) = \mu_r(a)b$ $(a, b \in A)$. If A has a BRAI (respectively, BLAI), then A can naturally be identified with a norm closed left (respectively, right) ideal in $LM(A)$ (respectively, in $RM(A)$). Any Banach algebra with a bounded approximate identity (BAI) has a faithful multiplication.

Some forms of the following lemma are known (cf. Dales [5, Theorem 2.9.49(iii)] and Lau and Ülger [28, Theorem 4.4]). For convenience, we include a complete proof here. We state only the $\langle A^*A\rangle$-version of these results.

Lemma 1. *Let A be a Banach algebra with a BRAI and E be a weak*-cluster point in A^{**} of a BRAI of A.*

(i) *The map $RM(A) \longrightarrow (A^{**}, \Box)$, $\mu \longmapsto \mu^{**}(E)$ is an injective algebra homomorphism with range contained in $A_R^{**} \cap (E \Box A^{**})$.*
(ii) *$(\langle A^*A\rangle^*, \Box)$ has an identity, and the map $\mu \longmapsto \mu' = \mu^{**}(E)|_{\langle A^*A\rangle}$ is a unital injective algebra homomorphism from $RM(A)$ into $\mathfrak{Z}_t(\langle A^*A\rangle^*)$ satisfying*

$$\langle \mu', f \cdot a\rangle = \langle \mu(a), f\rangle \quad (f \in A^*,\ a \in A).$$

Proof. It is easy to see that E is a right identity of $(A^{**}, \Box)$ satisfying

$$n \Box \mu^{**}(E) = \mu^{**}(n) \quad \text{and} \quad n \Diamond \mu^{**}(E) = \mu^{**}(n \Diamond E) \quad \big(\mu \in RM(A),\ n \in A^{**}\big). \qquad (\dagger)$$

(i) Let $\mu, \nu \in RM(A)$, $a \in A$, and $f \in A^*$. By (†), we have $\mu^{**}(\nu^{**}(E)) = \nu^{**}(E) \Box \mu^{**}(E)$, $(f \cdot a) \Diamond \mu^{**}(E) = f \cdot \mu(a)$, and $\mu^{**}(E) = E \Box \mu^{**}(E)$. Then the assertion follows.

(ii) Let $\mu \in RM(A)$. Note that $a \cdot \mu' = a \cdot \mu^{**}(E) = \mu(a) \in A$ for all $a \in A$. Then, for all $a \in A$, $f \in A^*$, and $p \in \langle A^*A\rangle^*$, we have

$$\langle \mu', f \cdot a\rangle = \langle \mu^{**}(E), f \cdot a\rangle = \langle \mu(a), f\rangle \quad \text{and} \quad \langle \mu' \Box p, f \cdot a\rangle = \langle p, f \cdot \mu(a)\rangle.$$

Thus $\nu \longmapsto \nu'$ maps $RM(A)$ injectively into $\mathfrak{Z}_t(\langle A^*A\rangle^*)$. Since $\nu \longmapsto \nu'$ is the composition of the map in (i) and the canonical quotient map $(A^{**}, \Box) \longrightarrow (\langle A^*A\rangle^*, \Box)$, it is an algebra homomorphism. Therefore, $(\langle A^*A\rangle^*, \Box)$ has an identity by [13, Theorem 4(i)]. Finally, it is easy to see that the map $RM(A) \longrightarrow (\langle A^*A\rangle^*, \Box)$, $\mu \longmapsto \mu'$ is unital. $\Box$

Lemma 1 modifies Dales [5, Theorem 2.9.49(iii)] and Lau and Ülger [28, Theorem 4.4], where A was assumed to have a BAI of norm 1, and an isometric embedding from $M(A)$ into $(A^{**}, \Box)$, respectively, from $RM(A)$ into $\mathfrak{Z}_t(\langle A^*A\rangle^*)$, was obtained.

We note that in Lemma 1(i), though $\mu \longmapsto \mu^{**}(E)$ maps $RM(A)$ into A_R^{**}, it is not an algebra homomorphism from $RM(A)$ to $(A_R^{**}, \Diamond)$ in general.

3. Topological centres of quotient algebras

Theorem 2. *Let A be a Banach algebra.*

(i) $\mathfrak{Z}_t(\langle A^*A\rangle^*) = \{m \in \langle A^*A\rangle_R^*\colon\ m \,\Box\, n = m \diamond n$ *for all* $n \in \langle A^*A\rangle^*\}$.
(ii) $\mathfrak{Z}_t(\langle AA^*\rangle^*) = \{m \in {}_L\langle AA^*\rangle^*\colon\ n \diamond m = n \,\Box\, m$ *for all* $n \in \langle AA^*\rangle^*\}$.

Proof. We prove (i); the proof of (ii) follows from similar arguments.

Obviously, if $m \in \langle A^*A\rangle_R^*$ and $m \,\Box\, n = m \diamond n$ for all $n \in \langle A^*A\rangle^*$, then the map $n \longmapsto m \,\Box\, n$ is weak*–weak* continuous on $\langle A^*A\rangle^*$; that is, $m \in \mathfrak{Z}_t(\langle A^*A\rangle^*)$.

Conversely, suppose that $m \in \mathfrak{Z}_t(\langle A^*A\rangle^*)$. Let $a \in A$, $n \in A^{**}$, and $p = n|_{\langle A^*A\rangle}$. Then $\langle (a \cdot m) \,\Box\, n, f\rangle = \langle m \,\Box\, p, f \cdot a\rangle$ for all $f \in A^*$. It follows that $a \cdot m \in \mathfrak{Z}_t(A^{**}, \Box)$. Hence,

$$\text{if } \mu \in \mathfrak{Z}_t\big(\langle A^*A\rangle^*\big), \quad \text{then } A \cdot \mu \subseteq \mathfrak{Z}_t(A^{**}, \Box). \tag{$\ddagger$}$$

It is known that

$$A^* \diamond \mathfrak{Z}_t(A^{**}, \Box) \subseteq \langle A^*A\rangle \quad \text{and} \quad \mathfrak{Z}_t(A^{**}, \diamond) \,\Box\, A^* \subseteq \langle AA^*\rangle. \tag{$\sharp$}$$

(Cf. Dales and Lau [6, Proposition 2.20] and Lau and Ülger [28, Lemma 3.1a)].) By the assertions ($\ddagger$) and ($\sharp$), we have

$$(A^*A) \diamond m \subseteq A^* \diamond (A \cdot m) \subseteq A^* \diamond \mathfrak{Z}_t(A^{**}, \Box) \subseteq \langle A^*A\rangle.$$

Therefore, $m \in \langle A^*A\rangle_R^*$. Since A is weak*-dense in $\langle A^*A\rangle^*$, we have $m \,\Box\, n = m \diamond n$ for all $n \in \langle A^*A\rangle^*$. $\Box$

It follows from Theorem 2 that $\mathfrak{Z}_t(\langle A^*A\rangle^*)$ is a subalgebra of $(\langle A^*A\rangle^*, \Box)$ and of $(\langle A^*A\rangle_R^*, \diamond)$, and $\mathfrak{Z}_t(\langle AA^*\rangle^*)$ is a subalgebra of $(\langle AA^*\rangle^*, \diamond)$ and of $({}_L\langle AA^*\rangle^*, \Box)$. Also, it is seen from ($\sharp$) that

$$\mathfrak{Z}_t(A^{**}, \Box) = \big\{m \in A_R^{**}\colon\ m \,\Box\, n = m \diamond n \text{ for all } n \in A^{**}\big\},$$

and

$$\mathfrak{Z}_t(A^{**}, \diamond) = \{m \in {}_LA^{**}\colon\ n \diamond m = n \,\Box\, m \text{ for all } n \in A^{**}\}.$$

Therefore, Theorem 2 shows that these descriptions of $\mathfrak{Z}_t(A^{**}, \Box)$ and $\mathfrak{Z}_t(A^{**}, \diamond)$ have their analogues for $\mathfrak{Z}_t(\langle A^*A\rangle^*)$ and $\mathfrak{Z}_t(\langle AA^*\rangle^*)$, respectively.

The corollary below generalizes Lau and Ülger [28, Lemma 3.1c)], where they assumed that A has a BAI.

Corollary 3. *Let A be a Banach algebra.*

(i) *For* $\mu \in \langle A^*A\rangle^*$, $\mu \in \mathfrak{Z}_t(\langle A^*A\rangle^*)$ *if and only if* $A \cdot \mu \subseteq \mathfrak{Z}_t(A^{**}, \Box)$.
(ii) *For* $\mu \in \langle AA^*\rangle^*$, $\mu \in \mathfrak{Z}_t(\langle AA^*\rangle^*)$ *if and only if* $\mu \cdot A \subseteq \mathfrak{Z}_t(A^{**}, \diamond)$.

Consequently, the canonical quotient algebra homomorphisms $(A_R^{**}, \diamond) \longrightarrow (\langle A^*A\rangle_R^*, \diamond)$ *and* $(A^{**}, \Box) \longrightarrow (\langle A^*A\rangle^*, \Box)$ *both map* $\mathfrak{Z}_t(A^{**}, \Box)$ *into* $\mathfrak{Z}_t(\langle A^*A\rangle^*)$.

Proof. We prove (i); the proof of (ii) is similar. Let $\mu \in \langle A^*A\rangle^*$. By assertion (‡) above, we suppose that $A \cdot \mu \subseteq \mathfrak{Z}_t(A^{**}, \Box)$ and show that $\mu \in \mathfrak{Z}_t(\langle A^*A\rangle^*)$.

First, for all $f \in A^*$ and $a \in A$, from assertion (♯) above, we have $(f \cdot a) \Diamond \mu = f \Diamond (a \cdot \mu) \in \langle A^*A\rangle$. Thus $\mu \in \langle A^*A\rangle^*_R$. Next, let $n \in \langle A^*A\rangle^*$ and $\widetilde{n} \in A^{**}$ be an extension of n. Then

$$\langle \mu \Box n, f \cdot a\rangle = \big\langle (a \cdot \mu) \Box \widetilde{n}, f\big\rangle = \big\langle (a \cdot \mu) \Diamond \widetilde{n}, f\big\rangle = \langle f \cdot a, \mu \Diamond n\rangle \quad (f \in A^*,\ a \in A).$$

Therefore, $\mu \Box n = \mu \Diamond n$ for all $n \in \langle A^*A\rangle^*$, and hence $\mu \in \mathfrak{Z}_t(\langle A^*A\rangle^*)$ by Theorem 2(i).

The final assertion follows from Theorem 2(i) and the surjectivity of the canonical quotient map $A^{**} \longrightarrow \langle A^*A\rangle^*$. □

By Corollary 3, we have immediately the following result, which will be needed in Section 5. The assertion (ii) below extends [28, Corollary 3.2], where Lau and Ülger showed that $A \cdot \mathfrak{Z}_t(A^{**}, \Box) = A \cdot \mathfrak{Z}_t(\langle A^*A\rangle^*)$ if A has a BAI (see also Dales and Lau [6, Theorem 5.12]). We note that the condition $\langle A^2\rangle = A$ in (i) below is satisfied by all quantum group algebras $L_1(\mathbb{G})$ (cf. Fact 1 in Section 4).

Corollary 4. *Let A be a Banach algebra.*

(i) *If $\langle A^2\rangle = A$, then $A \cdot \mathfrak{Z}_t(A^{**}, \Box) \subseteq A$ if and only if $A \cdot \mathfrak{Z}_t(\langle A^*A\rangle^*) \subseteq A$.*
(ii) *If A factors (that is, $A^2 = A$), in particular, if A has a BLAI or a BRAI, then $A \cdot \mathfrak{Z}_t(A^{**}, \Box) = A \cdot \mathfrak{Z}_t(\langle A^*A\rangle^*)$.*

Proof. (i) Note that $A \cdot \mathfrak{Z}_t(A^{**}, \Box) \subseteq A \cdot \mathfrak{Z}_t(\langle A^*A\rangle^*)$ since $a \cdot m = a \cdot p$ for all $a \in A$ and $m \in A^{**}$ with $p = m|_{\langle A^*A\rangle}$, and $p \in \mathfrak{Z}_t(\langle A^*A\rangle^*)$ if $m \in \mathfrak{Z}_t(A^{**}, \Box)$. Then the assertion follows from Corollary 3.

(ii) If $A^2 = A$, then, combining Corollary 3(i) with the inclusion above, we have

$$A \cdot \mathfrak{Z}_t\big(\langle A^*A\rangle^*\big) = A^2 \cdot \mathfrak{Z}_t\big(\langle A^*A\rangle^*\big) \subseteq A \cdot \mathfrak{Z}_t(A^{**}, \Box) \subseteq A \cdot \mathfrak{Z}_t\big(\langle A^*A\rangle^*\big).$$

Therefore, $A \cdot \mathfrak{Z}_t(A^{**}, \Box) = A \cdot \mathfrak{Z}_t(\langle A^*A\rangle^*)$. □

Let G be a locally compact group G. For $m \in LUC(G)^*$ and $f \in LUC(G)$, the bounded complex-valued function $m_r(f)$ on G is given by $m_r(f)(s) = \langle m, f_s\rangle$ $(s \in G)$, where f_s is the right translate of f by s. If $m_r(f) \in LUC(G)$ for all $f \in LUC(G)$, then, for each $n \in LUC(G)^*$, the product $m * n \in LUC(G)^*$ is defined by

$$\langle f, m * n\rangle = \big\langle m_r(f), n\big\rangle \quad \big(f \in LUC(G)\big).$$

(Cf. Berglund, Junghenn and Milnes [4, Definition 2.2.8] and Lau [25].) We note here that for $m \in L_\infty(G)^*$ and $f \in L_\infty(G)$, the function $s \longmapsto \langle m, f_s\rangle$ may be not even measurable on G (cf. Rudin [39], Talagrand [43], and Wells [46]). Following the notation used in [4], we let

$$Z_U(G) = \big\{m \in LUC(G)^*\colon\ m_r(f) \in LUC(G) \text{ for all } f \in LUC(G)\big\}.$$

Then $(Z_U(G), *)$ is a Banach algebra (cf. [4, Lemma 2.2.9]).

We use $LUC_{\ell_\infty}(G)$ to denote $LUC(G)$ when it is considered as a subspace of $\ell_\infty(G)$. Obviously, $LUC_{\ell_\infty}(G)$ is a closed $\ell_1(G)$-submodule of $\ell_\infty(G)$. Also, $LUC_{\ell_\infty}(G)$ is left introverted in $\ell_\infty(G)$ (cf. Dales and Lau [6, Theorem 7.19]). It is easy to see that for $f \in LUC_{\ell_\infty}(G)$ and $s \in G$, we have $\delta_s \cdot f = f_s$, where δ_s denotes the point mass at s. Let $\Diamond_{\ell_1}$ denote the right Arens product on $\ell_1(G)^{**}$. Then, for all $s \in G$, $m \in LUC_{\ell_\infty}(G)^*$, and $f \in LUC_{\ell_\infty}(G)$, we have

$$(f \Diamond_{\ell_1} m)(s) = \langle \delta_s, f \Diamond_{\ell_1} m \rangle = \langle \delta_s \cdot f, m \rangle = \langle m, f_s \rangle = m_r(f)(s).$$

It follows that

$$m_r(f) = f \Diamond_{\ell_1} m \quad \text{and} \quad q * m = q \Diamond_{\ell_1} m \quad \big(f \in LUC(G),\ m \in LUC(G)^*,\ q \in Z_U(G)\big).$$

Hence, we have

$$Z_U(G) = \big\{m \in LUC(G)^*\colon\ f \Diamond_{\ell_1} m \in LUC(G) \text{ for all } f \in LUC(G)\big\},$$

and

$$\big(Z_U(G), *\big) = \big(LUC_{\ell_\infty}(G)^*_R, \Diamond_{\ell_1}\big).$$

Therefore, $LUC(G)$ is two-sided introverted in $\ell_\infty(G)$ if and only if $Z_U(G) = LUC(G)^*$, and this is exactly the case when G is an SIN-group (i.e., the identity e_G of G has a basis consisting of compact sets invariant under inner automorphisms), which is also equivalent to that the left and the right uniformities on G coincide (cf. [4, Theorem 4.4.5] and [15, (4.14g)]).

For $m \in LUC(G)^*$ and $f \in LUC(G)$, let $m_l(f)(s) = \langle m, {}_sf \rangle$ $(s \in G)$, where ${}_sf$ is the left translate of f by s. According to Lau [23, Lemma 3], $m \Box f = m_l(f)$, or we can write it as $m \Box f = m \Box_{\ell_1} f$, where $\Box_{\ell_1}$ is the left Arens product on $\ell_1(G)^{**}$. It follows that

$$m \Box n = m \Box_{\ell_1} n \quad \big(m, n \in LUC(G)^*\big).$$

Consequently, we have

$$\big(LUC(G)^*, \Box\big) = \big(LUC_{\ell_\infty}(G)^*, \Box_{\ell_1}\big) \quad \text{and} \quad \mathfrak{Z}_t\big(LUC(G)^*\big) = \mathfrak{Z}_t\big(LUC_{\ell_\infty}(G)^*, \Box_{\ell_1}\big).$$

See Dales and Lau [6, Theorem 5.15] for the general Banach algebra case.

In [25, Lemma 2], Lau proved that

$$\mathfrak{Z}_t\big(LUC(G)^*\big) = \big\{m \in Z_U(G)\colon\ m \Box n = m * n \text{ for all } n \in LUC(G)^*\big\}.$$

This result was extended by Dales and Lau [6, Proposition 11.6] to all convolution Beurling algebras. Therefore, [25, Lemma 2] indeed characterizes $\mathfrak{Z}_t(LUC(G)^*)$ via the $\ell_1(G)$-module structure on $LUC(G)$ (or the group action of G on $LUC(G)$). More precisely, we have $\mathfrak{Z}_t(LUC(G)^*) = \mathfrak{Z}_t(LUC_{\ell_\infty}(G)^*, \Box_{\ell_1})$, and we even have the following description of $\mathfrak{Z}_t(LUC(G)^*)$ in the format of Theorem 2:

$$\mathfrak{Z}_t\big(LUC(G)^*\big) = \big\{m \in LUC_{\ell_\infty}(G)^*_R\colon\ m \Box_{\ell_1} n = m \Diamond_{\ell_1} n \text{ for all } n \in LUC_{\ell_\infty}(G)^*\big\}.$$

In Theorem 2(i) with $A = L_1(G)$, we consider $LUC(G)^*_R$ instead of $Z_U(G)$ so that the group action on $LUC(G)$ is replaced by the Banach $L_1(G)$-module action. Hence, Theorem 2 is a Banach algebraic version of [25, Lemma 2]. We point out that, in general, $(LUC(G)^*_R, \Diamond) \neq$

$(Z_U(G), *)$; that is, $(LUC(G)^*_R, \diamondsuit) \neq (LUC_{\ell_\infty}(G)^*_R, \diamondsuit_{\ell_1})$ (see Theorem 19 in Section 4). Therefore, even for $A = L_1(G)$, Theorem 2 does not follow from [25, Lemma 2].

Let X be either $\langle A^*A\rangle$ or $\langle AA^*\rangle$. For $x \in X$ and $m \in X^*$, we write

$$m_L(x) = m \,\Box\, x, \quad \text{and} \quad m_R(x) = x \diamondsuit m.$$

When $X = \langle A^*A\rangle$ and $m \in X^*$, we have $m_L \in B(X)$ and $m_R \in B(X, A^*)$, and $m_R \in B(X)$ if and only if $m \in X^*_R$. Similar assertions hold for $X = \langle AA^*\rangle$.

For brevity, for results in the rest of this section, we state only their $\langle A^*A\rangle$-versions.

Proposition 5. *Let A be a Banach algebra.*

(i) *If $m \in \mathfrak{Z}_t(\langle A^*A\rangle^*)$, then $m \in \langle A^*A\rangle^*_R$ and $m_R n_L = n_L m_R$ for all $n \in \langle A^*A\rangle^*$.*
(ii) *If $m \in \mathfrak{Z}_t(\langle A^*A\rangle^*_R)$, then $m_L n_R = n_R m_L$ for all $n \in \langle A^*A\rangle^*_R$.*

Proof. We prove (i); assertion (ii) can be proved similarly.

Let $m \in \mathfrak{Z}_t(\langle A^*A\rangle^*)$. By Theorem 2(i), $m \in \langle A^*A\rangle^*_R$ and $m \,\Box\, n = m \diamondsuit n$ for all $n \in \langle A^*A\rangle^*$. Let $n \in \langle A^*A\rangle^*$. Then, for all $x \in \langle A^*A\rangle$ and $a \in A$, we have

$$\bigl\langle m_R n_L(x), a\bigr\rangle = \bigl\langle a, (n \,\Box\, x) \diamondsuit m\bigr\rangle = \bigl\langle a \cdot (n \,\Box\, x), m\bigr\rangle = \bigl\langle (a \cdot n) \,\Box\, x, m\bigr\rangle = \bigl\langle m \,\Box\, (a \cdot n), x\bigr\rangle,$$

and

$$\bigl\langle n_L m_R(x), a\bigr\rangle = \bigl\langle n \,\Box\, (x \diamondsuit m), a\bigr\rangle = \langle x \diamondsuit m, a \cdot n\rangle = \bigl\langle x, m \diamondsuit (a \cdot n)\bigr\rangle = \bigl\langle m \,\Box\, (a \cdot n), x\bigr\rangle.$$

Therefore, $m_R n_L = n_L m_R$ for all $n \in \langle A^*A\rangle^*$. $\square$

Proposition 5 is motivated by [34, Proposition 1.2.13], where Neufang proved that if G is a locally compact group and $m \in LUC(G)^*$, then

$$m \in \mathfrak{Z}_t\bigl(LUC(G)^*\bigr) \quad \text{if and only if} \quad m \in Z_U(G) \quad \text{and}$$
$$m_r n_l = n_l m_r \quad \text{for all } n \in LUC(G)^*.$$

However, we note that for $m \in LUC(G)^*$, in general, $m_R \neq m_r$, though $m_L = m_l$. In fact, the converse of Proposition 5 is not true in general. For example, if G is non-SIN and $m \in LUC(G)^*$ is non-zero but vanishing on $UC(G)$, then $m_R = 0 \neq m_r$. In this situation, $m \in LUC(G)^*_R$, $m_R n_L = n_L m_R = 0$ for all $n \in LUC(G)^*$, but $m \notin \mathfrak{Z}_t(LUC(G)^*)$, since $\mathfrak{Z}_t(LUC(G)^*)$ is equal to the measure algebra $M(G)$ of G (cf. Lau [25]).

We observe that the identity e_G of G defines the identity δ_{e_G} of $(LUC(G)^*, \Box)$ and also gives the identity of $(Z_U(G), *)$. That is, δ_{e_G} is both an identity of $(LUC_{\ell_\infty}(G)^*, \Box_{\ell_1})$ and an identity of $(LUC_{\ell_\infty}(G)^*_R, \diamondsuit_{\ell_1})$. This fact plays a crucial rôle in the proof of the sufficiency part of [34, Proposition 1.2.13]. We are thus led to introducing the concept below for general Banach algebras.

Definition 6. Let A be a Banach algebra. An element e_0 of $\langle A^*A\rangle^*_R$ is a *strong identity* of $\langle A^*A\rangle^*$ if e_0 is a left identity of $(\langle A^*A\rangle^*, \Box)$ and a right identity of $(\langle A^*A\rangle^*_R, \diamondsuit)$. A strong identity of $\langle AA^*\rangle^*$ is defined similarly.

It is readily seen that if e_0 is a strong identity of $\langle A^*A\rangle^*$, then e_0 is both an identity of $(\langle A^*A\rangle^*, \Box)$ and an identity of $(\langle A^*A\rangle^*_R, \diamond)$.

Lemma 7. *Let A be a Banach algebra such that $\langle A^*A\rangle^*$ has a strong identity. Let $m \in \langle A^*A\rangle^*$.*

(i) *$m \in \mathfrak{Z}_t(\langle A^*A\rangle^*)$ if and only if $m \in \langle A^*A\rangle^*_R$ and $m_R n_L = n_L m_R$ for all $n \in \langle A^*A\rangle^*$.*
(ii) *$m \in \mathfrak{Z}_t(\langle A^*A\rangle^*_R)$ if and only if $m_L n_R = n_R m_L$ for all $n \in \langle A^*A\rangle^*_R$.*

Proof. We prove (i); the proof of (ii) is similar. By Proposition 5, we need only prove the sufficiency part.

Suppose that e_0 is a strong identity of $\langle A^*A\rangle$, $m \in \langle A^*A\rangle^*_R$, and $m_R n_L = n_L m_R$ for all $n \in \langle A^*A\rangle^*$. Let $x \in \langle A^*A\rangle$ and $n \in \langle A^*A\rangle^*$. Then $(n \Box x) \diamond m = n \Box (x \diamond m)$. Note that

$$\langle m \Box n, x\rangle = \langle m, n \Box x\rangle = \langle m \diamond e_0, n \Box x\rangle = \big\langle e_0, (n \Box x) \diamond m\big\rangle = \big\langle e_0, n \Box (x \diamond m)\big\rangle,$$

and

$$\langle m \diamond n, x\rangle = \langle x \diamond m, n\rangle = \langle e_0 \Box n, x \diamond m\rangle = \big\langle e_0, n \Box (x \diamond m)\big\rangle.$$

Therefore, we have $m \Box n = m \diamond n$ for all $n \in \langle A^*A\rangle^*$; so $m \in \mathfrak{Z}_t(\langle A^*A\rangle^*)$ by Theorem 2(i). $\Box$

From Theorem 2(i) and the proof of Proposition 5, it is seen that for $m \in \langle A^*A\rangle^*_R$,

$$\begin{aligned} & m_R n_L = n_L m_R \quad \text{for all } n \in \langle A^*A\rangle^* \\ \iff\quad & m \Box (a \cdot n) = m \diamond (a \cdot n) \quad \text{for all } n \in \langle A^*A\rangle^* \text{ and } a \in A \\ \iff\quad & (m \cdot a) \Box n = (m \cdot a) \diamond n \quad \text{for all } n \in \langle A^*A\rangle^* \text{ and } a \in A \\ \iff\quad & m \cdot A \subseteq \mathfrak{Z}_t\big(\langle A^*A\rangle^*\big). \end{aligned}$$

Therefore, combining these equivalences with Theorem 2 and Lemma 7, we have the following Banach algebraic extension of [34, Proposition 1.2.13].

Corollary 8. *Let A be a Banach algebra such that $\langle A^*A\rangle^*$ has a strong identity. Let $m \in \langle A^*A\rangle^*$. Then the following statements are equivalent.*

(i) *$m \in \mathfrak{Z}_t(\langle A^*A\rangle^*)$.*
(ii) *$m \in \langle A^*A\rangle^*_R$ and $m \Box n = m \diamond n$ for all $n \in \langle A^*A\rangle^*$.*
(iii) *$m \in \langle A^*A\rangle^*_R$ and $m_R n_L = n_L m_R$ for all $n \in \langle A^*A\rangle^*$.*
(iv) *$m \in \langle A^*A\rangle^*_R$ and $m \Box n = m \diamond n$ for all $n \in A \cdot \langle A^*A\rangle^*$.*
(v) *$\langle A^*A\rangle \diamond m \subseteq \langle A^*A\rangle$ and $m \cdot A \subseteq \mathfrak{Z}_t(\langle A^*A\rangle^*)$.*

4. Quotient algebras with a strong identity and SIN quantum groups

Let A be a Banach algebra. From the discussions in Section 3, it is natural to consider the question of when $\langle A^*A\rangle^*$ (respectively, $\langle AA^*\rangle^*$) has a strong identity. As mentioned earlier, for a locally compact group G, the identity δ_{e_G} of $(LUC(G)^*, \Box)$ is always an identity of $(Z_U(G), *)$. However, we will see from Theorem 19 below that δ_{e_G} may not be an identity of $(LUC(G)^*_R, \diamond)$.

Lemma 9. *Let A be a Banach algebra.*

(i) *If $(\langle A^*A\rangle_R^*, \Diamond)$ has a right identity, then $\langle A^*A\rangle = \langle AA^*A\rangle$.*
(ii) *Assume that A has a BRAI. If $\langle A^*A\rangle^*$ has a strong identity, then the map $\mu \longmapsto \mu'$ in Lemma 1(ii) maps $RM(A)$ into $\mathfrak{Z}_t(\langle A^*A\rangle^*) \cap \mathfrak{Z}_t(\langle A^*A\rangle_R^*)$.*

Proof. (i) Suppose that $m \in \langle A^*A\rangle^*$ and $m|_{\langle AA^*A\rangle} = 0$. Then $m \cdot a = 0$ for all $a \in A$. Thus $m \in \langle A^*A\rangle_R^*$, and $m \Diamond n = 0$ for all $n \in \langle A^*A\rangle^*$. Therefore, $\langle A^*A\rangle = \langle AA^*A\rangle$ if $(\langle A^*A\rangle_R^*, \Diamond)$ has a right identity.

(ii) Let e_0 be a strong identity of $\langle A^*A\rangle^*$. Then $e_0 = E|_{\langle A^*A\rangle}$, where E is a weak*-cluster point in A^{**} of a BRAI of A. Let $\mu \in RM(A)$. By Lemma 1(ii), we only have to show that $\mu' = \mu^{**}(E)|_{\langle A^*A\rangle} \in \mathfrak{Z}_t(\langle A^*A\rangle_R^*)$.

Let $m \in \langle A^*A\rangle_R^*$ and $\widetilde{m} \in A^{**}$ be an extension of m. By (†) in the proof of Lemma 1, $\widetilde{m} \Diamond \mu^{**}(E) = \mu^{**}(\widetilde{m} \Diamond E)$ and $\widetilde{m} \Box \mu^{**}(E) = \mu^{**}(\widetilde{m})$. For all $a \in A$ and $f \in A^*$, we have

$$\big\langle f \cdot a, \widetilde{m} \Diamond \mu^{**}(E)\big\rangle = \big\langle \mu^*(f \cdot a), \widetilde{m} \Diamond E\big\rangle = \big\langle \mu^*(f) \cdot a, m \Diamond e_0\big\rangle = \big\langle m, \mu^*(f) \cdot a\big\rangle,$$

and

$$\big\langle f \cdot a, \widetilde{m} \Box \mu^{**}(E)\big\rangle = \big\langle \mu^{**}(\widetilde{m}), f \cdot a\big\rangle = \big\langle \widetilde{m}, \mu^*(f \cdot a)\big\rangle = \big\langle \widetilde{m}, \mu^*(f) \cdot a\big\rangle = \big\langle m, \mu^*(f) \cdot a\big\rangle.$$

It follows that $(\widetilde{m} \Diamond (\mu^{**}(E))|_{\langle A^*A\rangle} = (\widetilde{m} \Box \mu^{**}(E))|_{\langle A^*A\rangle}$; that is,

$$m \Diamond \mu^{**}(E)\big|_{\langle A^*A\rangle} = m \Box \mu^{**}(E)\big|_{\langle A^*A\rangle}.$$

Therefore, $\mu^{**}(E)|_{\langle A^*A\rangle} \in \mathfrak{Z}_t(\langle A^*A\rangle_R^*)$. □

Theorem 10. *Let A be a Banach algebra. Then the following statements are equivalent.*

(i) *$\langle A^*A\rangle^*$ has a strong identity.*
(ii) *$(\langle A^*A\rangle^*, \Box)$ has an identity contained in $\mathfrak{Z}_t(\langle A^*A\rangle_R^*)$.*
(iii) *$(\langle A^*A\rangle^*, \Box)$ has a left identity and $\langle A^*A\rangle = \langle AA^*A\rangle$.*

Furthermore, if A satisfies $\langle A^2\rangle = A$, then each of the following statements is equivalent to (i)–(iii).

(iv) *$(\langle A^*A\rangle_R^*, \Diamond)$ is right unital.*
(v) *A has a BRAI and $\langle A^*A\rangle = \langle AA^*A\rangle$.*

Proof. (i) $\Longleftrightarrow$ (ii). This follows from Definition 6 and the equality

$$\mathfrak{Z}_t\big(\langle A^*A\rangle_R^*\big) = \big\{m \in \langle A^*A\rangle_R^*\colon\ n \Diamond m = n \Box m \text{ for all } n \in \langle A^*A\rangle_R^*\big\}.$$

(Cf. Section 2.)

(i) $\Longrightarrow$ (iii). This holds by Lemma 9(i).

(iii) $\Longrightarrow$ (i). Suppose that $\langle A^*A\rangle = \langle AA^*A\rangle$, and e_0 is a left identity of $(\langle A^*A\rangle^*, \Box)$. Then $e_0 \in \langle A^*A\rangle_R^*$. Let $m \in \langle A^*A\rangle_R^*$. Then, for $a \in A$ and $f \in \langle A^*A\rangle$,

$$\langle a \cdot f, m \Diamond e_0\rangle = \big\langle a \cdot (f \Diamond m), e_0\big\rangle = \langle e_0 \Box a, f \Diamond m\rangle = \langle a, f \Diamond m\rangle = \langle a \cdot f, m\rangle.$$

Since $\langle A^*A\rangle = \langle AA^*A\rangle$, we have $m \Diamond e_0 = m$. Therefore, e_0 is a right identity of $(\langle A^*A\rangle_R^*, \Diamond)$ and hence a strong identity of $\langle A^*A\rangle^*$.

Assume that $\langle A^2\rangle = A$. In this case, by [13, Theorem 4], $(\langle A^*A\rangle^*, \Box)$ is (right) unital if and only if A has a BRAI. Clearly, (i) $\Longrightarrow$ (iv), and [(i) and (iii)] $\Longrightarrow$ (v) $\Longrightarrow$ (iii). So, we only have to show that (iv) $\Longrightarrow$ (i).

(iv) $\Longrightarrow$ (i). Let e be a right identity of $(\langle A^*A\rangle_R^*, \Diamond)$. Since A is weak*-dense in $\langle A^*A\rangle^*$ and $A \subseteq \langle A^*A\rangle_R^*$, we see that e is also a right identity of $(\langle A^*A\rangle^*, \Box)$, and hence is an identity of $(\langle A^*A\rangle^*, \Box)$ by the paragraph above. Therefore, e is a strong identity of $\langle A^*A\rangle^*$.

We note here that the word "right" is missing in the conclusion of [13, Theorem 4(ii)], where it should read "A has a BRAI" rather than "A has a BAI". $\Box$

We consider below some conditions which are closely related to the existence of a strong identity of $\langle A^*A\rangle^*$.

(0) A is an ideal in A^{**}.

(1) A has a central approximate identity, i.e., A has an approximate identity from the algebraic centre of A.

(2) $\langle A^*A\rangle = \langle AA^*\rangle$.

(3) $\langle A^*A\rangle = \langle AA^*A\rangle$.

(4) $\langle A^*A\rangle \subseteq \langle AA^*\rangle$.

(5) $\langle A^*A\rangle$ is introverted in A^*.

(6) $\langle AA^*\rangle$ is introverted in A^*.

Proposition 11. *Let A be a Banach algebra. The following assertions hold.*

(i) $\langle AA^*A\rangle = \langle A^*A\rangle \cap \langle AA^*\rangle$ *in the following two cases*:
  (a) *A has a BRAI or a BLAI*;
  (b) *A is commutative satisfying $\langle A^2\rangle = A$.*

(ii) [(0) *or* (1)] $\Longrightarrow$ [(2) *and* (3)], [(2) *or* (3)] $\Longrightarrow$ (4)*, and* (2) $\Longrightarrow$ [(5) *and* (6)].

(iii) *If $\langle A^2\rangle = A$, then* [(0) *or* (1)] $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longleftrightarrow$ (4).

(iv) *If A is an involutive Banach algebra, then* (2) $\Longleftrightarrow$ (4)*, and* (5) $\Longleftrightarrow$ (6).

(v) *If A is an involutive Banach algebra satisfying $\langle A^2\rangle = A$, then*

$$\big[(0)\ \textit{or}\ (1)\big] \Longrightarrow (2) \Longleftrightarrow (3) \Longleftrightarrow (4) \Longrightarrow (5) \Longleftrightarrow (6).$$

(vi) *If A is the group algebra $L_1(G)$ of a locally compact group G, then* (1)–(6) *are all equivalent, and each of them is equivalent to that G is an SIN-group.*

Proof. (i) For case (a), suppose that (e_α) is a BRAI of A. Let $f \in \langle A^*A\rangle \cap \langle AA^*\rangle$. Then $f = \|\cdot\|\text{–}\lim f \cdot e_\alpha$ and thus $f \in \langle AA^*A\rangle$. Therefore, $\langle AA^*A\rangle = \langle A^*A\rangle \cap \langle AA^*\rangle$. When (e_α) is a BLAI of A, one just need replace $f \cdot e_\alpha$ above with $e_\alpha \cdot f$.

The assertion holds for case (b), since $\langle AA^*A\rangle = \langle A^*A^2\rangle = \langle A^*A\rangle = \langle AA^*\rangle$ if A is commutative satisfying $\langle A^2\rangle = A$.

(ii) This is clearly true.

(iii) Assume that $\langle A^2\rangle = A$. Suppose that $\langle A^*A\rangle \subseteq \langle AA^*\rangle$. Then

$$A^*A^2 \subseteq \langle A^*A\rangle A \subseteq \langle AA^*\rangle A \subseteq \langle AA^*A\rangle \subseteq \langle A^*A\rangle.$$

So, $\langle A^*A\rangle = \langle A^*A^2\rangle = \langle AA^*A\rangle$. Therefore, (4) $\Longrightarrow$ (3), and (2) $\Longrightarrow$ (3) $\Longleftrightarrow$ (4) by (ii).

(iv) Suppose that A is an involutive Banach algebra. Note that, in general, the involution on A cannot be extended to an involution on A^{**} with either Arens product (cf. [9]). For each $m \in A^{**}$, an element $m^* \in A^{**}$ can be defined by $m^*(f) = \overline{m(f^*)}$ $(f \in A^*)$, where $f^* \in A^*$ is given by $f^*(a) = \overline{f(a^*)}$ $(a \in A)$. It is easy to see that $(m \Box f)^* = f^* \Diamond m^*$ and $(f \Diamond m)^* = m^* \Box f^*$ $(m \in A^{**}, f \in A^*)$.

Note that $f \longmapsto f^*$ maps $\langle A^*A\rangle$ onto $\langle AA^*\rangle$, and $\langle AA^*\rangle$ onto $\langle A^*A\rangle$. Therefore, we have (2) $\Longleftrightarrow$ (4).

Assume that $\langle A^*A\rangle$ is introverted in A^*. Let $p \in \langle AA^*\rangle^*$ and $f \in \langle AA^*\rangle$. Let $m \in A^{**}$ be any extension of p. Then

$$p \Box f = m \Box f = (f^* \Diamond m^*)^* \in \langle AA^*\rangle,$$

since $f^* \in \langle A^*A\rangle$ and $f^* \Diamond m^* \in \langle A^*A\rangle$. Thus $\langle AA^*\rangle$ is introverted in A^*. Therefore, we have (5) $\Longrightarrow$ (6). Similarly, we have (6) $\Longrightarrow$ (5).

(v) This follows from (ii)–(iv).

(vi) Let G be a locally compact group and $A = L_1(G)$. It is known that condition (1) is equivalent to G being an SIN-group (cf. [33, Proposition]). By (v), we only have to show that G is an SIN-group if condition (6) is satisfied.

Assume that $RUC(G)$ is left introverted in $L_\infty(G)$. Applying [6, Theorem 5.15] with $X = RUC(G)$ and $B = \ell_1(G)$, we see that $RUC(G)$ is also left introverted in $\ell_\infty(G)$. It follows from [4, Theorem 4.4.5] and [31, Theorem 2] that G is an SIN-group. $\Box$

Remark 12. Let $A = A(F_2)$, where F_2 is the free group with two generators. Then A is commutative satisfying $\langle A^2\rangle = A$ (cf. Fact 1 below). Due to Proposition 11, we have $\langle A^*A\rangle = \langle AA^*A\rangle$ and $\langle A^*A\rangle^*_R = \langle A^*A\rangle^*$. But $(\langle A^*A\rangle^*_R, \Diamond)$ does not have a right identity by Theorem 10. Therefore, the converse of Lemma 9(i) is not true.

It is possible that a Banach algebra has a (not necessarily bounded) central approximate identity, but it has no bounded approximate identity. For example, as shown by De Cannière and Haagerup [8], the Fourier algebra $A(F_2)$ is weakly amenable. That is, $A(F_2)$ has a (central) approximate identity bounded with respect to the cb-multiplier norm on $M_{cb}A(F_2)$ (the completely bounded multiplier algebra of $A(F_2)$). However, $A(F_2)$ has no bounded approximate identity, since F_2 is a non-amenable group. We note that there are some groups which have even weaker approximation properties than weak amenability. For instance, it was shown by Haagerup and Kraus [14] that the semi-direct product $G = \mathbb{Z}^2 \rtimes_\rho SL(2,\mathbb{Z})$ is not weakly amenable (where ρ is the standard action of $SL(2,\mathbb{Z})$ on $\mathbb{Z}^2$), but it has the AP, i.e., there exists a net in $A(G)$ converging to 1_G in the weak* topology on $M_{cb}A(G)$.

As shown by Losert [30, Proposition 2], $\mathrm{span}\{A(G)^2\} = A(G)$ if and only if G is amenable. However, all $A(G)$ satisfy $\langle A^2\rangle = A$. This condition is indeed satisfied by all quantum group algebras as stated below in Fact 1. Before seeing this, we recall briefly the notion of locally compact quantum groups.

Let $\mathbb{G} = (M, \Gamma, \varphi, \psi)$ be a von Neumann algebraic locally compact quantum group in the sense of Kustermans and Vaes [21,22]. By definition, (M, Γ) is a Hopf–von Neumann algebra, φ is a normal semifinite faithful left invariant weight on (M, Γ), and ψ is a normal semifinite faithful right invariant weight on (M, Γ). Since the co-multiplication Γ is a normal isometric unital *-homomorphism from M into $M\bar{\otimes}M$, it is well known that its pre-adjoint $\Gamma_* : M_*\widehat{\otimes}M_* \longrightarrow M_*$ induces an associative completely contractive multiplication $\star$ on M_* (cf. Ruan [37,38]). Here, $\bar{\otimes}$ denotes the von Neumann algebra tensor product, and $\widehat{\otimes}$ denotes the operator space projective tensor product. In the two classical cases where M_* is $L_1(G)$ or $A(G)$, $\star$ is the usual convolution on $L_1(G)$ and the pointwise multiplication on $A(G)$, respectively. The reader is referred to Kustermans and Vaes [21,22] and van Daele [45] for more information on locally compact quantum groups.

Following the locally compact group case, the von Neumann algebra M is written as $L_\infty(\mathbb{G})$, and the Banach algebra M_* equipped with the multiplication $\star$ is denoted by $L_1(\mathbb{G})$. It is known that the quantum group algebra $L_1(\mathbb{G})$ is an involutive Banach algebra with a faithful multiplication (cf. [16]). The locally compact quantum group $\mathbb{G}$ is called *co-amenable* if $L_1(\mathbb{G})$ has a BAI. It turns out that $\mathbb{G}$ is co-amenable if and only if $L_1(\mathbb{G})$ has a BRAI if and only if $L_1(\mathbb{G})$ has a BLAI. We showed in [16, Theorem 2] that $\mathbb{G}$ is co-amenable if and only if $L_1(\mathbb{G})$ has a BAI consisting of normal states on $L_\infty(\mathbb{G})$.

Since the multiplication map $\Gamma_* : L_1(\mathbb{G})\widehat{\otimes}L_1(\mathbb{G}) \longrightarrow L_1(\mathbb{G})$ is a complete quotient map, we have the following

Fact 1. All quantum group algebras $L_1(\mathbb{G})$ satisfy $\langle L_1(\mathbb{G})^2\rangle = L_1(\mathbb{G})$.

Therefore, by Proposition 11(v), conditions (2)–(4) are equivalent for all quantum group algebras $L_1(\mathbb{G})$.

The Banach $L_1(\mathbb{G})$-modules $RUC(\mathbb{G})$ and $LUC(\mathbb{G})$ are defined to be $\langle L_1(\mathbb{G}) \star L_\infty(\mathbb{G})\rangle$ and $\langle L_\infty(\mathbb{G}) \star L_1(\mathbb{G})\rangle$, respectively, and $UC(\mathbb{G})$ denotes $LUC(\mathbb{G}) \cap RUC(\mathbb{G})$ (cf. Hu, Neufang and Ruan [16] and Runde [41]). It turns out that $RUC(\mathbb{G})$ and $LUC(\mathbb{G})$ are closed operator systems in $L_\infty(\mathbb{G})$ (cf. [41, Theorem 2.3]). Obviously, they are just the usual spaces $LUC(G)$ and $RUC(G)$ if $L_1(\mathbb{G})$ is the group algebra $L_1(G)$ of a locally compact group G. By Proposition 11(i), $UC(\mathbb{G}) = \langle L_1(\mathbb{G}) \star L_\infty(\mathbb{G}) \star L_1(\mathbb{G})\rangle$ if either $\mathbb{G}$ is co-amenable, or $\mathbb{G}$ is co-commutative which is precisely the case when $L_1(\mathbb{G}) = A(G)$ for some locally compact group G.

Recall that a locally compact group G is SIN if and only if $LUC(G) = RUC(G)$ (cf. Milnes [31]).

Definition 13. Let A be a Banach algebra and X be a Banach A-module. We say that the Banach A-module action on X is SIN if $\langle A \cdot X\rangle = \langle X \cdot A\rangle$.

A locally compact quantum group $\mathbb{G}$ is called an *SIN quantum group* if the canonical Banach $L_1(\mathbb{G})$-module action on $L_\infty(\mathbb{G})$ is SIN.

Note that the $\langle AA^*\rangle$-version of Theorem 10 holds. Therefore, we have the following immediate corollary of Theorem 10 and Proposition 11(v).

Corollary 14. *Let A be an involutive Banach algebra satisfying $\langle A^2\rangle = A$. Let X be either $\langle A^*A\rangle$ or $\langle AA^*\rangle$. Then X^* has a strong identity if and only if A has a BAI and the canonical Banach A-module action on A^* is SIN.*

For a Banach algebra A, let $B_A(A^*)$ be the Banach algebra of bounded right A-module homomorphisms on A^*, and $B_A^\sigma(A^*)$ be the normal part of $B_A(A^*)$ (i.e., consisting of all elements of $B_A(A^*)$ which are weak*–weak* continuous). Note that if $E \in A^{**}$ is a weak*-cluster point of a BRAI of A, then $T = T^*(E)_L$ for all $T \in B_A(A^*)$. It can be seen that we have the following

Fact 2. Let A be a Banach algebra.

(i) The map $RM(A) \longrightarrow B_A^\sigma(A^*)$, $\mu \longmapsto \mu^*$ is an isometric algebra isomorphism.
(ii) The map $(\langle A^*A\rangle^*, \Box) \longrightarrow B_A(A^*), m \longmapsto m_L$ is an injective contractive algebra homomorphism, and is a surjective isometry if A has a BRAI bounded by 1.

In the sequel, $RM(A)$ and $\langle A^*A\rangle^*$ will be compared via their canonical images in $B_A(A^*)$. Due to Lemma 1(ii), we have $RM(A) \subseteq \mathfrak{Z}_t(\langle A^*A\rangle^*)$ if A has a BRAI. The opposite inclusion is clearly equivalent to $A \cdot \mathfrak{Z}_t(\langle A^*A\rangle^*) \subseteq A$. Therefore, we have

Fact 3. $\mathfrak{Z}_t(\langle A^*A\rangle^*) \subseteq RM(A)$ if and only if $A \cdot \mathfrak{Z}_t(\langle A^*A\rangle^*) \subseteq A$.

As a consequence of Facts 1, 2, and Grosser and Losert [13, Theorem 4], we obtain the following characterizations of co-amenable locally compact quantum groups $\mathbb{G}$ in terms of $LUC(\mathbb{G})^*$.

Theorem 15. *Let $\mathbb{G}$ be a locally compact quantum group. Then the following statements are equivalent.*

(i) *$\mathbb{G}$ is co-amenable.*
(ii) *$LUC(\mathbb{G})^* \cong B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$ via the isometric algebra isomorphism $m \longmapsto m_L$.*
(iii) *$RM(L_1(\mathbb{G})) \subseteq \mathfrak{Z}_t(LUC(\mathbb{G})^*)$.*
(iv) *$id_{L_1(\mathbb{G})} \in \mathfrak{Z}_t(LUC(\mathbb{G})^*)$.*
(v) *$(LUC(\mathbb{G})^*, \Box)$ is unital.*
(vi) *$(LUC(\mathbb{G})^*, \Box)$ is right unital.*

Recall that a locally compact quantum group $\mathbb{G}$ is called *amenable* if there exists a left invariant mean on $L_\infty(\mathbb{G})$; that is, there exists $m \in L_\infty(\mathbb{G})^*$ such that $\|m\| = \langle m, 1\rangle = 1$ and $a \star m = \langle 1, a\rangle m$ ($a \in L_1(\mathbb{G})$). Right invariant means and (two-sided) invariant means on $L_\infty(\mathbb{G})$ are defined similarly. It is known that the existence of a right invariant mean and the existence of an invariant mean are both equivalent to $\mathbb{G}$ being amenable. It is well known that all co-commutative quantum groups are amenable.

In [24], Lau introduced and studied a large class of Banach algebras, called F-algebras, including all preduals of Hopf–von Neumann algebras. An F-algebra is a Banach algebra A which is the predual of a W^*-algebra M such that the identity 1 of M is a multiplicative linear functional on A. For an F-algebra A, let A_0 be the augmentation ideal in A; that is, $A_0 = \{a \in A: \langle a, 1\rangle = 0\}$.

Note that a quantum group algebra $L_1(\mathbb{G})$ is an involutive Banach algebra, and $L_1(\mathbb{G})_0$ is closed under the involution on $L_1(\mathbb{G})$. Thus $L_1(\mathbb{G})$ (respectively, $L_1(\mathbb{G})_0$) has a BRAI if and only if $L_1(\mathbb{G})$ (respectively, $L_1(\mathbb{G})_0$) has a BAI. Applying Lau [24, Theorem 4.10] to $L_1(\mathbb{G})$ (see also [24, Theorem 4.1]), we obtain the following proposition. We note that the equivalence below also follows by combining [24, Theorem 4.10] with its right-hand side version, which can be proved by interchanging the words "left" and "right" there.

Proposition 16. *Let $\mathbb{G}$ be a locally compact quantum group. Then $\mathbb{G}$ is amenable and co-amenable if and only if $L_1(\mathbb{G})_0$ has a BAI.*

In particular, if $\mathbb{G}$ is a co-commutative quantum group, then each of (i)–(vi) *in Theorem* 15 *is also equivalent to*

(vii) $L_1(\mathbb{G})_0$ *has a BAI.*

We point out that unlike in the case of $L_1(\mathbb{G})$, where $L_1(\mathbb{G})$ has a BAI (that is, $\mathbb{G}$ is co-amenable) if and only if $L_1(\mathbb{G})$ has a BAI of norm 1 (even, as shown in [16], consisting of states on $L_\infty(\mathbb{G})$), for any infinite-dimensional co-amenable co-commutative quantum group $\mathbb{G}$ (that is, $L_1(\mathbb{G}) = A(G)$ of an infinite amenable group G), $L_1(\mathbb{G})_0$ has a BAI bounded by 2, and 2 is the best possible norm bound for any BAI of $L_1(\mathbb{G})_0$ (cf. Kaniuth and Lau [19, Theorem 3.4]).

Remark 17. Recall that in the case where $L_1(\mathbb{G})$ is $L_1(G)$ or $A(G)$, the quantum group $\mathbb{G}$ is amenable and co-amenable if and only if G is an amenable group. Therefore, for all locally compact groups G, we have

$$G \text{ is amenable if and only if } L_1(G)_0 \text{ has a BAI if and only if } A(G)_0 \text{ has a BAI.}$$

It would be interesting to know whether for a general locally compact quantum group $\mathbb{G}$, we have

$$L_1(\mathbb{G})_0 \text{ has a BAI if and only if } L_1(\widehat{\mathbb{G}})_0 \text{ has a BAI.}$$

It is known by Bédos and Tuset [3, Theorem 3.2] that if $\widehat{\mathbb{G}}$ is co-amenable, then $\mathbb{G}$ is amenable. Therefore, to obtain the above equivalence, it suffices to know whether the amenability of $\mathbb{G}$ (respectively, the amenability and the co-amenability of $\mathbb{G}$) would imply the co-amenability of $\widehat{\mathbb{G}}$. This is still an open question, which is only known to be true if $\mathbb{G}$ is a locally compact group G (i.e., $L_1(\mathbb{G}) = L_1(G)$, cf. Leptin [29]), or $\mathbb{G}$ is a discrete quantum group (cf. Tomatsu [44], see also Ruan [38] for the discrete Kac algebra case).

By our definition, a locally compact quantum group $\mathbb{G}$ is SIN if and only if $LUC(\mathbb{G}) = RUC(\mathbb{G})$. Therefore, by Proposition 11(v), $\mathbb{G}$ is SIN if and only if $LUC(\mathbb{G}) = UC(\mathbb{G})$ if and only if $RUC(\mathbb{G}) = UC(\mathbb{G})$. It was proved recently by Runde [40] that $\mathbb{G}$ is compact if and only if $L_1(\mathbb{G})$ is an ideal in $L_1(\mathbb{G})^{**}$ (that is, $L_1(\mathbb{G})$ satisfies condition (0)). Thus, as in the locally compact group case, $\mathbb{G}$ is an SIN quantum group whenever $\mathbb{G}$ is compact, or $\mathbb{G}$ is discrete (that is, $L_1(\mathbb{G})$ has an identity), or $\mathbb{G}$ is co-commutative. Also, $\mathbb{G}$ is SIN if $L_1(\mathbb{G})$ has a central approximate identity (cf. Proposition 11).

As in Theorem 15, we identify $RM(L_1(\mathbb{G}))$ and $LUC(\mathbb{G})^*$ with their respective canonical images in $B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$. Then we obtain the following interesting counterpart of Theorem 15.

Theorem 18. *Let $\mathbb{G}$ be a locally compact quantum group. Then the following statements are equivalent.*

(i) $\mathbb{G}$ *is co-amenable and SIN.*
(ii) $RM(L_1(\mathbb{G})) \subseteq \mathfrak{Z}_t(LUC(\mathbb{G})^*_R)$.
(iii) $id_{L_1(\mathbb{G})} \in \mathfrak{Z}_t(LUC(\mathbb{G})^*_R)$.
(iv) $(LUC(\mathbb{G})^*_R, \diamondsuit)$ *is unital.*

(v) $(LUC(\mathbb{G})^*_R, \diamond)$ *is right unital.*
(vi) $LUC(\mathbb{G})^*$ *has a strong identity.*

Proof. (vi) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii) and (ii) $\Longrightarrow$ (iii) are trivial. (iii) $\Longrightarrow$ (vi) $\Longleftrightarrow$ (v) follows from Theorem 10. (i) $\Longleftrightarrow$ (vi) holds by Corollary 14. And (vi) $\Longrightarrow$ (ii) follows from Lemma 9(ii) and Theorem 15. $\square$

For a general co-amenable locally compact quantum group $\mathbb{G}$, we do not know whether $L_1(\mathbb{G})$ has a central BAI if $\mathbb{G}$ is SIN, though this is true when $\mathbb{G}$ is commutative or co-commutative. This is not clear even for co-amenable compact quantum groups. Also, it is not clear whether $\mathbb{G}$ is SIN if $\mathbb{G}$ is co-amenable and $LUC(\mathbb{G})$ is introverted in $L_\infty(\mathbb{G})$ (cf. Proposition 11(vi)).

Let G be a locally compact group. It is seen that if $f \in UC(G)$, then $m_r(f) = m_R(f) \in RUC(G)$ for all $m \in LUC(G)^*$ (cf. Lau [23, Lemma 3]). Also, if $m \in \mathfrak{Z}_t(LUC(G)^*) = M(G)$, then $m_r(f) = m_R(f)$ and $m \,\square\, n = m \diamond n = m * n$ for all $f \in LUC(G)$ and $n \in LUC(G)^*$.

Suppose that $LUC(G) = RUC(G)$. Then $LUC(G)^*_R = LUC(G)^* = RUC(G)^*$. As shown in Section 3 that $(LUC(G)^*, \square) = (LUC_{\ell_\infty}(G)^*, \square_{\ell_1})$, we also have

$$\big(RUC(G)^*, \diamond\big) = \big(RUC_{\ell_\infty}(G)^*, \diamond_{\ell_1}\big).$$

Recall that $(Z_U(G), *) = (LUC_{\ell_\infty}(G)^*_R, \diamond_{\ell_1})$ (cf. Section 3). Therefore, we have

$$\big(LUC(G)^*_R, \diamond\big) = \big(RUC(G)^*, \diamond\big) = \big(RUC_{\ell_\infty}(G)^*, \diamond_{\ell_1}\big) = \big(LUC_{\ell_\infty}(G)^*_R, \diamond_{\ell_1}\big) = \big(Z_U(G), *\big);$$

i.e., $(LUC(G)^*_R, \diamond) = (Z_U(G), *)$, or equivalently, $(LUC(G)^*_R, \diamond) = (LUC_{\ell_\infty}(G)^*_R, \diamond_{\ell_1})$.

Conversely, assume that $UC(G) \subsetneqq LUC(G)$. Then there exists a non-zero $m \in LUC(G)^*_R$ such that $m_R = 0$ (see the paragraph after the proof of Proposition 5). Obviously, $\delta_{e_G} \in Z_U(G) \cap LUC(G)^*_R$. Also, δ_{e_G} is an identity of $(Z_U(G), *)$, and δ_{e_G} is a left but not a right identity of $(LUC(G)^*_R, \diamond)$ (since $m \diamond \delta_{e_G} = 0$). Therefore, in this situation, $(LUC(G)^*_R, \diamond)$ cannot be a subalgebra of $(Z_U(G), *)$.

Recall that $RM(L_1(G)) = M(G)$, and δ_{e_G} is an identity of $(LUC(G)^*, \square)$. From these discussions together with Proposition 11(vi), Corollary 14, and Theorem 18, we obtain below several new characterizations of SIN locally compact groups.

Theorem 19. *Let G be a locally compact group. Then the following statements are equivalent.*

(i) *G is an SIN-group.*
(ii) $(LUC(G)^*_R, \diamond) = (LUC_{\ell_\infty}(G)^*_R, \diamond_{\ell_1})$.
(iii) $(LUC(G)^*_R, \diamond)$ *is a subalgebra of* $(LUC_{\ell_\infty}(G)^*_R, \diamond_{\ell_1})$.
(iv) $M(G) \subseteq \mathfrak{Z}_t(LUC(G)^*_R)$.
(v) $\delta_{e_G} \in \mathfrak{Z}_t(LUC(G)^*_R)$.
(vi) $(LUC(G)^*_R, \diamond)$ *is unital.*
(vii) $(LUC(G)^*_R, \diamond)$ *is right unital.*
(viii) $LUC(G)^*$ *has a strong identity.*
(ix) $LUC(G)$ *is two-sided introverted in* $L_\infty(G)$.

Remark 20. (a) The referee kindly informed us that the equivalence between (i) and (ii) in Theorem 19 may also be obtained by [32, Theorem 7] and [36, Lemma 5].

(b) Results obtained in this section illustrate that the SIN property is intrinsically related to topological centre problems.

(c) The Banach algebras $(LUC(G)^*_R, \diamond)$ and $(Z_U(G), *)$ are both used to describe the topological centre $\mathfrak{Z}_t(LUC(G)^*)$ and to characterize the SIN property of a locally compact group G. The approach by using $LUC(G)^*_R$ is of Banach algebraic flavor, that can be applied to general locally compact quantum groups. We note that $(LUC(G)^*_R, \diamond)$ cannot be replaced by $(Z_U(G), *)$ in Theorem 19(v)–(vii). We have other evidence showing advantages of $LUC(G)^*_R$ over $Z_U(G)$ for studying problems as characterizing the equality $LUC(\mathbb{G}) = WAP(\mathbb{G})$ for general quantum groups $\mathbb{G}$, where $WAP(\mathbb{G})$ is the space of weakly almost periodic functionals on $L_1(\mathbb{G})$. (Recall that a bounded linear functional f on a Banach algebra A is called weakly almost periodic if the map $A \longrightarrow A^*$, $a \longmapsto f \cdot a$ is weakly compact.)

(d) Suppose that G is amenable. Let $RIM(LUC(G))$ and $TRIM(LUC(G))$ be the sets of right translation invariant and topologically right invariant means on $LUC(G)$, respectively. Then

$$TRIM\big(LUC(G)\big) \subseteq RIM\big(LUC(G)\big) \subseteq Z_U(G), \quad \text{and} \quad TRIM\big(LUC(G)\big) \subseteq LUC(G)^*_R.$$

It follows that if $(Z_U(G), *)$ is a subalgebra of $(LUC(G)^*_R, \diamond)$, then $TRIM(LUC(G)) = RIM(LUC(G))$. We do not know whether the converse holds, that would be true if the above equality were equivalent to G being SIN. Also, it is not clear for us when we would have $LUC(G)^*_R = Z_U(G)$ as *subspaces* of $LUC(G)^*$.

Let $\mathbb{G}$ be a locally compact quantum group and $C_0(\mathbb{G})$ be the reduced C^*-algebra of $\mathbb{G}$. Then $C_0(\mathbb{G}) \subseteq WAP(\mathbb{G})$ (cf. Runde [41, Theorem 4.3]), $WAP(\mathbb{G})$ and $C_0(\mathbb{G})$ are introverted in $L_\infty(\mathbb{G})$, and the two Arens products on $WAP(\mathbb{G})^*$ and $C_0(\mathbb{G})^*$, respectively, coincide (cf. Dales and Lau [6, Propositions 3.11 and 5.7]). Therefore, $M(\mathbb{G}) = C_0(\mathbb{G})^*$ is a dual Banach algebra (i.e., the multiplication on $M(\mathbb{G})$ is separately weak*–weak* continuous). We point out that the Arens product on $M(\mathbb{G})$ is equal to the product on $C_0(\mathbb{G})^*$ as defined in Kustermans and Vaes [21], that is induced by the co-multiplication on $C_0(\mathbb{G})$.

It is known that $\mathbb{G}$ is co-amenable if and only if $C_0(\mathbb{G})^*$ is unital (cf. Bédos and Tuset [3, Theorem 3.1]). By Theorem 15, we see that the co-amenability of $\mathbb{G}$ is also equivalent to $WAP(\mathbb{G})^*$ being unital. It is interesting to compare these characterizations of co-amenable locally compact quantum groups (see also Theorem 15) with Theorem 18, where we show in particular that a quantum group $\mathbb{G}$ is co-amenable and SIN if and only if $(LUC(\mathbb{G})^*_R, \diamond)$ is unital.

Among these kinds of characterizations in terms of the existence of a unit, we point out here that Theorem 22 of the next section shows that when $L_1(\mathbb{G})$ is in the class of Banach algebras introduced by the authors in [16], in particular, when $L_1(\mathbb{G})$ is separable with $\mathbb{G}$ co-amenable, then $\mathbb{G}$ is discrete if and only if $L_1(G)^{**}$ is unital under either Arens product.

5. Interrelationships between strong Arens irregularity and quotient strong Arens irregularity

Let A be a Banach algebra. Recall that we say that A is left quotient strongly Arens irregular if $\mathfrak{Z}_t(\langle A^*A\rangle^*) \subseteq RM(A)$, which is equivalent to $\mathfrak{Z}_t(\langle A^*A\rangle^*) = RM(A)$ if A has a BRAI, where $RM(A)$ and $\langle A^*A\rangle^*$ are identified with their respective canonical images in $B_A(A^*)$ (cf. Section 1). In this section, we consider how the inclusion $\mathfrak{Z}_t(\langle A^*A\rangle^*) \subseteq RM(A)$ is related to the left strong Arens irregularity of A. For brevity, most results in this section are stated in their one-sided versions. We remind the reader that here $RM(A)$ is the opposite right multiplier algebra of A.

Recall that a Banach space X is weakly sequentially complete (WSC) if every weakly Cauchy sequence in X is weakly convergent. It is well known that the predual of a von Neumann algebra is WSC (cf. Takesaki [42, Corollary III.5.2]).

First, using the connection between $RM(A)$ and $\mathfrak{Z}_t(\langle A^*A\rangle^*)$ as shown in Lemma 1(ii), we give the following generalization of Lau and Ülger [28, Theorem 2.6], where they proved that if A is WSC with a sequential BAI, then $\langle A^*A\rangle = A^*$ if and only if $\langle AA^*\rangle = A^*$ if and only if A is unital. Lau and Ülger asked there whether one can drop the word "sequential" above. See Baker, Lau and Pym [2, Corollary 2.3] for some related results.

To present results in this section, we need recall the definition of the class of Banach algebras introduced by the authors in [16].

Definition 21. (See [16].) Let A be a Banach algebra with a BAI. Then A is said to be of type (RM) if for every $\mu \in RM(A)$, there is a closed subalgebra B of A with a BAI such that

(I) $\mu|_B \in RM(B)$;
(II) $f|_B \in BB^*$ for all $f \in AA^*$;
(III) there is a family $\{B_i\}$ of closed right ideals in B satisfying (i) each B_i is WSC with a sequential BAI, (ii) for all i, there exists a left B_i-module projection from B onto B_i, and (iii) $\mu \in A$ if $\mu|_{B_i} \in B_i$ for all i.

Similarly, Banach algebras of type (LM) are defined. A is said to be of type (M) if A is both of type (LM) and of type (RM).

Obviously, a Banach algebra A is of type (M) if A is WSC with a sequential BAI. This is the case when A is $L_1(\mathbb{G})$ of a co-amenable quantum group $\mathbb{G}$ over a separable Hilbert space. It is shown in [16] that all convolution Beurling algebras $L_1(G,\omega)$ with $\omega \geqslant 1$, in particular, all group algebras $L_1(G)$, are of type (M). And so are Fourier algebras $A(G)$ of amenable locally compact groups G.

Theorem 22. *Let A be a Banach algebra.*

(i) *Suppose that A is of type (LM). Then $\langle A^*A\rangle = A^*$ if and only if A is unital.*
(ii) *Suppose that A is of type (RM). Then $\langle AA^*\rangle = A^*$ if and only if A is unital.*

Consequently, if A is of type (M), then A is unital if and only if A^{**} is unital under either Arens product.

In particular, if $\mathbb{G}$ is a locally compact quantum group with $L_1(\mathbb{G})$ of type (M), then $LUC(\mathbb{G}) = L_\infty(\mathbb{G})$ if and only if $\mathbb{G}$ is discrete if and only if $L_1(\mathbb{G})^{**}$ is unital.

Proof. We prove assertion (i); similar arguments establish (ii).

Clearly, $\langle A^*A\rangle = A^*$ if A is unital. Conversely, assume that $\langle A^*A\rangle = A^*$. Then we have $(\langle A^*A\rangle^*, \Box) = (A^{**}, \Box)$ and $\mathfrak{Z}_t(\langle A^*A\rangle^*) = \mathfrak{Z}_t(A^{**}, \Box)$. Let $(\mu, \nu) \in M(A)$ and E be a weak*-cluster point in A^{**} of a BAI of A. By Lemma 1(ii), $\nu^{**}(E) \in \mathfrak{Z}_t(\langle A^*A\rangle^*) = \mathfrak{Z}_t(A^{**}, \Box)$. It can be seen that $\nu^{**}(E) \cdot a = \mu(a)$ for all $a \in A$. Thus $\nu^{**}(E) \cdot A \subseteq A$. By [16, Theorem 32(i)],

$\nu^{**}(E) = a_0$ for some $a_0 \in A$. It follows that $\mu(a) = a_0 a$ and $\nu(a) = a a_0$ $(a \in A)$. We conclude that A is unital by taking $(\mu, \nu) = (id_A, id_A)$.

By (i), (ii), and [28, Proposition 2.2], we conclude that A is unital if and only if A^{**} is unital under either Arens product. $\square$

It is interesting to compare Theorem 23 below with [16, Theorem 32], where we proved that if A is of type (LM), then

$$A \text{ is left strongly Arens irregular if and only if } \mathfrak{Z}_t(A^{**}, \Box) \cdot A \subseteq A;$$

and if A is of type (RM), then

$$A \text{ is right strongly Arens irregular if and only if } A \cdot \mathfrak{Z}_t(A^{**}, \Diamond) \subseteq A.$$

Theorem 23. *Let A be a Banach algebra satisfying $\langle A^2 \rangle = A$.*

(i) *A is left quotient strongly Arens irregular if and only if $A \cdot \mathfrak{Z}_t(A^{**}, \Box) \subseteq A$.*
(ii) *A is right quotient strongly Arens irregular if and only if $\mathfrak{Z}_t(A^{**}, \Diamond) \cdot A \subseteq A$.*

Consequently, $\mathfrak{Z}_t(A^{**}, \Box) \cap \mathfrak{Z}_t(A^{**}, \Diamond) = A$ in the following two cases:

(a) *A is of type (RM) and $\mathfrak{Z}_t(\langle A^* A \rangle^*) = RM(A)$;*
(b) *A is of type (LM) and $\mathfrak{Z}_t(\langle A A^* \rangle^*) = LM(A)$.*

Proof. Assertion (i) follows from Corollary 4(i) and Fact 3. Similarly, assertion (ii) holds.

For the final assertion, let $m \in \mathfrak{Z}_t(A^{**}, \Box) \cap \mathfrak{Z}_t(A^{**}, \Diamond)$. In case (a), we have $A \cdot m \subseteq A$ by (i), and since $m \in \mathfrak{Z}_t(A^{**}, \Diamond)$, we have $m \in A$ by [16, Theorem 32(ii)]. Therefore, $\mathfrak{Z}_t(A^{**}, \Box) \cap \mathfrak{Z}_t(A^{**}, \Diamond) = A$. The proof for case (b) is similar. $\square$

Remark 24. Suppose that A has a BRAI, and $\langle A^* A \rangle^*$ has a strong identity. It is seen that

$$\big[\mathfrak{Z}_t\big(\langle A^* A \rangle^*\big) \cap \mathfrak{Z}_t\big(\langle A^* A \rangle_R^*\big) = RM(A)\big] \Longrightarrow \big[A \cdot \big(\mathfrak{Z}_t(A^{**}, \Box) \cap \mathfrak{Z}_t(A^{**}, \Diamond)\big) \subseteq A\big].$$

(Cf. Lemma 9(ii)].) Hence, the equality $\mathfrak{Z}_t(\langle A^* A \rangle^*) \cap \mathfrak{Z}_t(\langle A^* A \rangle_R^*) = RM(A)$ implies that $\mathfrak{Z}_t(A^{**}, \Box) \cap \mathfrak{Z}_t(A^{**}, \Diamond) = A$ if A is of type (RM) (comparing with case (a) in Theorem 23). In this situation, $[\mathfrak{Z}_t(\langle A^* A \rangle^*) = RM(A)] \Longrightarrow [\mathfrak{Z}_t(\langle A^* A \rangle^*) \subseteq \mathfrak{Z}_t(\langle A^* A \rangle_R^*)]$.

Due to Theorem 23, we have the following theorem on topological centres.

Theorem 25. *Let A be a Banach algebra. Consider the following statements.*

(i) $\mathfrak{Z}_t(\langle A^* A \rangle^*) = RM(A)$.
(ii) $\mathfrak{Z}_t(A^{**}, \Box) = A$.
(iii) $\mathfrak{Z}_t(A^{**}, \Box) \subseteq \mathfrak{Z}_t(A^{**}, \Diamond)$.

If A is of type (RM), then any two of (i)–(iii) *imply the third.*

Proof. By Theorem 23(i), we just need to show that [(i) and (iii)] $\Longrightarrow$ (ii).

Assume that $\mathfrak{Z}_t(\langle A^*A\rangle^*) = RM(A)$ and $\mathfrak{Z}_t(A^{**}, \Box) \subseteq \mathfrak{Z}_t(A^{**}, \Diamond)$. Then, by case (a) in Theorem 23, we have $\mathfrak{Z}_t(A^{**}, \Box) = \mathfrak{Z}_t(A^{**}, \Box) \cap \mathfrak{Z}_t(A^{**}, \Diamond) = A$; that is, (ii) holds. $\Box$

Combining Theorem 23 with Theorem 25 gives the corollary below.

Corollary 26. *Let A be a Banach algebra of type (M). If $\mathfrak{Z}_t(A^{**}, \Box) = \mathfrak{Z}_t(A^{**}, \Diamond)$ (e.g., A is commutative), then the following statements are equivalent.*

(i) $\mathfrak{Z}_t(\langle A^*A\rangle^*) = RM(A)$.
(ii) $\mathfrak{Z}_t(\langle AA^*\rangle^*) = LM(A)$.
(iii) $\mathfrak{Z}_t(A^{**}, \Box) = A$.
(iv) $\mathfrak{Z}_t(A^{**}, \Box) \cdot A \subseteq A$.
(v) $A \cdot \mathfrak{Z}_t(A^{**}, \Box) \subseteq A$.

Remark 27. When A is the Fourier algebra of an amenable locally compact group, Lau and Losert proved (i) $\Longrightarrow$ (iii) in [27, Theorem 6.4].

Remark 28. In [28, Remark 5.2.3°], Lau and Ülger observed the asymmetry between the equality "$A \cdot \mathfrak{Z}_t(A^{**}, \Box) = A \cdot \mathfrak{Z}_t(\langle A^*A\rangle^*)$" as stated in Corollary 4(ii), and the inclusion "$\mathfrak{Z}_t(A^{**}, \Box) \cdot A \subseteq \mathfrak{Z}_t(A^{**}, \Box)$" considered in [28, Theorem 5.1]—the topological centre $\mathfrak{Z}_t(A^{**}, \Box)$ is treated as a *left* A-module in the former but a *right* A-module in the latter. One may also compare the condition "$A \cdot \mathfrak{Z}_t(A^{**}, \Box) \subseteq A$" in Theorem 23(i) with the condition "$\mathfrak{Z}_t(A^{**}, \Box) \cdot A \subseteq A$" considered in [16, Theorem 32(i)]. These asymmetries may be explained as follows.

In the equality $A \cdot \mathfrak{Z}_t(A^{**}, \Box) = A \cdot \mathfrak{Z}_t(\langle A^*A\rangle^*)$, the topological centre $\mathfrak{Z}_t(A^{**}, \Box)$ is linked to the opposite right multiplier algebra $RM(A)$ through the embedding $RM(A) \hookrightarrow \mathfrak{Z}_t(\langle A^*A\rangle^*)$.

On the other hand, in $\mathfrak{Z}_t(A^{**}, \Box) \cdot A \subseteq \mathfrak{Z}_t(A^{**}, \Box)$, one relates $\mathfrak{Z}_t(A^{**}, \Box)$ to the left multiplier algebra $LM(A)$ via the map from $LM(A)$ into $(A^{**}, \Diamond)$ as given in Lemma 1(i): the product $\mathfrak{Z}_t(A^{**}, \Box) \cdot A$ taken here should be recognized as the product $\mathfrak{Z}_t(A^{**}, \Box) \Diamond A$ though they are equal.

Assume that A is of type (M). If $\mathfrak{Z}_t(A^{**}, \Box) = \mathfrak{Z}_t(A^{**}, \Diamond)$, then, by Corollary 26, the strong Arens irregularity of A is equivalent to the quotient strong Arens irregularity of A. However, there do exist involutive Banach algebras A of type (M) such that $\mathfrak{Z}_t(A^{**}, \Box) \neq \mathfrak{Z}_t(A^{**}, \Diamond)$ (cf. Propositions 34 and 36 in Section 6). In this situation, Theorem 25 shows that the assertion

$$\left[\mathfrak{Z}_t\left(\langle A^*A\rangle^*\right) = RM(A)\right] \Longrightarrow \left[\mathfrak{Z}_t(A^{**}, \Box) = A\right] \tag{1}$$

is equivalent to

$$\left[\mathfrak{Z}_t\left(\langle A^*A\rangle^*\right) = RM(A)\right] \Longrightarrow \left[\mathfrak{Z}_t(A^{**}, \Box) \subseteq \mathfrak{Z}_t(A^{**}, \Diamond)\right], \tag{2}$$

which is also equivalent to

$$\left[A \cdot \mathfrak{Z}_t(A^{**}, \Box) \subseteq A\right] \Longrightarrow \left[\mathfrak{Z}_t(A^{**}, \Box) \cdot A \subseteq A\right] \tag{3}$$

by Theorem 23(i) and [16, Theorem 32(i)].

Therefore, even in order to obtain the left strong Arens irregularity of A through the left quotient strong Arens irregularity of A, one may have to consider both topological centres of A^{**} and their relationship. As noted in Remark 28, $\mathfrak{Z}_t(A^{**}, \Box)$ is intrinsically related to both $LM(A)$ and $RM(A)$. In other words, $LM(A)$ and $RM(A)$ are each related to both $\mathfrak{Z}_t(A^{**}, \Box)$ and $\mathfrak{Z}_t(A^{**}, \Diamond)$. Moreover, the equivalence between (1) and (3) shows that implication (1) holds precisely when A is not a "wrong" sided ideal in $\mathfrak{Z}_t(A^{**}, \Box)$; that is, A cannot be only a right but not a left ideal in $\mathfrak{Z}_t(A^{**}, \Box)$. All of these facts (see also Remark 28) illustrate the complex nature of topological centre problems.

Next, we consider the case where A is an involutive Banach algebra. In this situation, there exists a closer connection between $\mathfrak{Z}_t(A^{**}, \Box)$ and $\mathfrak{Z}_t(A^{**}, \Diamond)$ (respectively, $\mathfrak{Z}_t(\langle A^*A\rangle^*)$ and $\mathfrak{Z}_t(\langle AA^*\rangle^*)$). Let $\tau : A^{**} \longrightarrow A^{**}, m \longmapsto m^*$ be the unique weak*–weak* continuous extension of the involution on A (cf. the proof of Proposition 11(iv)). Then τ is usually not an involution on A^{**} with either Arens product (cf. Farhadi and Ghahramani [9]) but a *linear involution*; that is, $\tau(\tau(m)) = m$ and $\tau(\alpha m + \beta n) = \bar{\alpha}\tau(m) + \bar{\beta}\tau(n)$ $(m, n \in A^{**}, \alpha, \beta \in \mathbb{C})$. It can be seen that $(m \Box n)^* = n^* \Diamond m^*$ and $(m \Diamond n)^* = n^* \Box m^*$ for all $m, n \in A^{**}$ (cf. Dales and Lau [6, Chapter 2]). So, $\tau(\mathfrak{Z}_t(A^{**}, \Box)) = \mathfrak{Z}_t(A^{**}, \Diamond)$ and $\tau(\mathfrak{Z}_t(A^{**}, \Diamond)) = \mathfrak{Z}_t(A^{**}, \Box)$. Thus,

$$\mathfrak{Z}_t(A^{**}, \Box) = A \quad \text{if and only if} \quad \mathfrak{Z}_t(A^{**}, \Diamond) = A,$$

and $A \cdot \mathfrak{Z}_t(A^{**}, \Box) \subseteq A$ if and only if $\mathfrak{Z}_t(A^{**}, \Diamond) \cdot A \subseteq A$. Therefore, when A satisfies $\langle A^2\rangle = A$, by Theorem 23, we have

$$\mathfrak{Z}_t\big(\langle A^*A\rangle^*\big) \subseteq RM(A) \quad \text{if and only if} \quad \mathfrak{Z}_t\big(\langle AA^*\rangle^*\big) \subseteq LM(A).$$

It is routine to check that A is of type (LM) if and only if A is of type (RM), and hence if and only if A is of type (M) (cf. [16]). The theorem below is immediate by Theorem 25 and the relation between the assertions (1), (2), and (3).

Theorem 29. *Let A be an involutive Banach algebra of type (M). Consider the following statements.*

(i) $\mathfrak{Z}_t(\langle A^*A\rangle^*) = RM(A)$.
(ii) $\mathfrak{Z}_t(A^{**}, \Box) = A$.
(iii) $\mathfrak{Z}_t(A^{**}, \Box) = \mathfrak{Z}_t(A^{**}, \Diamond)$.

Then any two of (i)–(iii) *imply the third. Consequently, the following statements are equivalent.*

(a) $[\mathfrak{Z}_t(\langle A^*A\rangle^*) = RM(A)] \Longrightarrow [\mathfrak{Z}_t(A^{**}, \Box) = A]$.
(b) $[\mathfrak{Z}_t(\langle A^*A\rangle^*) = RM(A)] \Longrightarrow [\mathfrak{Z}_t(A^{**}, \Box) = \mathfrak{Z}_t(A^{**}, \Diamond)]$.
(c) $[A \cdot \mathfrak{Z}_t(A^{**}, \Box) \subseteq A] \Longrightarrow [A \cdot \mathfrak{Z}_t(A^{**}, \Diamond) \subseteq A]$.
(d) $[\mathfrak{Z}_t(A^{**}, \Diamond) \cdot A \subseteq A] \Longrightarrow [\mathfrak{Z}_t(A^{**}, \Box) \cdot A \subseteq A]$.

Finally, we relate the multiplier algebra $M(A)$ to topological centres. Let A be a Banach algebra with a BAI. Since the maps $(\mu_l, \mu_r) \longmapsto \mu_l$ and $(\mu_l, \mu_r) \longmapsto \mu_r$ are unital injective algebra homomorphisms from $M(A)$ to $LM(A)$ and $RM(A)$, respectively, we have two unital injective algebra homomorphisms

$$M(A) \longrightarrow \mathfrak{Z}_t\big(\langle AA^*\rangle^*\big) \quad \text{and} \quad M(A) \longrightarrow \mathfrak{Z}_t\big(\langle A^*A\rangle^*\big).$$

(Cf. Lemma 1(ii).) The theorem below shows that if A is of type (LM) and $\mathfrak{Z}_t(\langle A^*A\rangle^*) = M(A)$ (i.e., the first embedding above is onto), then the left strong Arens irregularity of A can be determined by testing elements of $\mathfrak{Z}_t(A^{**}, \Box)$ against one particular element of A^{**}, rather than verifying conditions like $\mathfrak{Z}_t(A^{**}, \Box) \cdot A \subseteq A$, or $\mathfrak{Z}_t(A^{**}, \Box) \Box A^* \subseteq \langle AA^*\rangle$ (comparing with [16, Theorem 18]).

We recall that an element E of A^{**} is called a mixed identity of A^{**} if $m \Box E = E \diamond m = m$ for all $m \in A^{**}$. It is known that E is a mixed identity of A^{**} if and only if E is a weak*-cluster point of a BAI of A (cf. Dales [5, Proposition 2.9.16(iii)]).

Theorem 30. *Let A be a Banach algebra with a mixed identity E of A^{**}.*

(i) *Assume that A is of type (LM) and $\mathfrak{Z}_t(\langle A^*A\rangle^*) = M(A)$. Let $m \in \mathfrak{Z}_t(A^{**}, \Box)$. Then $m \in A$ if and only if $E \Box m = m$.*
(ii) *Assume that A is of type (RM) and $\mathfrak{Z}_t(\langle AA^*\rangle^*) = M(A)$. Let $m \in \mathfrak{Z}_t(A^{**}, \diamond)$. Then $m \in A$ if and only if $m \diamond E = m$.*

Proof. We consider only assertion (i). A similar argument shows (ii).

Obviously, $E \Box m = E \diamond m = m$ if $m \in A$. Conversely, suppose that $E \Box m = m$. By the assumption, $M(A) \hookrightarrow RM(A)$ and $RM(A) \hookrightarrow \mathfrak{Z}_t(\langle A^*A\rangle^*)$ are both surjective. By Theorem 23(i), we have $A \cdot \mathfrak{Z}_t(A^{**}, \Box) \subseteq A$. Hence, $a \longmapsto a \cdot m$ defines a $\mu_r \in RM(A)$. Then, there exists $\mu_l \in LM(A)$ such that $(\mu_l, \mu_r) \in M(A)$. For all $a, b \in A$, we have

$$b \cdot \mu_l(a) = \mu_r(b) \cdot a = (b \cdot m) \cdot a = b \cdot (m \cdot a).$$

Thus $n \cdot \mu_l(a) = n \Box (m \cdot a)$ for all $n \in A^{**}$ and $a \in A$. It follows that, for all $a \in A$,

$$m \cdot a = (E \Box m) \cdot a = E \Box (m \cdot a) = E \cdot \mu_l(a) = \mu_l(a) \in A;$$

that is, $m \cdot A \subseteq A$. Therefore, $m \in A$ by [16, Theorem 32(i)]. $\Box$

We have the following immediate corollary of Theorem 30, which shows that for a Banach algebra A of type (LM), if A has a central BAI, then $\mathfrak{Z}_t(\langle A^*A\rangle^*) = M(A)$ does imply that A is left strongly Arens irregular without the need of any testing point.

Corollary 31. *Let A be a Banach algebra with a central BAI. Assume that A is of type (LM). If $\mathfrak{Z}_t(\langle A^*A\rangle^*) = M(A)$, then $\mathfrak{Z}_t(A^{**}, \Box) = A$. In particular, when A satisfies $RM(A) = M(A)$, then $\mathfrak{Z}_t(\langle A^*A\rangle^*) = M(A)$ if and only if $\mathfrak{Z}_t(A^{**}, \Box) = A$.*

Proof. Assume that A is of type (LM) and $\mathfrak{Z}_t(\langle A^*A\rangle^*) = M(A)$. Let E be a weak*-cluster point in A^{**} of a central BAI. Then $E \Box m = m \diamond E$ for all $m \in A^{**}$. Therefore, if $m \in \mathfrak{Z}_t(A^{**}, \Box)$, then $E \Box m = m \diamond E = m \Box E = m$. Hence, we have $\mathfrak{Z}_t(A^{**}, \Box) = A$ by Theorem 30(i).

The second assertion follows from Theorem 23(i). $\Box$

With group algebras $L_1(G)$ and the Banach algebra given in Proposition 36 of the next section, we see that the two conditions "$\mathfrak{Z}_t(A^{**}, \Box) = \mathfrak{Z}_t(A^{**}, \diamond)$" and "$A$ has a central BAI" are independent, which are required in Corollaries 26 and 31, respectively.

Recall that the condition $RM(A) = M(A)$ (cf. Corollary 31) is satisfied by convolution Beurling algebras $L_1(G,\omega)$ and Fourier algebras $A(G)$. This condition is also satisfied by the quantum group algebra $L_1(\mathbb{G})$ of any co-amenable quantum group $\mathbb{G}$.

In fact, if $\mathbb{G}$ is a co-amenable locally compact quantum group, then we have the canonical isometric algebra isomorphisms

$$M(\mathbb{G}) \cong M\big(L_1(\mathbb{G})\big) \cong RM\big(L_1(\mathbb{G})\big) \cong LM\big(L_1(\mathbb{G})\big).$$

(Cf. Hu, Neufang and Ruan [17].) As shown by Kraus and Ruan [20, Proposition 3.1] for Kac algebras, if $\mathbb{G}$ is co-amenable, every left (respectively, right) multiplier on $L_1(\mathbb{G})$ is completely bounded. In this situation, the subscript "*cb*" can be added to the above algebras of multipliers and all the identifications there become completely isometric isomorphisms. We will study multipliers on co-amenable locally compact quantum groups in the subsequent work [17]. See Junge, Neufang and Ruan [18] for representations of *cb*-multipliers over general locally compact quantum groups.

It is known that $L_1(\mathbb{G})$ is a two-sided ideal in $M(\mathbb{G})$. We showed in [16, Proposition 1] that the multiplication on $L_1(\mathbb{G})$ is faithful. By Kustermans and Vaes [21, Corollary 6.11] and modifying the arguments used in the proof of [16, Proposition 1], one can show that the multiplication on $M(\mathbb{G})$ is also faithful. Thus $M(\mathbb{G})$ can be canonically identified with a subalgebra of $RM(L_1(\mathbb{G}))$ via $\nu \longmapsto \nu_r$, where $\nu_r(a) = a \star \nu$ $(a \in L_1(\mathbb{G}))$. Therefore, $RM(L_1(\mathbb{G})) \subseteq \mathfrak{Z}_t(LUC(\mathbb{G})^*)$ implies that $M(\mathbb{G}) \subseteq \mathfrak{Z}_t(LUC(\mathbb{G})^*)$. In general, the converse implication is not true, since when $L_1(\mathbb{G}) = A(G)$, the latter always holds (cf. Lau and Losert [27, Proposition 4.5]), but the former holds precisely when G is amenable (cf. Theorem 15).

However, we show below that the two inclusions $\mathfrak{Z}_t(LUC(\mathbb{G})^*) \subseteq RM(L_1(\mathbb{G}))$ and $\mathfrak{Z}_t(LUC(\mathbb{G})^*) \subseteq M(\mathbb{G})$ are equivalent.

Theorem 32. *Let $\mathbb{G}$ be a locally compact quantum group. Then the following statements are equivalent.*

(i) $L_1(\mathbb{G})$ *is quotient strongly Arens irregular.*
(ii) $\mathfrak{Z}_t(LUC(\mathbb{G})^*) \subseteq RM(L_1(\mathbb{G}))$.
(iii) $\mathfrak{Z}_t(LUC(\mathbb{G})^*) \subseteq M(\mathbb{G})$.
(iv) $L_1(\mathbb{G}) \star \mathfrak{Z}_t(L_1(\mathbb{G})^{**}, \Box) \subseteq L_1(\mathbb{G})$.
(v) $L_1(\mathbb{G}) \star \mathfrak{Z}_t(LUC(\mathbb{G})^*) \subseteq L_1(\mathbb{G})$.

Proof. Note that $L_1(\mathbb{G})$ is an involutive Banach algebra satisfying $\langle L_1(\mathbb{G})^2\rangle = L_1(\mathbb{G})$ (cf. [16] and Fact 1). By Corollary 4(i) and Theorem 23(i) together with the discussions before Theorem 29, we see that (i), (ii), (iv), and (v) are equivalent. Clearly, (iii) $\Longrightarrow$ (ii). So, we only have to show that (ii) $\Longrightarrow$ (iii).

Assume that $\mathfrak{Z}_t(LUC(\mathbb{G})^*) \subseteq RM(L_1(\mathbb{G}))$. Let $m \in \mathfrak{Z}_t(LUC(\mathbb{G})^*)$. Then there exists $\mu \in RM(L_1(\mathbb{G}))$ such that $m_L = \mu^*$; that is,

$$\langle m, f \star a\rangle = \big\langle f, \mu(a)\big\rangle \quad \big(f \in L_\infty(\mathbb{G}),\ a \in L_1(\mathbb{G})\big).$$

It is known that $C_0(\mathbb{G}) \subseteq LUC(\mathbb{G})$ (cf. [41, Theorem 2.3]). Let $\nu = m|_{C_0(\mathbb{G})}$. Then $\nu \in C_0(\mathbb{G})^* = M(\mathbb{G})$. Let $a \in L_1(\mathbb{G})$. Then, for all $f \in C_0(\mathbb{G})$, we have

$$\langle f, a \star \nu\rangle = \langle \nu, f \star a\rangle = \langle m, f \star a\rangle = \big\langle m_L(f), a\big\rangle = \big\langle \mu^*(f), a\big\rangle = \big\langle f, \mu(a)\big\rangle.$$

Thus $\langle f, a \star \nu\rangle = \langle f, \mu(a)\rangle$ for all $f \in L_\infty(\mathbb{G})$, since $C_0(\mathbb{G})$ is weak*-dense in $L_\infty(\mathbb{G})$. It follows that $\mu(a) = a \star \nu$. Therefore, $\mu = \nu_r$, and hence $m_L = \mu^* = (\nu_r)^*$; that is, $m \in M(\mathbb{G})$ (cf. Fact 2). $\square$

Finally, with Theorems 18 and 32, we are able to characterize quantum groups $\mathbb{G}$ satisfying $\mathfrak{Z}_t(LUC(\mathbb{G})^*_R) = RM(L_1(\mathbb{G}))$.

Theorem 33. *Let $\mathbb{G}$ be a locally compact quantum group. Then the following statements are equivalent.*

(i) $\mathfrak{Z}_t(LUC(\mathbb{G})^*_R) = RM(L_1(\mathbb{G}))$.
(ii) $\mathbb{G}$ *is co-amenable and SIN, and* $L_1(\mathbb{G})$ *is quotient strongly Arens irregular.*

Proof. (i) $\Longrightarrow$ (ii). Due to Theorem 18, $\mathbb{G}$ is co-amenable and SIN.

In this situation, we have $LUC(\mathbb{G}) = RUC(\mathbb{G}) = L_1(\mathbb{G}) \star L_\infty(\mathbb{G}) \star L_1(\mathbb{G})$, and $RM(L_1(\mathbb{G})) \cong LM(L_1(\mathbb{G}))$ (cf. [17]). Thus it can be seen that the identity map $LUC(\mathbb{G})^* \longrightarrow RUC(\mathbb{G})^*$ maps $\mathfrak{Z}_t(LUC(\mathbb{G})^*_R)$ onto $\mathfrak{Z}_t(RUC(\mathbb{G})^*)$ and $RM(L_1(\mathbb{G}))$ onto $LM(L_1(\mathbb{G}))$. It follows that $\mathfrak{Z}_t(RUC(\mathbb{G})^*) = LM(L_1(\mathbb{G}))$, which is equivalent to

$$\mathfrak{Z}_t\big(LUC(\mathbb{G})^*\big) = RM\big(L_1(\mathbb{G})\big)$$

(see the paragraphs before Theorem 29). Therefore, by Theorem 32, $L_1(\mathbb{G})$ is quotient strongly Arens irregular.

Similar arguments will establish (ii) $\Longrightarrow$ (i). $\square$

6. Some examples of Arens irregular Banach algebras

We start this section with an example related to an open question in [28]. For a Banach algebra A with a BAI, Lau and Ülger asked whether "$A \cdot \mathfrak{Z}_t(A^{**}, \Box) \subseteq A$" implies that "$\mathfrak{Z}_t(A^{**}, \Diamond) \cdot A \subseteq A$" (see [28, question 6e)]). By Theorem 23, this is equivalent to asking whether "$\mathfrak{Z}_t(\langle A^*A\rangle^*) = RM(A)$" implies that "$\mathfrak{Z}_t(\langle AA^*\rangle^*) = LM(A)$". This question was answered in the negative by Ghahramani, McClure, and Meng in [11, Theorem 3]. However, the proof of [11, Theorem 3] used an identification of $\mathfrak{Z}_t(K(c_0)^{**}, \Diamond)$ from [28], that is incorrect as pointed out by Dales and Lau in [6, Example 6.2].

In [6, Example 4.5], Dales and Lau constructed an LSAI Banach algebra which is not RSAI. It is seen that this Banach algebra has a BLAI. An earlier example of an LSAI Banach algebra with a BRAI which is not RSAI was given by Neufang (see, e.g., [6, p. 41]). Note that these two Banach algebras are both WSC. By taking the unitization and applying [11, Lemma 1], one can obtain a unital WSC Banach algebra A such that $\mathfrak{Z}_t(A^{**}, \Box) = A$ but $\mathfrak{Z}_t(A^{**}, \Diamond) \neq A$.

We give below a non-unital WSC Banach algebra A with a sequential BAI which is LSAI but far from being RSAI or right quotient strongly Arens irregular. In this situation, the two Banach algebras A^{**} and $\langle A^*A\rangle^*$ do not coincide, and the topological centre problems for $\langle AA^*\rangle^*$ and $\langle A^*A\rangle^*$ are distinct. Recall that for all locally compact groups G, we have $\mathfrak{Z}_t(L_1(G)^{**}, \Box) = \mathfrak{Z}_t(L_1(G)^{**}, \Diamond) = L_1(G)$ (cf. Lau and Losert [26]).

Proposition 34. *There exists a non-unital WSC Banach algebra A with a sequential BAI such that*

(i) $\mathfrak{Z}_t(A^{**}, \Box) = A$ *but* $\mathfrak{Z}_t(A^{**}, \Diamond) \neq A$;
(ii) $A \cdot \mathfrak{Z}_t(A^{**}, \Box) = \mathfrak{Z}_t(A^{**}, \Box) \cdot A = A$, *but* $\mathfrak{Z}_t(A^{**}, \Diamond) \cdot A \nsubseteq A$ *and* $A \cdot \mathfrak{Z}_t(A^{**}, \Diamond) \nsubseteq A$;
(iii) $\mathfrak{Z}_t(\langle A^*A\rangle^*) = RM(A)$ *but* $\mathfrak{Z}_t(\langle AA^*\rangle^*) \neq LM(A)$.

Proof. As mentioned above, there exists a unital WSC Banach algebra B satisfying $\mathfrak{Z}_t(B^{**}, \Box) = B$ and $\mathfrak{Z}_t(B^{**}, \Diamond) \neq B$. Let $C = L_1(\mathbb{R})$ and $A = B \oplus_1 C$. Then A is a WSC Banach algebra under the usual multiplication, and is non-unital with a sequential BAI. It is clear that

$$\mathfrak{Z}_t(A^{**}, \Box) = \mathfrak{Z}_t(B^{**}, \Box) \oplus \mathfrak{Z}_t(C^{**}, \Box) = B \oplus C = A,$$

and

$$\mathfrak{Z}_t(A^{**}, \Diamond) = \mathfrak{Z}_t(B^{**}, \Diamond) \oplus \mathfrak{Z}_t(C^{**}, \Diamond) = \mathfrak{Z}_t(B^{**}, \Diamond) \oplus C \supsetneqq B \oplus C = A.$$

Therefore, (i) holds.

Note that B is unital and $\mathfrak{Z}_t(B^{**}, \Diamond) \neq B$. We have $\mathfrak{Z}_t(B^{**}, \Diamond) \cdot B \nsubseteq B$, and hence

$$\mathfrak{Z}_t(A^{**}, \Diamond) \cdot A = \big(\mathfrak{Z}_t(B^{**}, \Diamond) \cdot B\big) \oplus C^2 = \big(\mathfrak{Z}_t(B^{**}, \Diamond) \cdot B\big) \oplus C \nsubseteq B \oplus C = A.$$

Since A is of type (M) and $\mathfrak{Z}_t(A^{**}, \Diamond) \neq A$, we have $A \cdot \mathfrak{Z}_t(A^{**}, \Diamond) \nsubseteq A$ by [16, Theorem 32(ii)]. Therefore, (ii) holds. Finally, (iii) follows from (ii) and Theorem 23. $\Box$

For a Banach algebra as in Proposition 34, taking the opposite algebra, one obtains a non-unital WSC Banach algebra A with a (sequential) BAI satisfying $\mathfrak{Z}_t(\langle A^*A\rangle^*) \neq RM(A)$. This answers [28, question 6f)] in the negative, where it was asked whether $\mathfrak{Z}_t(\langle A^*A\rangle^*) = RM(A)$ if A is such a Banach algebra (see also [16, Proposition 34]).

The proposition below shows that the answer to [28, question 6k)] is also negative, where Lau and Ülger asked whether $\mathfrak{Z}_t(\langle A^*A\rangle^*)$ is a dual Banach space when A is a Banach algebra with a BAI.

Proposition 35. *There exists a Banach algebra A with a BAI such that neither $\langle A^*A\rangle$ nor $\mathfrak{Z}_t(\langle A^*A\rangle^*)$ is a dual Banach space.*

Proof. Let B be the unitization of the group algebra $L_1(\mathbb{T})$. By [11, Lemma 1], we have $\mathfrak{Z}_t(B^{**}, \Box) = \mathfrak{Z}_t(L_1(\mathbb{T})^{**}, \Box) \oplus \mathbb{C} = L_1(\mathbb{T}) \oplus \mathbb{C}$. Let $A = L_1(\mathbb{T}) \oplus_1 B$. Then

$$A^* = L_\infty(\mathbb{T}) \oplus B^*, \qquad \langle A^*A\rangle = C(\mathbb{T}) \oplus B^*, \quad \text{and} \quad \langle A^*A\rangle^* = C(\mathbb{T})^* \oplus B^{**}.$$

Therefore, $\mathfrak{Z}_t(\langle A^*A\rangle^*) = Z(C(\mathbb{T})^*) \oplus \mathfrak{Z}_t(B^{**}, \Box) = M(\mathbb{T}) \oplus (L_1(\mathbb{T}) \oplus \mathbb{C})$. In this situation, neither $\langle A^*A\rangle$ nor $\mathfrak{Z}_t(\langle A^*A\rangle^*)$ is a dual Banach space, since $C(\mathbb{T})$ and $L_1(\mathbb{T})$ are both not dual Banach spaces. $\Box$

For an involutive Banach algebra A, as before, we let $\tau : A^{**} \longrightarrow A^{**}$ denote the unique weak*–weak* continuous extension of the involution on A. Note that $\mathfrak{Z}_t(A^{**}, \Box) = \mathfrak{Z}_t(A^{**}, \Diamond)$ if and only if $\tau(\mathfrak{Z}_t(A^{**}, \Box)) = \mathfrak{Z}_t(A^{**}, \Box)$. Therefore,

$$\mathfrak{Z}_t(A^{**}, \Box) = \mathfrak{Z}_t(A^{**}, \Diamond) \quad \text{if and only if} \quad \big(\mathfrak{Z}_t(A^{**}, \Box), \tau\big) \text{ is an involutive Banach algebra.}$$

In [6], Dales and Lau constructed some interesting involutive Banach algebras C with $\mathfrak{Z}_t(C^{**}, \Box) \neq \mathfrak{Z}_t(C^{**}, \Diamond)$, where either a convolution Beurling algebra $\ell_1(F_2, \omega)$ or the C^*-algebra $K(c_0)$ was used in their constructions. It is seen that these Banach algebras are either unital or non-WSC. In the proposition below, using group algebras, we define a non-unital WSC separable involutive Banach algebra A with a BAI such that $\mathfrak{Z}_t(A^{**}, \Box) \neq \mathfrak{Z}_t(A^{**}, \Diamond)$.

Proposition 36. *There exists a non-unital WSC separable involutive Banach algebra A with a central BAI such that $\mathfrak{Z}_t(A^{**}, \Box) \neq \mathfrak{Z}_t(A^{**}, \Diamond)$, $\mathfrak{Z}_t(\langle A^*A\rangle^*) \neq RM(A)$, and A^* is a von Neumann algebra.*

Proof. We will combine and modify some constructions provided in [6, Examples 4.4 and 4.5]. Take $B = \ell_1(\mathbb{Z}) \oplus_1 \ell_1(\mathbb{Z})$ with the multiplication given by

$$(f_1, g_1)(f_2, g_2) = (f_1 * f_2, f_1 * g_2) \quad \big(f_1, f_2, g_1, g_2 \in \ell_1(\mathbb{Z})\big).$$

Then B is a WSC Banach algebra satisfying $B = \mathfrak{Z}_t(B^{**}, \Box) \neq \mathfrak{Z}_t(B^{**}, \Diamond)$ (cf. [6, Example 4.5]). Obviously, the multiplication on B is not faithful.

For $(f, g) \in B$, let $\overline{(f, g)} = (\overline{f}, \overline{g})$, where $\overline{f}(x) = \overline{f(x)}$ (the complex conjugate of $f(x)$). This defines a linear involution on B satisfying $\overline{b_1 b_2} = \overline{b_1}\,\overline{b_2}$ $(b_1, b_2 \in B)$. Replacing B by its unitization, we can obtain a unital WSC Banach algebra B with a linear involution as above such that $B = \mathfrak{Z}_t(B^{**}, \Box) \neq \mathfrak{Z}_t(B^{**}, \Diamond)$.

Following the same arguments as used in [6, Example 4.4], we let $C = B \oplus_1 B^{op}$ with the usual multiplication, and define $(b_1, b_2)^* = (\overline{b_2}, \overline{b_1})$ $(b_1, b_2 \in B)$. Then C is a unital WSC involutive Banach algebra such that

$$B \oplus \mathfrak{Z}_t(B^{**}, \Diamond) = \mathfrak{Z}_t(C^{**}, \Box) \neq \mathfrak{Z}_t(C^{**}, \Diamond) = \mathfrak{Z}_t(B^{**}, \Diamond) \oplus B,$$

since $\mathfrak{Z}_t(B^{**}, \Diamond) \neq B$.

Finally, as in the proof of Proposition 34, we let $A = C \oplus_1 L_1(\mathbb{R})$. Clearly, A^* is a von Neumann algebra. Under the canonical multiplication and involution, A is a non-unital WSC separable involutive Banach algebra with a central BAI such that $\mathfrak{Z}_t(A^{**}, \Box) \neq \mathfrak{Z}_t(A^{**}, \Diamond)$. Since B is unital and $\mathfrak{Z}_t(B^{**}, \Diamond) \neq B$, it can be seen that $A \cdot \mathfrak{Z}_t(A^{**}, \Box) \not\subseteq A$. Therefore, $\mathfrak{Z}_t(\langle A^*A\rangle^*) \neq RM(A)$ by Theorem 23(i). $\Box$

Remark 37. Comparing with Theorem 29 and Proposition 36, we note that if A is taken to be $A(SU(3) \times \mathbb{Z})$, then we have $\mathfrak{Z}_t(A^{**}, \Box) = \mathfrak{Z}_t(A^{**}, \Diamond) \neq A$, and $\mathfrak{Z}_t(\langle A^*A\rangle^*) \neq RM(A)$ (cf. [16, Proposition 34]).

Recall that we say that a Banach algebra A is left quotient Arens regular if $\mathfrak{Z}_t(\langle A^*A\rangle^*) = \langle A^*A\rangle^*$ (see Section 2). It is seen that if A is left quotient Arens regular, and B is a closed subalgebra of A such that the restriction map $A^* \longrightarrow B^*$ maps $\langle A^*A\rangle$ onto $\langle B^*B\rangle$, then B is also left quotient Arens regular. Clearly, if A is Arens regular, then A is quotient Arens regular. Examples below illustrate that the converse does not hold even for the two classical quantum group algebras.

Example 38. Let G be a locally compact group.

(i) It is known that $L_1(G)$ has a BAI, and $L_1(G)$ is not Arens regular unless G is finite (cf. Young [47]). Note that $\mathfrak{Z}_t(LUC(G)^*) = \mathfrak{Z}_t(RUC(G)^*) = M(G) = C_0(G)^*$ (cf. Lau [25]). Therefore, $L_1(G)$ is quotient Arens regular if and only if G is compact. In particular, $L_1(G)$ is quotient Arens regular but not Arens regular precisely when G is infinite and compact.

(ii) It is also known that $B_\rho(G) \subseteq \mathfrak{Z}_t(UC(\widehat{G})^*) \subseteq UC(\widehat{G})^*$, and $B_\rho(G) = UC(\widehat{G})^*$ if and only if G is discrete, where $B_\rho(G)$ is the reduced Fourier–Stieltjes algebra of G and $UC(\widehat{G}) = \langle A(G)VN(G)\rangle$ is the C^*-algebra of uniformly continuous functionals on $A(G)$ (cf. Lau and Losert [27]). Then $A(F_2)$ is quotient Arens regular since F_2 is discrete. However, $A(F_2)$ is not Arens regular (cf. Forrest [10]). Let L be a non-amenable second countable connected group, and let $H = \mathbb{R} \times L$. By Lau and Losert [27, Corollary 5.9], we have $\mathfrak{Z}_t(UC(\widehat{H})^*) = B_\rho(H)$. However, $B_\rho(H) \subsetneqq UC(\widehat{H})^*$ since H is non-discrete. Therefore, $A(H)$ is not quotient Arens regular, and hence is not Arens regular. Note that the Fourier algebras $A(F_2)$ and $A(H)$ both do not have a BAI.

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