

MEAN FIELD SOCIAL OPTIMIZATION: FEEDBACK PERSON-BY-PERSON OPTIMALITY AND THE DYNAMIC PROGRAMMING EQUATION*

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Abstract. We consider mean field social optimization in nonlinear diffusion models. By dynamic programming with a representative agent employing cooperative optimizer selection, we derive a new Hamilton–Jacobi–Bellman (HJB) equation to be called the master equation of the value function. Under some regularity conditions, we establish ϵ -person-by-person optimality of the master equation–based control laws, which may be viewed as a necessary condition for nearly attaining the social optimum. A major challenge in the analysis is to obtain tight estimates, within an error of $O(1/N)$, of the social cost having order $O(N)$. This will be accomplished by multiscale analysis via constructing two auxiliary master equations. We illustrate explicit solutions of the master equations for the linear-quadratic (LQ) case and give an application to systemic risk.

Key words. controlled diffusion, mean field social optimization, person-by-person optimality, dynamic programming, master equation

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1. Introduction. Mean field social optimization studies decision problems involving a large number of agents which have a common optimization objective and interact through coupling in their individual dynamics or costs, or both [27]. These problems are also referred to as large population optimal control [20]. The cooperative behavior of the agents differs from the noncooperative behavior of the agents in mean field games [7]. The reader is referred to [3, 26, 27, 31, 41] for the analysis of social optima in a linear-quadratic (LQ) framework. McKean–Vlasov optimal control has been studied in [5, 8, 11, 21, 32, 34, 39, 42] and may be heuristically interpreted as a limit form of large population optimal control. Cooperative mean field control has applications in economic theory [33], collective choice problems [36], multi-agent flocking [35], and power systems [17]. The notion of social optima is also useful for measuring (in)efficiency of mean field games [4, 10].

In this paper we analyze social optimization for a class of nonlinear models. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be the underlying filtered probability space satisfying the usual condition. Consider a population of agents $\mathcal{A}_i$, $1 \leq i \leq N$, satisfying the following stochastic differential equations (SDEs):

$$(1.1) \quad dX_t^i = f(X_t^i, u_t^i, \mu_t^{-i})dt + \sigma dW_t^i + \sigma_0 dW_t^0, \quad 1 \leq i \leq N, \quad t \geq 0,$$

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where X_t^i and u_t^i are agent $\mathcal{A}_i$'s state and control, respectively, and

$$\mu_t^{-i} := \frac{1}{N-1} \sum_{1 \leq j \leq N, j \neq i} \delta_{X_t^j}$$

is the empirical distribution of all other $N-1$ agents, with δ_x being the Dirac measure at $x \in \mathbb{R}^n$. The initial states $\{X_0^i, 1 \leq i \leq N\}$ are $\mathcal{F}_0$ -measurable and are independent with finite second moment. The $N+1$ $\mathcal{F}_t$ -adapted standard Brownian motions $\{W^j, 0 \leq j \leq N\}$, as the individual noises and the common noise, respectively, are independent and also independent of the initial states. The dimensions of X_t^i , u_t^i , W_t^i , and W_t^0 are n , n_1 , n_2 , and n_3 , respectively, and f , σ , and σ_0 have compatible dimensions. We restrict ourselves to the case of constant noise coefficients. Our approach can deal with state and control dependent noise.

Agent $\mathcal{A}_i$ has its own cost

$$(1.2) \quad J_i(u^i(\cdot), \mathbf{u}^{-i}(\cdot)) = \mathbb{E} \int_0^T L(X_t^i, u_t^i, \mu_t^{-i}) dt + \mathbb{E} g(X_T^i, \mu_T^{-i}),$$

where $\mathbf{u}^{-i}(\cdot)$ denotes the controls of all agents other than $\mathcal{A}_i$. To avoid heavy notation, we do not include t as an argument in f, σ, σ_0 , and L but can treat the t -dependent case without further difficulty. The social cost is given by

$$(1.3) \quad J_{\text{soc}}^{(N)}(\mathbf{u}) = \sum_{i=1}^N J_i, \quad \mathbf{u} := (u^1, \dots, u^N).$$

1.1. Method, analytical challenges, and contributions. One may attempt to minimize the social cost by directly solving an optimal control problem. This approach, however, becomes infeasible when N is large. Instead, we develop our solution by studying the optimizing behavior of a representative agent. We exploit a simple but useful idea in team decision theory called person-by-person (PbP) optimality. The reader is referred to [24, 37, 18] for its characterization and to [15] for its feedback form in decentralized stochastic control. To explain the idea, let $\mathcal{J}(u^1, \dots, u^N)$ be the team cost of N agents with strategies u^i , $1 \leq i \leq N$. Under a given information structure, if the team attains its optimum by a joint strategy $(\hat{u}^1, \dots, \hat{u}^N)$, then no agent can unilaterally take a new strategy to improve for the team. PbP optimality is a necessary condition for team optimality.

Our PbP optimality-based approach studies the value function $V(t, x, \mu)$ of a representative agent and applies dynamic programming in an extended state space including an individual state x and a measure μ describing the mean field. The value function will be identified by a specific procedure (see section 2.1) but not by simply infimizing a cost functional as in standard stochastic optimal control. Our method enables the agent to choose its optimizer in a feedback form $\hat{u}^i = \phi(t, x, \mu)$, which differs from [38], where the control perturbation is a nonanticipative process. In the end we obtain a special Hamilton–Jacobi–Bellman (HJB) equation, which will be called the master equation for the value function; some heuristic derivation has appeared in [29] without performance analysis. Master equations have been widely studied in mean field games [6, 9, 12] and also mean field optimal control [5, 6, 21, 34].

In the setting of social optimization (or large population optimal control), our approach above differs from some existing works, which approximate the N -agent optimal control problem by a McKean–Vlasov optimal control problem (see, e.g., [20, 21,

32, 42]). As a result, their value function takes the form $V_{mv}(t, \mu)$, only characterizing the performance of the population as a whole. However, in practical cooperative decision scenarios, each constituent agent still desires to know its own performance; our approach responds to this need and specifies an individual's value function $V(t, x, \mu)$. The authors in [5] formulate an optimal control problem combining a representative agent's state equation with the McKean–Vlasov dynamics, both controlled by the same process. They prove a dynamic programming principle without common noise, and the drift term in their master equation is structured differently from ours.

When the control laws $(\hat{u}^1, \dots, \hat{u}^N)$ of N agents have been determined from the master equation, an issue of central importance is their performance. For certain classes of models, the value function of the controlled McKean–Vlasov dynamics has been shown to be the limit of the value function of the N -agent optimal control problem as $N \rightarrow \infty$ [8, 21]. However, to our best knowledge, except for LQ cases [22, 27, 28, 41], there has existed no past work on social optimization addressing the performance of the infinite population-based control laws when they are applied by finite populations. As it turns out, our PbP optimality-based approach provides a tractable framework for such performance analysis. We will establish approximate PbP optimality for the master equation-based control laws in optimizing $J_{soc}^{(N)}$, which may be viewed as a necessary condition for achieving near social optimality. Naturally, it is desirable to bound the gap between the attained performance and exact PbP optimality as tightly as possible. A loose bound would not tell much about performance.

PbP optimality may be viewed as a special case of a Nash equilibrium where all agents take the social cost as their individual costs. For this reason, our performance estimate in terms of approximate PbP optimality is similar to the ϵ -Nash equilibrium analysis in mean field games [7]. However, we confront a far more difficult problem, especially under model nonlinearity. We may think of $J_{soc}^{(N)}$ as a quantity of order $O(N)$ while expecting the agent in question to minimize it up to an accuracy of $o(1)$. The estimate of the performance loss becomes very intricate since we face quantities of such drastically different scales as $O(N)$ and $o(1)$. In contrast, the ϵ -Nash equilibrium analysis involves quantities of scales $O(1)$ and $o(1)$ for an agent's cost and its performance loss, respectively. Under some regularity conditions pertaining to the solution of the master equation, we prove that $\hat{u}^i$ is nearly PbP optimal, with an inadequacy (or regret) of at most $O(1/N)$ for optimizing $J_{soc}^{(N)}$.

Our performance analysis depends on exploiting the multiscale feature when an individual attempts to optimize the social cost, where the key idea is to decompose the social cost roughly as the sum of a much larger macroscopic term and another term that carries information on the representative agent and so will be called the instrumental value function. Meanwhile, we need to introduce two auxiliary master equations. Such combined usage of functions describing phenomena of different scales, together with the construction of their equations, has got much of its inspiration from the physics literature on computing tiny quantities which are superposed to very large ones [1, pp. 199–209], [19, pp. 195–198] and also from perturbation methods of dynamical systems [25, Chap. 3].

The paper is organized as follows. Section 2 introduces the master equation of the value function. To prepare for performance analysis, section 3 introduces the instrumental value function and two auxiliary master equations. The asymptotic PbP optimality theorem is proved in section 4. Section 5 illustrates explicit solutions of the master equations by linear-quadratic (LQ) models and a systemic risk example.

1.2. Notation. The Frobenius norm of a vector or matrix v is denoted by $|v|$. For the function $\psi(x)$ from $\mathbb{R}^n$ to $\mathbb{R}$, we denote the partial derivatives $\partial_{x_i}\psi$ and $\partial_{x_i x_j}\psi := \partial_{x_i}(\partial_{x_j}\psi)$. Denote $\partial_x\psi := (\partial_{x_1}\psi, \dots, \partial_{x_n}\psi)$ as a row vector. The Hessian matrix is $\partial_x^2\psi := (\partial_{x_i x_j}\psi)_{1 \leq i, j \leq n}$, where $\partial_{x_i x_j}\psi$ is the (i, j) th entry of the matrix. If $\psi(x, y)$ is a scalar function with $x, y \in \mathbb{R}^n$, we denote

$$\partial_{xy}\psi(x, y) := (\partial_{x_i y_j}\psi(x, y))_{1 \leq i, j \leq n}.$$

The evaluation $\psi(x, y)|_{y=x}$ gives $\psi(x, x)$. When there is only one space variable x in ψ (as in $\psi(t, x, \mu)$), we use the short notation $\psi_x = \partial_x\psi$ and $\psi_{xx} = \partial_x^2\psi$.

Let $\mathcal{P}_2(\mathbb{R}^n)$ be the space of Borel probability measures on $\mathbb{R}^n$ with finite second moment. On $\mathcal{P}_2(\mathbb{R}^n)$, we endow the Wasserstein distance $W_2(\mu, \nu)$ so that it becomes a complete metric space [2]. Let $\mathcal{P}_{\text{em}}^k(\mathbb{R}^n)$ consist of all probability measures μ on $\mathbb{R}^n$ such that $\mu = \frac{1}{k} \sum_{i=1}^k \delta_{x^i}$ for some $\mathbf{x} = (x^1, \dots, x^k)$ with $x^i \in \mathbb{R}^n$, $1 \leq i \leq k$.

For a function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ and a probability measure μ , we write the integral $\int_{\mathbb{R}^n} \psi(x) \mu(dx)$ as $\langle \mu, \psi \rangle$ or $\langle \psi \rangle_\mu$. We sometimes write such a form $\langle \mu(dy), \psi(t, y, \mu) \rangle$ to explicitly indicate the variable of integration. For two probability measures μ_1 and μ_2 , $\langle \mu_1 - \mu_2, \psi \rangle := \langle \mu_1, \psi \rangle - \langle \mu_2, \psi \rangle$.

Denote the agent index set $\mathcal{N} = \{1, \dots, N\}$, and denote $\mathcal{N}_{-i} = \mathcal{N} - \{i\}$. Throughout the paper, the agent indices $i, j \in \mathcal{N}$, etc., appearing in $(X_t^i, u_t^i, W_t^j, x^i, u^j, \text{etc.})$ are always interpreted as a superscript and not as an exponent. We follow [9] (which uses the notation $\delta V / \delta \mu$) to define the measure differentiation for a function $V(t, x, \mu)$; for fixed (t, x) , the derivative $\delta_\mu V$, if existing, is a function from $\mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n \ni (\mu, y)$ to $\mathbb{R}$, denoted by $\delta_\mu V(t, x, \mu; y)$, where y after the semicolon is the new variable arising from the differentiation, such that (i) $\delta_\mu V$ is a Lebesgue measurable function of y for any fixed (t, x, μ) , (ii) for all $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^n)$, it holds that

$$\lim_{s \rightarrow 0^+} \frac{1}{s} [V(t, x, (1-s)\mu + s\mu') - V(t, x, \mu)] = \int_{\mathbb{R}^n} \delta_\mu V(t, x, \mu; y) (\mu'(dy) - \mu(dy)),$$

and (iii) the normalization condition $\langle \mu(dy), \delta_\mu V(t, x, \mu; y) \rangle = 0$ holds. We make this normalization convention in (iii), which is needed in Proposition A.5. Higher-order derivatives are similarly introduced and denoted in the form $\delta_{\mu\mu} V(t, x, \mu; y, z)$, etc. A function (such as $\psi(t, x, \mu)$) is said to be jointly continuous (or simply to be continuous) if it is continuous under the product topology of $[0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)$.

Unless otherwise indicated, the partial derivative will be based on the actual appearance of the variables. For example, ∂_x in $\partial_x \delta_\mu V(t, y, \mu; x)$ means differentiation with respect to the variable after μ , even if we initially introduce the form $\delta_\mu V(t, x, \mu; y)$. The functions $\chi, \chi_1, \chi_2, \dots$ (such as $\chi(t, \mu), \chi_1(t, x, \mu)$, etc.) are reserved as a normalizing term (see section 5.1 for examples) when differentiating a function with respect to μ . In various estimates we use $C, C_1, C_2, \dots$ as generic constants which do not depend on N and may change from place to place.

2. The value function and its master equation. Let $\mathbb{U}$ be a nonempty closed subset of $\mathbb{R}^{n_1}$. Here and hereafter, we reuse C_a as a generic constant in various assumptions; we follow section 1.2 to define continuity of a function with several arguments by using the product topology of the domain. We make the following standing assumptions:

(A1) The map $f : \mathbb{R}^n \times \mathbb{U} \times \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R}^n$ is continuous, and Lipschitz continuous in (x, μ) , uniformly with respect to u . In addition, $|f(x, u, \mu)| \leq C_a(1 + |x| + |u| + \langle |y| \rangle_\mu)$.

(A2) $L: \mathbb{R}^n \times \mathbb{U} \times \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R}_+$ and $g: \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R}_+$ are continuous, and

$$L(x, u, \mu), \quad g(x, \mu) \leq C_a(1 + |x|^2 + |u|^2 + \langle |y| \rangle_\mu^2)$$

for all $(x, u, \mu) \in \mathbb{R}^n \times \mathbb{U} \times \mathcal{P}_2(\mathbb{R}^n)$.

2.1. The ϕ -value function. To explain the idea underlying the derivation of the master equation, we analyze a family of (N, ϕ) -indexed control problems for agent $\mathcal{A}_i$ in a population of N agents. Consider a set of control laws $\phi(t, X_t^k, \mu_t^{-k})$, $1 \leq k \leq N$, with a continuous function ϕ from $[0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)$ to $\mathbb{U}$ that ensures a unique solution to the SDE system (1.1).

For each ϕ , let $V^{N, \phi}(t, x^i, \mu^{-i})$ denote the cost

$$J_i(t, x^i, \mu^{-i}, \hat{\mathbf{u}}(\cdot)) = \mathbb{E} \int_t^T L(X_s^i, u_s^i, \mu_s^{-i}) ds + \mathbb{E} g(X_T^i, \mu_T^{-i})$$

of agent $\mathcal{A}_i$ when all N agents apply the control laws $\hat{u}_s^k = \phi(s, X_s^k, \mu_s^{-k})$, $1 \leq k \leq N$, on $[t, T]$ with $X_t^i = x^i$ and $\mu_t^{-i} = \mu^{-i} \in \mathcal{P}_{\text{em}}^{N-1}(\mathbb{R}^n)$. The choice of $\mu^{-i} \in \mathcal{P}_{\text{em}}^{N-1}(\mathbb{R}^n)$ is arbitrary, which may be matched by appropriate initial states of all other agents $\mathcal{A}_j$, $j \neq i$ at time t . Then $V^{N, \phi}$ is a well-defined function on $[0, T] \times \mathbb{R}^n \times \mathcal{P}_{\text{em}}^{N-1}(\mathbb{R}^n)$. It is clear that this cost depends on N . For notational simplicity, we will just write it as $V^\phi(t, x^i, \mu)$ by dropping the superscript N . We may denote the cost of $\mathcal{A}_i$ in this form due to symmetry of the other $N - 1$ agents. Specifically, if we permute the other agents' initial states, the cost of $\mathcal{A}_i$ remains the same. We shall call V^ϕ the ϕ -value function for $\mathcal{A}_i$.

Now let the N agents be assigned the initial time-state values (t, x^j, μ^{-j}) , $1 \leq j \leq N$, respectively, which are subject to the constraint $\frac{1}{N} \delta_{x^j} + \frac{N-1}{N} \mu^{-j} = \frac{1}{N} \sum_{k=1}^N \delta_{x^k}$ for all j . For the initial time $t \in [0, T]$, denote the controlled state processes for agents $\mathcal{A}_j$, $j \neq i$, by

$$(2.1) \quad dX_s^j = f(X_s^j, \phi(s, X_s^j, \mu_s^{-j}), \mu_s^{-j}) ds + \sigma dW_s^j + \sigma_0 dW_s^0, \quad s \geq t.$$

For agent $\mathcal{A}_i$, we have

$$(2.2) \quad dX_s^i = f(X_s^i, u_s^i, \mu_s^{-i}) ds + \sigma dW_s^i + \sigma_0 dW_s^0, \quad t \leq s \leq T,$$

where we take $u_s^i \equiv u^i \in \mathbb{U}$ for $s \in [t, t + \epsilon]$ and $u_s^i = \phi(s, X_s^i, \mu_s^{-i})$ on $[t + \epsilon, T]$. Under (2.1)–(2.2), denote

$$(2.3) \quad \begin{aligned} J_i(t, x^i, \mu^{-i}, u^i(\cdot), \hat{\mathbf{u}}^{-i}(\cdot)) &:= \mathbb{E} \int_t^T L(X_s^i, u_s^i, \mu_s^{-i}) ds + \mathbb{E} g(X_T^i, \mu_T^{-i}) \\ &= \mathbb{E} \left[\int_t^{t+\epsilon} L(X_s^i, u^i, \mu_s^{-i}) ds + V^\phi(t + \epsilon, X_{t+\epsilon}^i, \mu_{t+\epsilon}^{-i}) \right]. \end{aligned}$$

Using dynamic programming, we select $u^i \in \mathbb{U}$ on the small interval $[t, t + \epsilon]$ to minimize

$$\mathbb{E} \left[\int_t^{t+\epsilon} L(X_s^i, u^i, \mu_s^{-i}) ds + \sum_{k=1}^N V^\phi(t + \epsilon, X_{t+\epsilon}^k, \mu_{t+\epsilon}^{-k}) \right]$$

and next take $N \rightarrow \infty$, where the term $\sum_{k \neq i} \mathbb{E} \int_t^{t+\epsilon} L(X_s^k, u_s^k, \mu_s^{-k}) ds$ is not needed since u^i has too little impact on it. This allows us to formally derive the master equation of the value function $V(t, x, \mu)$, as the limiting form of J_i , after finding the minimizer $\hat{u}^i$, which is further required to be equal to ϕ ; see details in [30, Appx. A]. The selection of $\hat{u}^i$ is cooperative, in that it takes into account its impact on both J_i and the sum of all other agents' costs.

2.2. The dynamic programming equation (or master equation). Denote the matrices $\Sigma := \sigma\sigma^\top$, $\Sigma_0 := \sigma_0\sigma_0^\top$. We interpret (x, u^i) as the state and control of the representative agent $\mathcal{A}_i$ in an infinite population. The master equation of the value function $V(t, x, \mu)$ takes the form

$$\begin{aligned} (2.4) \quad -\partial_t V(t, x, \mu) &= V_x(t, x, \mu)f(x, \hat{u}^i, \mu) + \frac{1}{2}\text{Tr}[V_{xx}(t, x, \mu)(\Sigma + \Sigma_0)] \\ &\quad + L(x, \hat{u}^i, \mu) + \langle \mu(dy), \partial_y \delta_\mu V(t, x, \mu; y)f(y, \phi(t, y, \mu), \mu) \rangle \\ &\quad + \frac{1}{2}\langle \mu(dy), \text{Tr}[\partial_y^2 \delta_\mu V(t, x, \mu; y)(\Sigma + \Sigma_0)] \rangle \\ &\quad + \langle \mu(dy), \text{Tr}[\partial_{xy} \delta_\mu V(t, x, \mu; y)\Sigma_0] \rangle \\ &\quad + \frac{1}{2}\langle \mu^{\otimes 2}(dydz), \text{Tr}[\partial_{yz} \delta_{\mu\mu} V(t, x, \mu; y, z)\Sigma_0] \rangle, \\ (2.5) \quad V(T, x, \mu) &= g(x, \mu), \quad (t, x, \mu) \in [0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n), \end{aligned}$$

where

$$(2.6) \quad \hat{u}^i := \phi(t, x, \mu)$$

is selected with ϕ fulfilling the condition

$$(2.7) \quad \Phi(t, x, \phi(t, x, \mu), \mu, V(\cdot)) = \min_{u^i \in \mathbb{U}} \Phi(t, x, u^i, \mu, V(\cdot))$$

for

$$\begin{aligned} \Phi(t, x, u^i, \mu, V(\cdot)) &:= V_x(t, x, \mu)f(x, u^i, \mu) + L(x, u^i, \mu) \\ &\quad + \langle \mu(dy), \partial_x \delta_\mu V(t, y, \mu; x)f(x, u^i, \mu) \rangle. \end{aligned}$$

Remark 2.1. The control law ϕ has a dual role. While $\phi(t, y, \mu)$ with a generic state y is interpreted as the control of other agents from the continuum population, $\phi(t, x, \mu)$ applied by agent $\mathcal{A}_i$ is required to possess the minimizer property in (2.7).

Recall section 1.2 for (joint) continuity of a function with several arguments.

DEFINITION 2.2. We call the pair (V, ϕ) , as mappings from $[0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)$ to $\mathbb{R}$ and $\mathbb{U}$, respectively, a solution of (2.4) if the following conditions are satisfied:

- (i) V , $\partial_t V$, V_x , and V_{xx} are each (jointly) continuous on $[0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)$;
- (ii) $\delta_\mu V(t, x, \mu; y)$, $\delta_{\mu\mu} V(t, x, \mu; y, z)$, $\partial_y \delta_\mu V(t, x, \mu; y)$, $\partial_y^2 \delta_\mu V(t, x, \mu; y)$, $\partial_{xy} \delta_\mu V(t, x, \mu; y)$, $\partial_{yz} \delta_{\mu\mu} V(t, x, \mu; y, z)$ are jointly continuous in $t \in [0, T]$, $x, y, z \in \mathbb{R}^n$, and $\mu \in \mathcal{P}_2(\mathbb{R}^n)$;
- (iii) ϕ is continuous on $[0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)$ with

$$|\phi(t, x, \mu)| \leq C_\phi(1 + |x| + \langle |y| \rangle_\mu);$$

- (iv) there exists a constant C_a such that for all (t, x, y, z, μ) ,

$$(2.8) \quad \begin{cases} |\delta_\mu V(t, x, \mu; y)|, |\delta_{\mu\mu} V(t, x, \mu; y, z)| \leq C_a(1 + |x|^2 + |y|^2 + |z|^2 + \langle |y| \rangle_\mu^2), \\ |\partial_y \delta_\mu V(t, x, \mu; y)| \leq C_a(1 + |x| + |y| + \langle |y| \rangle_\mu), \\ |\partial_y^2 \delta_\mu V(t, x, \mu; y)|, |\partial_{xy} \delta_\mu V(t, x, \mu; y)| \leq C_a, \\ |\partial_{yz} \delta_{\mu\mu} V(t, x, \mu; y, z)| \leq C_a; \end{cases}$$

- (v) (V, ϕ) satisfies (2.4) with terminal condition (2.5).

For the subsequent analysis, it is necessary to look for a solution of (2.4) with further restrictions on V . We introduce the following class of functions $\mathcal{C}_V$ consisting of all functions V from $[0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)$ to $\mathbb{R}$ such that (a) V fulfills conditions (i) and (ii) in Definition 2.2 and, furthermore, $\delta_{\mu\mu\mu}V(t, x, \mu; y, z, w)$, $\partial_x\delta_\mu V(t, x, \mu; y)$, $\partial_x^2\delta_\mu V$, $\partial_t\delta_\mu V$, $\partial_y\delta_{\mu\mu}V(t, x, \mu; y, z)$, $\partial_x\delta_{\mu\mu}V$, $\partial_y^2\delta_{\mu\mu}V$, $\partial_{xy}\delta_{\mu\mu}V$, $\partial_y\delta_{\mu\mu\mu}V$, $\partial_{yz}\delta_{\mu\mu\mu}V$ are jointly continuous for $t \in [0, T]$, $x, y, z, w \in \mathbb{R}^n$, $\mu \in \mathcal{P}_2(\mathbb{R}^n)$; (b) V satisfies (2.8) and, moreover,

$$(2.9) \quad \left\{ \begin{array}{l} |V(t, x, \mu)| \leq C_a(1 + |x|^2 + \langle |y| \rangle_\mu^2), \\ |V_x(t, x, \mu)| \leq C_a(1 + |x| + \langle |y| \rangle_\mu), \\ |V_{xx}(t, x, \mu)| \leq C_a, \\ |\partial_t\delta_\mu V(t, x, \mu; y)| \leq C_a(1 + |x|^2 + |y|^2 + \langle |y| \rangle_\mu^2), \\ |\partial_x\delta_\mu V(t, x, \mu; y)| \leq C_a(1 + |x| + |y| + \langle |y| \rangle_\mu), \\ |\partial_x^2\delta_\mu V(t, x, \mu; y)| \leq C_a, \\ |\partial_y\delta_{\mu\mu}V(t, x, \mu; y, z)| \leq C_a(1 + |x| + |y| + |z| + \langle |y| \rangle_\mu), \\ |\partial_y^2\delta_{\mu\mu}V(t, x, \mu; y, z)|, |\partial_{xy}\delta_{\mu\mu}V(t, x, \mu; y, z)| \leq C_a, \\ |\delta_{\mu\mu\mu}(t, x, \mu; y, z, w)| \leq C_a(1 + |x|^2 + |y|^2 + |z|^2 + |w|^2 + \langle |y| \rangle_\mu^2), \\ |\partial_y\delta_{\mu\mu\mu}V(t, x, \mu; y, z, w)| \leq C_a(1 + |x| + |y| + |z| + |w| + \langle |y| \rangle_\mu), \\ |\partial_{yz}\delta_{\mu\mu\mu}V(t, x, \mu; y, z, w)| \leq C_a. \end{array} \right.$$

The above additional growth conditions in (2.9) are mainly for ensuring well-defined coefficients in the equation of U in section 3 and for constructing a solution of U , which involves differentiating both sides of (2.4) and leads to various higher-order derivatives.

3. Closed-loop systems and auxiliary master equations. We begin by introducing the following assumption.

Assumption 3.1. The master equation (2.4) has a solution pair (V, ϕ) with $V \in \mathcal{C}_V$.

A question of central interest is what kind of social performance can be achieved when the master equation-based control law ϕ in (2.6) is used by all N agents. Note that $\phi(t, x, \mu)$ involves the measure-valued variable μ . For its implementation, we will use the actual process μ_t^{-i} in the control laws of the N agents:

$$(3.1) \quad \hat{u}_t^i = \phi(t, X_t^i, \mu_t^{-i}), \quad 1 \leq i \leq N.$$

Under the set of control laws $\hat{u}_t^i$, $1 \leq i \leq N$, the closed-loop system takes the form

$$(3.2) \quad dX_t^i = f(X_t^i, \phi(t, X_t^i, \mu_t^{-i}), \mu_t^{-i})dt + \sigma dW_t^i + \sigma_0 dW_t^0, \quad 0 \leq t \leq T, \\ 1 \leq i \leq N.$$

To analyze the performance of the control law ϕ in terms of PbP optimality, we need to consider a general control u_t^1 for agent $\mathcal{A}_1$ while the other agents take

$$(3.3) \quad \hat{u}_t^k = \phi(t, X_t^k, \mu_t^{-k}), \quad 2 \leq k \leq N.$$

The key question is by how much the social cost may be reduced by optimizing u_t^1 . This leads to a control problem with social cost $J_{\text{soc}}^{(N)}(u^1(\cdot), \hat{\mathbf{u}}^{-1}(\cdot))$ and dynamics

$$(3.4) \quad dX_t^1 = f(X_t^1, u_t^1, \mu_t^{-1})dt + \sigma dW_t^1 + \sigma_0 dW_t^0,$$

$$(3.5) \quad dX_t^k = f(X_t^k, \phi(t, X_t^k, \mu_t^{-k}), \mu_t^{-k})dt + \sigma dW_t^k + \sigma_0 dW_t^0, \\ 0 \leq t \leq T, \quad 2 \leq k \leq N.$$

To facilitate the performance analysis, we need to examine two functions U and $\bar{U}$ below, which are related to the asymptotic behavior of the social costs attained by ϕ as $N \rightarrow \infty$.

The social cost under the control law ϕ in (2.6) for all agents will serve as a benchmark performance level. Based on (3.2), we consider the state processes on $[s, T]$:

$$(3.6) \quad dX_t^i = f(X_t^i, \phi(t, X_t^i, \mu_t^{-i}), \mu_t^{-i})dt + \sigma dW_t^i + \sigma_0 dW_t^0, \quad s \leq t \leq T, \\ 1 \leq i \leq N,$$

which are assigned the initial condition $(s, x^1, \dots, x^N)$. The use of general initial conditions will enable us to derive a partial differential equation of the cost

$$(3.7) \quad J_{\text{soc}, \phi}^{(N)}(s, x^1, \dots, x^N) := \sum_{i=1}^N J_i(s, x^i, \mu^{-i}, \phi)$$

subject to dynamics (3.6). Here $J_i(s, x^i, \mu^{-i}, \phi)$ means that the cost is evaluated with initial time s , initial state (x^i, μ^{-i}) , and the same control law ϕ for N agents.

We will look for a suitable representation of $J_{\text{soc}, \phi}^{(N)}$. By symmetry of all other agents, agent $\mathcal{A}_1$ may write $J_{\text{soc}, \phi}^{(N)}$ in (3.7) as $U_{\text{soc}}^N(s, x^1, \mu^{-1})$, where the initial condition μ^{-1} , as the empirical probability distribution of the state values $(x^2, \dots, x^N)$, is sufficient for describing the behavior of all other agents as a whole.

We take a decomposition of the form

$$(3.8) \quad U_{\text{soc}}^N(t, x^1, \mu^{-1}) = U^N(t, x^1, \mu^{-1}) + (N-1)\bar{U}(t, \mu^{-1}),$$

where the function $\bar{U}$, to be identified later, is defined on $[0, T] \times \mathcal{P}_2(\mathbb{R}^n)$ and does not depend on N . Given $U_{\text{soc}}^N(t, x^1, \mu^{-1})$, one might choose different pairs $(U^N, \bar{U})$ for the right-hand side of (3.8). However, an appropriate choice of $\bar{U}$ is crucial in order to prevent uncontrolled growth of U^N as $N \rightarrow \infty$. We will take

$$(3.9) \quad \bar{U}(t, \mu) := \langle \mu(dy), V(t, y, \mu) \rangle, \quad \mu \in \mathcal{P}_2(\mathbb{R}^n),$$

which is well-defined in view of (2.9). Our next step is to get the limit form of $U^N(t, x^1, \mu^{-1})$, denoted by $U(t, x, \mu)$, and determine the equation of U . Since U aggregates the effect of a particular agent's state x on the social cost $J_{\text{soc}, \phi}^{(N)}$ which itself approaches infinity as $N \rightarrow \infty$, we shall call U the instrumental value function (IVF) to distinguish it from the value function V .

3.1. Auxiliary master equations. For $(t, x, \mu) \in [0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)$, define

$$f^*(t, x, \mu) := f(x, \phi(t, x, \mu), \mu), \quad L^*(t, x, \mu) := L(x, \phi(t, x, \mu), \mu).$$

We introduce the following assumption.

Assumption 3.2. Each of the functions $\delta_\mu f^*$, $\delta_{\mu\mu} f^*$, $\delta_\mu L^*$, $\delta_{\mu\mu} L^*$, $\delta_\mu g$, and $\delta_{\mu\mu} g$ is jointly continuous in its arguments, and there exists a constant C_a such that

$$\begin{aligned} |\delta_\mu f^*(t, x, \mu; y)| &\leq C_a(1 + |x| + |y| + \langle |y| \rangle_\mu), \\ |\delta_{\mu\mu} f^*(t, x, \mu; y, z)| &\leq C_a(1 + |x| + |y| + |z| + \langle |y| \rangle_\mu), \\ |\delta_\mu L^*(t, x, \mu; y)| &\leq C_a(1 + |x|^2 + |y|^2 + \langle |y| \rangle_\mu^2), \\ |\delta_{\mu\mu} L^*(t, x, \mu; y, z)| &\leq C_a(1 + |x|^2 + |y|^2 + |z|^2 + \langle |y| \rangle_\mu^2), \\ |\delta_\mu g(x, \mu; y)| &\leq C_a(1 + |x|^2 + |y|^2 + \langle |y| \rangle_\mu^2), \\ |\delta_{\mu\mu} g(x, \mu; y, z)| &\leq C_a(1 + |x|^2 + |y|^2 + |z|^2 + \langle |y| \rangle_\mu^2) \end{aligned}$$

for all $t \in [0, T]$, $x, y, z \in \mathbb{R}^n$.

We introduce the following master equation for the instrumental value function $U(t, x, \mu)$:

$$\begin{aligned}
 (3.10) \quad 0 = & \partial_t U(t, x, \mu) + U_x(t, x, \mu) f^*(t, x, \mu) \\
 & + \frac{1}{2} \text{Tr}[U_{xx}(t, x, \mu)(\Sigma + \Sigma_0)] + L^*(t, x, \mu) \\
 & + \langle \mu(dy), \partial_y \delta_\mu U(t, x, \mu; y) f^*(t, y, \mu) \rangle \\
 & + \frac{1}{2} \langle \mu(dy), \text{Tr}[\partial_y^2 \delta_\mu U(t, x, \mu; y)(\Sigma + \Sigma_0)] \rangle \\
 & + \langle \mu(dy), \text{Tr}[\partial_{xy} \delta_\mu U(t, x, \mu; y) \Sigma_0] \rangle \\
 & + \frac{1}{2} \langle \mu^{\otimes 2}(dydz), \text{Tr}[\partial_{yz} \delta_{\mu\mu} U(t, x, \mu; y, z) \Sigma_0] \rangle \\
 & + \langle \mu(dy), \delta_\mu L^*(t, y, \mu; x) - \delta_\mu L^*(t, y, \mu; y) \rangle \\
 & + \langle \mu(dy), \partial_y \delta_\mu \bar{U}(t, \mu; y) [\delta_\mu f^*(t, y, \mu; x) - \delta_\mu f^*(t, y, \mu; y)] \rangle \\
 & + \frac{1}{2} \langle \mu(dy), \text{Tr}\{\partial_{yz} \delta_{\mu\mu} \bar{U}(t, \mu; y, z)|_{z=y} \Sigma\} \rangle, \\
 (3.11) \quad U(T, x, \mu) = & g(x, \mu) + \langle \mu(dy), \delta_\mu g(y, \mu; x) - \delta_\mu g(y, \mu; y) \rangle, \\
 & (t, x, \mu) \in [0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n),
 \end{aligned}$$

where $\bar{U}(t, \mu)$ is given by (3.9).

It will be helpful to explain how (3.10) is derived. Our method is to follow the idea underlying the Feynman–Kac formula to derive an equation for $U_{\text{soc}}^N(t, x^1, \mu^{-1})$ using the dynamics of (X_t^1, μ_t^{-1}) in (3.6). Subsequently, we separate terms of two different scales, depending on whether they are multiplied by $N - 1$, to formally obtain two equations for U^N and $\bar{U}$, respectively. Finally, taking $N \rightarrow \infty$, we obtain a limit form U for U^N and its equation (3.10). See [30, Appx. D] for details. By this procedure, we have in fact identified the equation of $\bar{U}$ (see (3.14)) first and will next show that (3.9) gives a solution.

Our next step is to construct a particular solution to master equation (3.10). To do this, we need to introduce another function $M: [0, T] \times \mathcal{P}_2(\mathbb{R}^2) \rightarrow \mathbb{R}$ by the following equation:

$$\begin{aligned}
 (3.12) \quad 0 = & \partial_t M(t, \mu) + \langle \mu(dy), \partial_y \delta_\mu M(t, \mu; y) f^*(t, y, \mu) \rangle \\
 & + \frac{1}{2} \langle \mu(dy), \text{Tr}[\partial_y^2 \delta_\mu M(t, \mu; y)(\Sigma + \Sigma_0)] \rangle \\
 & + \frac{1}{2} \langle \mu^{\otimes 2}(dydz), \text{Tr}[\partial_{yz} \delta_{\mu\mu} M(t, \mu; y, z) \Sigma_0] \rangle \\
 & + \langle \mu^{\otimes 2}(dydz), \partial_y \delta_\mu V(t, z, \mu; y) [\delta_\mu f^*(t, y, \mu; z) - \delta_\mu f^*(t, y, \mu; y)] \rangle \\
 & + \frac{1}{2} \langle \mu^{\otimes 2}(dydw), \text{Tr}\{\partial_{yz} \delta_{\mu\mu} V(t, w, \mu; y, z)|_{z=y} \Sigma\} \rangle, \\
 (3.13) \quad M(T, \mu) = & 0, \quad (t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^n).
 \end{aligned}$$

We have constructed this equation so as to use M as an adjustment term in the representation of U (see (3.21) below). With V known, we may view (3.12) as a linear equation.

Let the class $\mathcal{C}_M$ consist of all functions $M(t, \mu)$ from $[0, T] \times \mathcal{P}_2(\mathbb{R}^n)$ to $\mathbb{R}$ such that (i) $M(t, \mu)$, $\partial_t M(t, \mu)$, $\delta_\mu M(t, \mu; y)$, $\delta_{\mu\mu} M(t, \mu; y, z)$, $\partial_y \delta_\mu M(t, \mu; y)$, $\partial_y^2 \delta_\mu M(t, \mu; y)$, $\partial_{yz} \delta_{\mu\mu} M(t, \mu; y, z)$ are jointly continuous; (ii) for some constant C_a ,

$$\begin{aligned} |\delta_\mu M(t, \mu; y)|, |\delta_{\mu\mu} M(t, \mu; y, z)| &\leq C_a(1 + |x|^2 + |y|^2 + \langle |y|^2 \rangle_\mu), \\ |\partial_y \delta_\mu M(t, \mu; y)| &\leq C_a(1 + |y| + \langle |y| \rangle_\mu), \\ |\partial_y^2 \delta_\mu M(t, \mu; y)| &\leq C_a, \quad |\partial_{yz} \delta_{\mu\mu} M(t, \mu; y, z)| \leq C_a. \end{aligned}$$

We further introduce the following assumption.

Assumption 3.3. There exists a solution $M \in \mathcal{C}_M$ to (3.12).

THEOREM 3.4. *Under Assumption 3.1, $\bar{U}$ defined by (3.9) satisfies the following equation:*

$$\begin{aligned} (3.14) \quad 0 &= \partial_t \bar{U}(t, \mu) + \langle \mu, L^*(t, y, \mu) \rangle + \langle \mu, \partial_y \delta_\mu \bar{U}(t, \mu; y) f^*(t, y, \mu) \rangle \\ &\quad + \frac{1}{2} \langle \mu, \text{Tr}[\partial_y^2 \delta_\mu \bar{U}(t, \mu; y)(\Sigma + \Sigma_0)] \rangle \\ &\quad + \frac{1}{2} \langle \mu^{\otimes 2}(dydz), \text{Tr}[\partial_{yz} \delta_{\mu\mu} \bar{U}(t, \mu; y, z) \Sigma_0] \rangle, \\ (3.15) \quad \bar{U}(T, \mu) &= \langle \mu, g(y, \mu) \rangle, \quad (t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^n). \end{aligned}$$

Proof. For $\bar{U}$ defined in (3.9), we have

$$(3.16) \quad \delta_\mu \bar{U}(t, \mu; y) = V(t, y, \mu) + \chi(t, \mu) + \langle \mu(dw), \delta_\mu V(t, w, \mu; y) \rangle,$$

where χ is a normalizing term. We have

$$(3.17) \quad \begin{aligned} \delta_{\mu\mu} \bar{U}(t, \mu; y, z) &= \delta_\mu V(t, y, \mu; z) + \delta_\mu \chi(t, \mu; z) + \delta_\mu V(t, z, \mu; y) + \chi_1(t, \mu, y) \\ &\quad + \langle \mu(dw), \delta_{\mu\mu} V(t, w, \mu; y, z) \rangle. \end{aligned}$$

Therefore,

$$(3.18) \quad \begin{aligned} \partial_{yz} \delta_{\mu\mu} \bar{U}(t, \mu; y, z) &= \partial_{yz} \delta_\mu V(t, y, \mu; z) + \partial_{yz} \delta_\mu V(t, z, \mu; y) \\ &\quad + \langle \mu(dw), \partial_{yz} \delta_{\mu\mu} V(t, w, \mu; y, z) \rangle. \end{aligned}$$

The partial differentiation in the last term can get inside the integration by an application of the dominated convergence theorem.

Substituting these expressions into the right-hand side of (3.14), we obtain

$$\begin{aligned} (3.19) \quad \Xi &:= \partial_t \langle \mu, V(t, y, \mu) \rangle + \langle \mu, L^*(t, y, \mu) \rangle + \langle \mu, V_y(t, y, \mu) f^*(t, y, \mu) \rangle \\ &\quad + \langle \mu^{\otimes 2}(dydw), \partial_y \delta_\mu V(t, w, \mu; y) f^*(t, y, \mu) \rangle \\ &\quad + \frac{1}{2} \langle \mu, \text{Tr}[V_{yy}(t, y, \mu)(\Sigma + \Sigma_0)] \rangle \\ &\quad + \frac{1}{2} \langle \mu^{\otimes 2}(dydw), \text{Tr}[\partial_y^2 \delta_\mu V(t, w, \mu; y)(\Sigma + \Sigma_0)] \rangle \\ &\quad + \frac{1}{2} \langle \mu^{\otimes 2}(dydz), \text{Tr}\{[\partial_{yz} \delta_\mu V(t, y, \mu; z) + \partial_{yz} \delta_\mu V(t, z, \mu; y)] \Sigma_0\} \rangle \\ &\quad + \frac{1}{2} \langle \mu^{\otimes 3}(dydzdw), \text{Tr}\{\partial_{yz} \delta_{\mu\mu} V(t, w, \mu; y, z) \Sigma_0\} \rangle. \end{aligned}$$

By a change of variables, we further apply Lemma A.2 to write the constituent term

$$(3.20) \quad \langle \mu^{\otimes 2}(dydz), \text{Tr}[\partial_{yz} \delta_\mu V(t, z, \mu; y) \Sigma_0] \rangle = \langle \mu^{\otimes 2}(dydz), \text{Tr}[\partial_{yz} \delta_\mu V(t, y, \mu; z) \Sigma_0] \rangle.$$

Now, after substituting (3.20) into (3.19), we show $\Xi = 0$ by integrating both sides of (2.4) using $\mu(dx)$, where we have $\langle \mu(dx), \partial_t V(t, x, \mu) \rangle = \partial_t \langle \mu(dx), V(t, x, \mu) \rangle$. $\square$

We now take

$$(3.21) \quad \begin{aligned} U(t, x, \mu) &= V(t, x, \mu) + M(t, \mu) \\ &\quad + \langle \mu(dy), \delta_\mu V(t, y, \mu; x) - \delta_\mu V(t, y, \mu; y) \rangle \end{aligned}$$

for $(t, x, \mu) \in [0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)$ and proceed to show that this function satisfies (3.10).

Remark 3.5. We explain the idea behind the representation (3.21). We write $U_{\text{soc}}^N(t, x^1, \mu^{-1}) = \sum_{i=1}^N V^\phi(t, x^i, \mu^{-i})$ and next expand $V^\phi(t, x^i, \mu^{-i})$ around μ^{-1} . The resulting sum is compared with (3.8) to suggest the construction in (3.21).

THEOREM 3.6. *Under Assumptions 3.1, 3.2, and 3.3, U defined by (3.21) is a solution to master equation (3.10) with terminal condition (3.11).*

Proof. We give a sketch proof; a detailed proof is provided in [30, Appx. C]. We differentiate both sides of (2.4) to get an equation of $\delta_\mu \partial_t V(t, x, \mu; y)$, which in view of Lemma A.4 leads to the equation of $\partial_t \langle \mu(dy), \delta_\mu V(t, y, \mu; x) \rangle$ after switching x and y and next integrating with respect to $\mu(dy)$. Next, we easily get the equation of $\partial_t \langle \mu(dy), \delta_\mu V(t, y, \mu; y) \rangle$. By adding up the three equations of $\partial_t V(t, x, \mu)$, $\partial_t M(t, \mu)$ and $\partial_t \langle \mu(dy), \delta_\mu V(t, y, \mu; x) - \delta_\mu V(t, y, \mu; y) \rangle$, we further show that the right-hand side of this summed equation may be equivalently written as the right-hand side of (3.10) when U takes the form (3.21). In the reduction of the summed equation to the equation of U , we need to use Lemma A.2 and the semi-symmetry property (A.12). $\square$

3.2. A minimizer property for ϕ . The next lemma gives a useful inequality resulting from taking u^i in place of $\phi(t, x, \mu)$ in (3.10) for U .

LEMMA 3.7. *Suppose Assumptions 3.1, 3.2, and 3.3 hold, and let U be given by (3.21). Then, for each $u \in \mathbb{U}$, we have*

$$(3.22) \quad \begin{aligned} 0 \leq & \partial_t U(t, x, \mu) + U_x(t, x, \mu) f(x, u, \mu) \\ & + \frac{1}{2} \text{Tr}[U_{xx}(t, x, \mu)(\Sigma + \Sigma_0)] + L(x, u, \mu) \\ & + \langle \mu(dy), \partial_y \delta_\mu U(t, x, \mu; y) f^*(t, y, \mu) \rangle \\ & + \frac{1}{2} \langle \mu(dy), \text{Tr}[\partial_y^2 \delta_\mu U(t, x, \mu; y)(\Sigma + \Sigma_0)] \rangle \\ & + \langle \mu(dy), \text{Tr}[\partial_{xy} \delta_\mu U(t, x, \mu; y) \Sigma_0] \rangle \\ & + \frac{1}{2} \langle \mu^{\otimes 2}(dydz), \text{Tr}[\partial_{yz} \delta_{\mu\mu} U(t, x, \mu; y, z) \Sigma_0] \rangle \\ & + \langle \mu(dy), \delta_\mu L^*(t, y, \mu; x) - \delta_\mu L^*(t, y, \mu; y) \rangle \\ & + \langle \mu(dy), \partial_y \delta_\mu \bar{U}(t, \mu; y) [\delta_\mu f^*(t, y, \mu; x) - \delta_\mu f^*(t, y, \mu; y)] \rangle \\ & + \frac{1}{2} \langle \mu(dy), \text{Tr}\{[\partial_{yz} \delta_{\mu\mu} \bar{U}(t, \mu; y, z)]|_{z=y} \Sigma\} \rangle, \end{aligned}$$

where $\bar{U}$ is given by (3.9).

Proof. There are only two terms within (3.22) that depend on u . By (3.21), we have

$$(3.23) \quad U_x(t, x, \mu) = V_x(t, x, \mu) + \langle \mu(dy), \partial_x \delta_\mu V(t, x, \mu; y) \rangle,$$

where differentiation ∂_x can go inside the integral by dominated convergence. We have

$$\begin{aligned} U_x(t, x, \mu) f(x, u, \mu) + L(x, u, \mu) \\ = [V_x(t, x, \mu) + \langle \mu(dy), \partial_x \delta_\mu V(t, x, \mu; y) \rangle] f(x, u, \mu) + L(x, u, \mu) \\ \geq [V_x(t, x, \mu) + \langle \mu(dy), \partial_x \delta_\mu V(t, x, \mu; y) \rangle] f^*(t, x, \mu) + L^*(t, x, \mu), \end{aligned}$$

where the inequality is due to the choice of ϕ in (2.6). On the other hand, U satisfies (3.10) by Theorem 3.6. The lemma follows from (3.10) and the above inequality. $\square$

4. Person-by-person optimality. We introduce the following assumptions:

Assumption 4.1. $\phi(t, x, \mu)$ in the solution pair (V, ϕ) for (2.4) is Lipschitz continuous in $(x, \mu) \in \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)$, uniformly with respect to t .

Assumption 4.2. There exists a constant C_X such that $\sup_N \max_{1 \leq i \leq N} \mathbb{E}|X_0^i|^2 \leq C_X$.

Under the standing assumption (A1) and Assumptions 3.1 and 4.1, $f^*(t, x, \mu)$ is Lipschitz continuous in (x, μ) with linear growth; i.e., for some constant $C_{f,\phi}$,

$$(4.1) \quad |f^*(t, x, \mu) - f^*(t, y, \nu)| \leq C_{f,\phi}(|x - y| + W_2(\mu, \nu)),$$

$$(4.2) \quad |f^*(t, x, \mu)| \leq C_{f,\phi}(1 + |x| + \langle |y| \rangle_\mu).$$

PROPOSITION 4.3. *Under Assumptions 3.1, 3.2, 3.3, 4.1, and 4.2, for $(X_t^1, \dots, X_t^N)$ given by (3.2), we have*

$$\sup_N \max_{1 \leq i \leq N} \sup_{0 \leq t \leq T} \mathbb{E}|X_t^i|^2 \leq C_{T,X},$$

where the constant $C_{T,X}$ only depends on $(T, C_X, C_{f,\phi}, \Sigma, \Sigma_0)$.

Proof. Consider the closed-loop system

$$(4.3) \quad dX_t^i = f^*(t, X_t^i, \mu_t^{-i}) dt + \sigma dW_t^i + \sigma_0 dW_t^0, \quad 1 \leq i \leq N.$$

Take $\mu_1 = \frac{1}{k} \sum_{j=1}^k \delta_{x^j}$ and $\mu_2 = \frac{1}{k} \sum_{j=1}^k \delta_{y^j}$ with $x^j, y^j \in \mathbb{R}^n$. Taking a particular coupling of μ_1 and μ_2 as the distribution assigning probability $1/k$ at each point $(x^j, y^j) \in \mathbb{R}^{2n}$, we obtain $W_2(\mu_1, \mu_2) \leq (\frac{1}{k} \sum_j |x^j - y^j|^2)^{1/2} \leq \sum_j |x^j - y^j|$. So, in view of (4.1), $f^*(t, X_t^i, \mu_t^{-i})$ is Lipschitz continuous in $(X_t^1, \dots, X_t^N)$, ensuring that the SDE system (4.3) has a unique solution on $[0, T]$. Applying Itô's formula to (4.3) and next taking expectation, we have

$$\begin{aligned} \mathbb{E}|X_t^i|^2 &= \mathbb{E}|X_0^i|^2 + \mathbb{E} \int_0^t [X_s^{i\top} f^*(s, X_s^i, \mu_s^{-i}) + \text{Tr}(\Sigma + \Sigma_0)] ds \\ &\leq \mathbb{E}|X_0^i|^2 + T \text{Tr}(\Sigma + \Sigma_0) + C_{f,\phi} \mathbb{E} \int_0^t |X_s^i| \left(1 + |X_s^i| + \frac{1}{N-1} \sum_{k \neq i} |X_s^k| \right) ds \\ &\leq \mathbb{E}|X_0^i|^2 + T \text{Tr}(\Sigma + \Sigma_0) + C_1 \int_0^t \left(1 + \mathbb{E}|X_s^i|^2 + \frac{1}{N-1} \sum_{k \neq i} \mathbb{E}|X_s^k|^2 \right) ds. \end{aligned}$$

Denote $z_t = \max_{1 \leq i \leq N} \mathbb{E}|X_t^i|^2$. We obtain $z_t \leq C_2 + C_1 \int_0^t (1 + 2z_s) ds$. By Grönwall's inequality, we obtain the desired estimate with $C_{T,X}$ as specified. $\square$

For any fixed constant $K_0 > 0$, let $\mathcal{U}_{\mathcal{F}^N}^{K_0}$ consist of all $\mathbb{R}^{n_1}$ -valued processes $u_t^1 := u^1(t, \omega)$ that satisfy the following conditions: (i) $u^1(t, \omega)$ is adapted to the filtration $\mathcal{F}_t^N := \sigma(X_0^k, W_s^k, W_s^0, k = 1, \dots, N, s \leq t)$. (ii) $\mathbb{E} \int_0^T |u^1(t, \omega)|^2 dt \leq K_0$.

Remark 4.4. Given $\hat{u}^{-1}$, if $u^1(t, x_1, \dots, x_N)$ is a Lipschitz feedback control law yielding a unique closed-loop state process $(X_t^1, \dots, X_t^N)$, we may identify such a control law as the process $u^1(t, X_t^1, \dots, X_t^N)$, which is adapted to the filtration $\mathcal{F}_t^N$.

Denote $\hat{u}_t^1 = \phi(t, X_t^1, \mu_t^{-1})$ in (3.2). Then, by Proposition 4.3, $\hat{u}_t^1$ belongs to $\mathcal{U}_{\mathcal{F}^N}^{K_0}$ if K_0 is sufficiently large.

THEOREM 4.5 (asymptotic person-by-person optimality). *Under Assumptions 3.1, 3.2, 3.3, 4.1, and 4.2, for (3.4)–(3.5) and any fixed $K_0 > 0$, we have*

$$J_{\text{soc}}^{(N)}(\hat{u}^1(\cdot), \hat{u}^2(\cdot), \dots, \hat{u}^N(\cdot)) \leq \inf_{u^1(\cdot) \in \mathcal{U}_{\mathcal{F}^N}^{K_0}} J_{\text{soc}}^{(N)}(u^1(\cdot), \hat{u}^2(\cdot), \dots, \hat{u}^N(\cdot)) + \epsilon_N,$$

where $\epsilon_N = O(1/N)$ and $J_{\text{soc}}^{(N)}$ is defined by (1.3).

4.1. Proof of Theorem 4.5. Throughout this subsection, we suppose that all assumptions in Theorem 4.5 hold. Using Itô's formula, we obtain the following lemma.

LEMMA 4.6. *Let the processes $(X_t^1, \dots, X_t^N)$ be given by (3.4)–(3.5) with initial states $(X_0^1, \dots, X_0^N)$ at $t = 0$ and $u_t^1 \in \mathcal{U}_{\mathcal{F}^N}^{K_0}$ as in Theorem 4.5 ensuring a unique strong solution to (3.4)–(3.5), and denote the derivatives $\delta_\mu U(t, x, \mu; y)$, $\delta_{\mu\mu} U(t, x, \mu; y, z)$, $\delta_\mu \bar{U}(t, \mu; y)$, $\delta_{\mu\mu} \bar{U}(t, \mu; y, z)$ for U and $\bar{U}$ defined by (3.21) and (3.9), respectively. Then we have*

$$\begin{aligned} (4.4) \quad & \mathbb{E}[U(T, X_T^1, \mu_T^{-1}) - U(0, X_0^1, \mu_0^{-1})] \\ &= \mathbb{E} \int_0^T \left\{ \partial_t U(t, X_t^1, \mu_t^{-1}) + U_x(t, X_t^1, \mu_t^{-1}) f(X_t^1, u_t^1, \mu_t^{-1}) \right. \\ &\quad + \frac{1}{2} \text{Tr}[U_{xx}(t, X_t^1, \mu_t^{-1})(\Sigma + \Sigma_0)] \\ &\quad + \frac{1}{N-1} \sum_{j \geq 2} \partial_y \delta_\mu U(t, X_t^1, \mu_t^{-1}; X_t^j) f^*(t, X_t^j, \mu_t^{-j}) \\ &\quad + \frac{1}{2(N-1)} \sum_{j \geq 2} \text{Tr}[\partial_y^2 \delta_\mu U(t, X_t^1, \mu_t^{-1}; X_t^j)(\Sigma + \Sigma_0)] \\ &\quad + \frac{1}{N-1} \sum_{j \geq 2} \text{Tr}[\partial_{xy} \delta_\mu U(t, X_t^1, \mu_t^{-1}; X_t^j) \Sigma_0] \\ &\quad + \frac{1}{2(N-1)^2} \sum_{j \geq 2, k \geq 2} \text{Tr}[\partial_{yz} \delta_{\mu\mu} U(t, X_t^1, \mu_t^{-1}; X_t^k, X_t^j) \Sigma_0] \\ &\quad \left. + \frac{1}{2(N-1)^2} \sum_{k \geq 2} \text{Tr}[\partial_{yz} \delta_{\mu\mu} U(t, X_t^1, \mu_t^{-1}; X_t^k, X_t^k) \Sigma] \right\} dt \end{aligned}$$

and

$$\begin{aligned}
 (4.5) \quad & \mathbb{E}[\bar{U}(T, \mu_T^{-1}) - \bar{U}(0, \mu^{-1})] \\
 &= \mathbb{E} \int_0^T \left\{ \partial_t \bar{U}(t, \mu_t^{-1}) + \frac{1}{N-1} \sum_{j \geq 2} \partial_y \delta_\mu \bar{U}(t, \mu_t^{-1}; X_t^j) f^*(t, X_t^j, \mu_t^{-j}) \right. \\
 &\quad + \frac{1}{2(N-1)} \sum_{j \geq 2} \text{Tr}[\partial_y^2 \delta_\mu \bar{U}(t, \mu_t^{-1}; X_t^j)(\Sigma + \Sigma_0)] \\
 &\quad + \frac{1}{2(N-1)^2} \sum_{j \geq 2, k \geq 2} \text{Tr}[\partial_{yz} \delta_{\mu\mu} \bar{U}(t, \mu_t^{-1}; X_t^k, X_t^j) \Sigma_0] \\
 &\quad \left. + \frac{1}{2(N-1)^2} \sum_{k \geq 2} \text{Tr}[\partial_{yz} \delta_{\mu\mu} \bar{U}(t, \mu_t^{-1}; X_t^k, X_t^k) \Sigma] \right\} dt.
 \end{aligned}$$

Proof. For agent $\mathcal{A}_1$, by Itô's formula involving the measure flow $\{\mu_t^{-1}, 0 \leq t \leq T\}$ (see, e.g., [9, 12, 14]), we have

$$\begin{aligned}
 dU(t, X_t^1, \mu_t^{-1}) &= [\partial_t U(t, X_t^1, \mu_t^{-1}) + U_x(t, X_t^1, \mu_t^{-1}) f(X_t^1, u_t^1, \mu_t^{-1})] dt \\
 &\quad + \frac{1}{2} \text{Tr}[U_{xx}(t, X_t^1, \mu_t^{-1})(\Sigma + \Sigma_0)] dt \\
 &\quad + U_x(t, X_t^1, \mu_t^{-1})(\sigma dW_t^1 + \sigma_0 dW_t^0) \\
 &\quad + \frac{1}{N-1} \sum_{j \geq 2} \partial_y \delta_\mu U(t, X_t^1, \mu_t^{-1}; X_t^j) f^*(t, X_t^j, \mu_t^{-j}) dt \\
 &\quad + \frac{1}{2(N-1)} \sum_{j \geq 2} \text{Tr}[\partial_y^2 \delta_\mu U(t, X_t^1, \mu_t^{-1}; X_t^j)(\Sigma + \Sigma_0)] dt \\
 &\quad + \frac{1}{N-1} \sum_{j \geq 2} \partial_y \delta_\mu U(t, X_t^1, \mu_t^{-1}; X_t^j)(\sigma dW_t^j + \sigma_0 dW_t^0) \\
 &\quad + \frac{1}{N-1} \sum_{k \geq 2} \text{Tr}[\partial_{xy} \delta_\mu U(t, X_t^1, \mu_t^{-1}; X_t^k) \Sigma_0] dt \\
 &\quad + \frac{1}{2(N-1)^2} \sum_{j \geq 2, k \geq 2} \text{Tr}[\partial_{yz} \delta_{\mu\mu} U(t, X_t^1, \mu_t^{-1}; X_t^k, X_t^j) \Sigma_0] dt \\
 &\quad + \frac{1}{2(N-1)^2} \sum_{k \geq 2} \text{Tr}[\partial_{yz} \delta_{\mu\mu} U(t, X_t^1, \mu_t^{-1}; X_t^k, X_t^k) \Sigma] dt.
 \end{aligned}$$

Integrating both sides on $[0, T]$ and taking expectation, we obtain (4.4). The proof of (4.5) is similar and uses the following relation:

$$\begin{aligned}
 (4.6) \quad & d\bar{U}(t, \mu_t^{-1}) = \partial_t \bar{U}(t, \mu_t^{-1}) dt + \frac{1}{N-1} \sum_{j \geq 2} \partial_y \delta_\mu \bar{U}(t, \mu_t^{-1}; X_t^j) f^*(t, X_t^j, \mu_t^{-j}) dt \\
 &\quad + \frac{1}{2(N-1)} \sum_{j \geq 2} \text{Tr}[\partial_y^2 \delta_\mu \bar{U}(t, \mu_t^{-1}; X_t^j)(\Sigma + \Sigma_0)] dt \\
 &\quad + \frac{1}{N-1} \sum_{j \geq 2} \partial_y \delta_\mu \bar{U}(t, \mu_t^{-1}; X_t^j)(\sigma dW_t^j + \sigma_0 dW_t^0) \\
 &\quad + \frac{1}{2(N-1)^2} \sum_{j \geq 2, k \geq 2} \text{Tr}[\partial_{yz} \delta_{\mu\mu} \bar{U}(t, \mu_t^{-1}; X_t^k, X_t^j) \Sigma_0] dt \\
 &\quad + \frac{1}{2(N-1)^2} \sum_{k \geq 2} \text{Tr}[\partial_{yz} \delta_{\mu\mu} \bar{U}(t, \mu_t^{-1}; X_t^k, X_t^k) \Sigma] dt. \quad \square
 \end{aligned}$$

To obtain the crucial lower bound in Theorem 4.7 below, we need to slightly rewrite the integrands in (4.4) and (4.5) to relate to (3.10) and (3.14) for U and $\bar{U}$, respectively. We proceed to introduce the following error terms:

$$\begin{aligned}\xi_{1t}^a &:= \frac{1}{N-1} \sum_{j \geq 2} \partial_y \delta_\mu U(t, X_t^1, \mu_t^{-1}; X_t^j) [f^*(t, X_t^j, \mu_t^{-j}) - f^*(t, X_t^j, \mu_t^{-1})], \\ \xi_{2t}^a &:= \frac{1}{2(N-1)^2} \sum_{k \geq 2} \text{Tr}[\partial_{yz} \delta_{\mu\mu} U(t, X_t^1, \mu_t^{-1}; X_t^k, X_t^k) \Sigma]\end{aligned}$$

and

$$\begin{aligned}\xi_t^b &:= \sum_{j \geq 2} \partial_y \delta_\mu \bar{U}(t, \mu_t^{-1}; X_t^j) \left\{ f^*(t, X_t^j, \mu_t^{-j}) - f^*(t, X_t^j, \mu_t^{-1}) \right. \\ &\quad \left. - \frac{1}{N-1} [\delta_\mu f^*(t, X_t^j, \mu_t^{-1}; X_t^1) - \delta_\mu f^*(t, X_t^j, \mu_t^{-1}; X_t^j)] \right\}.\end{aligned}$$

Moreover, for $2 \leq j \leq N$, we denote

$$\begin{aligned}\xi_{jt}^L &:= L^*(t, X_t^j, \mu_t^{-j}) - L^*(t, X_t^j, \mu_t^{-1}) \\ &\quad - \frac{1}{N-1} [\delta_\mu L^*(t, X_t^j, \mu_t^{-1}; X_t^1) - \delta_\mu L^*(t, X_t^j, \mu_t^{-1}; X_t^j)], \\ \xi_j^g &:= g(X_T^j, \mu_T^{-j}) - g(X_T^j, \mu_T^{-1}) \\ &\quad - \frac{1}{N-1} [\delta_\mu g(X_T^j, \mu_T^{-1}; X_T^1) - \delta_\mu g(X_T^j, \mu_T^{-1}; X_T^j)].\end{aligned}$$

THEOREM 4.7. *For (3.4)–(3.5) and every control u^1 as in Theorem 4.5, we have*

$$\begin{aligned}(4.7) \quad J_{\text{soc}}^{(N)}(u^1(\cdot), \hat{u}^{-1}(\cdot)) &\geq \mathbb{E}[U(0, X_0^1, \mu_0^{-1}) + (N-1)\bar{U}(0, \mu_0^{-1})] \\ &\quad + \mathbb{E} \int_0^T \left(\xi_{1t}^a + \xi_{2t}^a + \xi_t^b + \sum_{j=2}^N \xi_{jt}^L \right) dt + \mathbb{E} \sum_{j=2}^N \xi_j^g,\end{aligned}$$

where the equality holds if $u_t^1 = \phi(t, X_t^1, \mu_t^{-1})$ for all $t \in [0, T]$ in (3.4) for agent $\mathcal{A}_1$.

Proof. Denote

$$\begin{aligned}\Theta_t^a &:= \partial_t U(t, X_t^1, \mu_t^{-1}) + U_x(t, X_t^1, \mu_t^{-1}) f(X_t^1, u_t^1, \mu_t^{-1}) \\ &\quad + \frac{1}{2} \text{Tr}[U_{xx}(t, X_t^1, \mu_t^{-1})(\Sigma + \Sigma_0)] \\ &\quad + \frac{1}{N-1} \sum_{j \geq 2} \partial_y \delta_\mu U(t, X_t^1, \mu_t^{-1}; X_t^j) f^*(t, X_t^j, \mu_t^{-1}) \\ &\quad + \frac{1}{2(N-1)} \sum_{j \geq 2} \text{Tr}[\partial_y^2 \delta_\mu U(t, X_t^1, \mu_t^{-1}; X_t^j)(\Sigma + \Sigma_0)] \\ &\quad + \frac{1}{N-1} \sum_{j \geq 2} \text{Tr}[\partial_{xy} \delta_\mu U(t, X_t^1, \mu_t^{-1}; X_t^j) \Sigma_0] \\ &\quad + \frac{1}{2(N-1)^2} \sum_{j \geq 2, k \geq 2} \text{Tr}[\partial_{yz} \delta_{\mu\mu} U(t, X_t^1, \mu_t^{-1}; X_t^k, X_t^j) \Sigma_0],\end{aligned}$$

which consists of the first six lines in the integrand of (4.4) but replaces μ_t^{-j} by μ_t^{-1} within f^* on the third line. Then, from Lemma 4.6, we have the relation

$$\mathbb{E}[U(T, X_T^1, \mu_T^{-1}) - U(0, X_0^1, \mu_0^{-1})] = \mathbb{E} \int_0^T (\Theta_t^a + \xi_{1t}^a + \xi_{2t}^a) dt.$$

Similarly, based on (4.5), we define

$$\begin{aligned} \Theta_t^b := & (N-1) \left\{ \partial_t \bar{U}(t, \mu_t^{-1}) + \frac{1}{N-1} \sum_{j \geq 2} \partial_y \delta_\mu \bar{U}(t, \mu_t^{-1}; X_t^j) f^*(t, X_t^j, \mu_t^{-1}) \right. \\ & + \frac{1}{2(N-1)} \sum_{j \geq 2} \text{Tr}[\partial_y^2 \delta_\mu \bar{U}(t, \mu_t^{-1}; X_t^j)(\Sigma + \Sigma_0)] \\ & \left. + \frac{1}{2(N-1)^2} \sum_{j \geq 2, k \geq 2} \text{Tr}[\partial_{yz} \delta_{\mu\mu} \bar{U}(t, \mu_t^{-1}; X_t^k, X_t^j) \Sigma_0] \right\} \end{aligned}$$

and

$$\begin{aligned} \bar{\Theta}_{1t}^b &:= \sum_{j \geq 2} \partial_y \delta_\mu \bar{U}(t, \mu_t^{-1}; X_t^j) \left[f^*(t, X_t^j, \mu_t^{-j}) - f^*(t, X_t^j, \mu_t^{-1}) \right], \\ \Theta_{2t}^b &:= \frac{1}{2(N-1)} \sum_{k \geq 2} \text{Tr}[\partial_{yz} \delta_{\mu\mu} \bar{U}(t, \mu_t^{-1}; X_t^k, X_t^k) \Sigma]. \end{aligned}$$

Using (4.5), we obtain

$$(N-1) \mathbb{E}[\bar{U}(T, \mu_T^{-1}) - \bar{U}(0, \mu_0^{-1})] = \mathbb{E} \int_0^T (\Theta_t^b + \bar{\Theta}_{1t}^b + \Theta_{2t}^b) dt.$$

Subsequently, we obtain

$$\begin{aligned} (4.8) \quad & \mathbb{E}[U(T, X_T^1, \mu_T^{-1}) + (N-1) \bar{U}(T, \mu_T^{-1})] \\ & - \mathbb{E}[U(0, X_0^1, \mu_0^{-1}) + (N-1) \bar{U}(0, \mu_0^{-1})] \\ & = \mathbb{E} \int_0^T (\Theta_t^a + \xi_{1t}^a + \xi_{2t}^a + \Theta_t^b + \bar{\Theta}_{1t}^b + \Theta_{2t}^b) dt. \end{aligned}$$

We need to further decompose $\bar{\Theta}_{1t}^b$ as the sum of a main component and a small error term. For this purpose, denote

$$\Theta_{1t}^b := \frac{1}{N-1} \sum_{j \geq 2} \partial_y \delta_\mu \bar{U}(t, \mu_t^{-1}; X_t^j) [\delta_\mu f^*(t, X_t^j, \mu_t^{-1}; X_t^1) - \delta_\mu f^*(t, X_t^j, \mu_t^{-1}; X_t^j)].$$

Then we have $\bar{\Theta}_{1t}^b = \Theta_{1t}^b + \xi_t^b$, where ξ_t^b is viewed as a small error term. By (3.14), we have

$$\Theta_t^b = -(N-1) \langle \mu_t^{-1}(dy), L^*(t, y, \mu_t^{-1}) \rangle.$$

For (4.8), denote $\Theta_t^* := \Theta_t^a + \xi_{1t}^a + \xi_{2t}^a + \Theta_t^b + \bar{\Theta}_{1t}^b + \Theta_{2t}^b$. Now we have

$$\begin{aligned} \Theta_t^* &= \Theta_t^a + \Theta_t^b + \Theta_{1t}^b + \Theta_{2t}^b + \xi_{1t}^a + \xi_{2t}^a + \xi_t^b \\ &= \left\{ \partial_t U(t, X_t^1, \mu_t^{-1}) + U_x(t, X_t^1, \mu_t^{-1}) f(X_t^1, u_t^1, \mu_t^{-1}) \right. \\ &\quad + \frac{1}{2} \text{Tr}[U_{xx}(t, X_t^1, \mu_t^{-1})(\Sigma + \Sigma_0)] \\ &\quad + \frac{1}{N-1} \sum_{j \geq 2} \partial_y \delta_\mu U(t, X_t^1, \mu_t^{-1}; X_t^j) f^*(t, X_t^j, \mu_t^{-1}) \\ &\quad + \frac{1}{2(N-1)} \sum_{j \geq 2} \text{Tr}[\partial_y^2 \delta_\mu U(t, X_t^1, \mu_t^{-1}; X_t^j)(\Sigma + \Sigma_0)] \\ &\quad + \frac{1}{N-1} \sum_{j \geq 2} \text{Tr}[\partial_{xy} \delta_\mu U(t, X_t^1, \mu_t^{-1}; X_t^j) \Sigma_0] \\ &\quad + \frac{1}{2(N-1)^2} \sum_{j \geq 2, k \geq 2} \text{Tr}[\partial_{yz} \delta_{\mu\mu} U(t, X_t^1, \mu_t^{-1}; X_t^k, X_t^j) \Sigma_0] \\ &\quad + \frac{1}{N-1} \sum_{j \geq 2} \partial_y \delta_\mu \bar{U}(t, \mu_t^{-1}; X_t^j) [\delta_\mu f^*(t, X_t^j, \mu_t^{-1}; X_t^1) \\ &\quad \quad \quad - \delta_\mu f^*(t, X_t^j, \mu_t^{-1}; X_t^j)] \\ &\quad \left. + \frac{1}{2(N-1)} \sum_{k \geq 2} \text{Tr}[\partial_{yz} \delta_{\mu\mu} \bar{U}(t, \mu_t^{-1}; X_t^k, X_t^k) \Sigma] \right\} \quad (=:\Upsilon) \\ &= \Upsilon - (N-1) \langle \mu_t^{-1}(dy), L^*(t, y, \mu_t^{-1}) \rangle + \xi_{1t}^a + \xi_{2t}^a + \xi_t^b \\ &= \Upsilon - (N-1) \langle \mu_t^{-1}(dy), L^*(t, y, \mu_t^{-1}) \rangle + \xi_{1t}^a + \xi_{2t}^a + \xi_t^b. \end{aligned}$$

Now Lemma 3.7 implies, after setting $x = X_t^1$, $u = u_t^1$, and $\mu = \mu_t^{-1}$ in (3.22), that

$$(4.9) \quad \Upsilon + L(X_t^1, u_t^1, \mu_t^{-1}) + \langle \mu_t^{-1}(dy), \delta_\mu L^*(t, y, \mu_t^{-1}; X_t^1) - \delta_\mu L^*(t, y, \mu_t^{-1}; y) \rangle \geq 0,$$

where the equality holds if $u_t^1 = \phi(t, X_t^1, \mu_t^{-1})$ in both Υ and L . It follows from (4.9) that

$$(4.10) \quad \begin{aligned} \Theta_t^* &\geq -L(X_t^1, u_t^1, \mu_t^{-1}) - \langle \mu_t^{-1}(dy), \delta_\mu L^*(t, y, \mu_t^{-1}; X_t^1) - \delta_\mu L^*(t, y, \mu_t^{-1}; y) \rangle \\ &\quad - (N-1) \langle \mu_t^{-1}(dy), L^*(t, y, \mu_t^{-1}) \rangle + \xi_{1t}^a + \xi_{2t}^a + \xi_t^b. \end{aligned}$$

We have

$$(4.11) \quad \begin{aligned} \sum_{j=2}^N L^*(t, X_t^j, \mu_t^{-j}) &= (N-1) \langle \mu_t^{-1}(dy), L^*(t, y, \mu_t^{-1}) \rangle \\ &\quad + \langle \mu_t^{-1}(dy), \delta_\mu L^*(t, y, \mu_t^{-1}; X_t^1) - \delta_\mu L^*(t, y, \mu_t^{-1}; y) \rangle + \sum_{j=2}^N \xi_{jt}^L \end{aligned}$$

and, in view of the terminal conditions of U and $\bar{U}$,

$$(4.12) \quad \sum_{j=1}^N g(X_T^j, \mu_T^{-j}) = U(T, X_T^1, \mu_T^{-1}) + (N-1) \bar{U}(T, \mu_T^{-1}) + \sum_{j=2}^N \xi_{jt}^g.$$

Combining (4.10) and (4.11) yields

$$(4.13) \quad \Theta_t^* \geq -L(X_t^1, u_t^1, \mu_t^{-1}) - \sum_{j=2}^N L^*(t, X_t^j, \mu_t^{-j}) + \xi_{1t}^a + \xi_{2t}^a + \xi_t^b + \sum_{j=2}^N \xi_{jt}^L.$$

By (4.8) and (4.13), we have

$$(4.14) \quad \begin{aligned} & \mathbb{E}[U(T, X_T^1, \mu_T^{-1}) + (N-1)\bar{U}(T, \mu_T^{-1})] \\ & \quad - \mathbb{E}[U(0, X_0^1, \mu_0^{-1}) + (N-1)\bar{U}(0, \mu_0^{-1})] \\ & \geq -\mathbb{E} \int_0^T \left[L(X_t^1, u_t^1, \mu_t^{-1}) + \sum_{j=2}^N L^*(t, X_t^j, \mu_t^{-j}) \right] dt \\ & \quad + \mathbb{E} \int_0^T \left(\xi_{1t}^a + \xi_{2t}^a + \xi_t^b + \sum_{j=2}^N \xi_{jt}^L \right) dt. \end{aligned}$$

Adding up (4.12) and (4.14), and rearranging all terms involving (L, L^*, g) on the left-hand side, we obtain (4.7), where the equality holds if $u_t^1 = \phi(t, X_t^1, \mu_t^{-1})$ for all t in (3.4). $\square$

LEMMA 4.8. For (3.4)–(3.5) and $j \geq 2$, $t \in [0, T]$, we have

$$(4.15) \quad |\xi_{1t}^a|, |\xi_t^b| \leq \frac{C}{N-1} (1 + |X_t^1|^2 + \langle |y|^2 \rangle_{\mu_t^{-1}}),$$

$$(4.16) \quad |\xi_{2t}^a| \leq \frac{C}{N-1},$$

$$(4.17) \quad |\xi_{jt}^L| \leq \frac{C}{|N-1|^2} (1 + |X_t^1|^2 + |X_t^j|^2 + \langle |y|^2 \rangle_{\mu_t^{-1}}),$$

$$(4.18) \quad |\xi_j^g| \leq \frac{C}{|N-1|^2} (1 + |X_T^1|^2 + |X_T^j|^2 + \langle |y|^2 \rangle_{\mu_T^{-1}}).$$

Proof. By (3.21), we calculate $\partial_y \delta_\mu U(t, x, \mu; y)$ and use Assumption 3.1 to obtain

$$(4.19) \quad |\partial_y \delta_\mu U(t, x, \mu; y)| \leq C(1 + |x| + |y| + \langle |y| \rangle_\mu).$$

Next, for $\mu^{-j} := \frac{1}{N-1} \sum_{k \neq j, k=1}^N \delta_{x^k}$, we have

$$(4.20) \quad \begin{aligned} & f^*(t, x^j, \mu^{-j}) - f^*(t, x^j, \mu^{-1}) \\ & = \int_0^1 \int_y \delta_\mu f^*(t, x^j, \mu^{-1} + s(\mu^{-j} - \mu^{-1}); y) (\mu^{-j} - \mu^{-1})(dy) ds. \end{aligned}$$

Denote $\hat{\mu}_s := \mu^{-1} + s(\mu^{-j} - \mu^{-1})$ and $\Delta^{j,1} \mu := \mu^{-j} - \mu^{-1}$. Assumption 3.2 implies

$$\begin{aligned} \left| \int_y \delta_\mu f^*(t, x^j, \hat{\mu}_s; y) \Delta^{j,1} \mu(dy) \right| &= \frac{1}{N-1} |\delta_\mu f^*(t, x^j, \hat{\mu}_s; x^1) - \delta_\mu f^*(t, x^j, \hat{\mu}_s; x^j)| \\ &\leq \frac{C}{N-1} \left(1 + |x^j| + |x^1| + \frac{1}{N-1} \sum_{k \geq 2} |x^k| \right). \end{aligned}$$

Subsequently, by (4.19) and (4.20), we have

$$(4.21) \quad \begin{aligned} |\xi_{1t}^a| &\leq \frac{C}{N-1} \left(1 + |X_t^1|^2 + \frac{1}{N-1} \sum_{j \geq 2} |X_t^j|^2 + \langle |y| \rangle_{\mu_t^{-1}}^2 \right) \\ &\leq \frac{C}{N-1} (1 + |X_t^1|^2 + \langle |y|^2 \rangle_{\mu_t^{-1}}). \end{aligned}$$

Next, we use the growth conditions of V , as specified by $\mathcal{C}_V$, and the semi-symmetry property (A.12) to establish $|\partial_{yz} \delta_{\mu\mu} U(t, X_t^1, \mu_t^{-1}; X_t^k, X_t^k)| \leq C$, which implies (4.16). To get the bound of $|\xi_t^b|$, we write

$$(4.22) \quad \begin{aligned} f^*(t, x^j, \mu^{-j}) &= f^*(t, x^j, \mu^{-1}) + \langle \Delta^{j,1} \mu(dy), \delta_{\mu} f^*(t, x^j, \mu^{-1}; y) \rangle \\ &\quad + \int_0^1 \int_0^1 \int_{y,z} s \delta_{\mu\mu} f^*(t, x^j, \mu^{-1} + s\tau(\mu^{-j} - \mu^{-1}); y, z) \\ &\quad \cdot \Delta^{j,1} \mu(dy) \Delta^{j,1} \mu(dz) d\tau ds. \end{aligned}$$

Denote $\hat{\mu}_{s,\tau} := \mu^{-1} + s\tau(\mu^{-j} - \mu^{-1})$. Then

$$\begin{aligned} &\int_{y,z} \delta_{\mu\mu} f^*(t, x^j, \hat{\mu}_{s,\tau}; y, z) \Delta^{j,1} \mu(dy) \Delta^{j,1} \mu(dz) \\ &= \frac{1}{N-1} \int_z [\delta_{\mu\mu} f^*(t, x^j, \hat{\mu}_{s,\tau}; x^1, z) - \delta_{\mu\mu} f^*(t, x^j, \hat{\mu}_{s,\tau}; x^j, z)] \Delta^{j,1} \mu(dz) \\ &= \frac{1}{(N-1)^2} \left\{ [\delta_{\mu\mu} f^*(t, x^j, \hat{\mu}_{s,\tau}; x^1, x^1) - \delta_{\mu\mu} f^*(t, x^j, \hat{\mu}_{s,\tau}; x^j, x^1)] \right. \\ &\quad \left. - [\delta_{\mu\mu} f^*(t, x^j, \hat{\mu}_{s,\tau}; x^1, x^j) - \delta_{\mu\mu} f^*(t, x^j, \hat{\mu}_{s,\tau}; x^j, x^j)] \right\}. \end{aligned}$$

Consequently, under Assumption 3.2, we have

$$(4.23) \quad \begin{aligned} &\left| \int_{y,z} \delta_{\mu\mu} f^*(t, x^j, \hat{\mu}_{s,\tau}; y, z) \Delta^{j,1} \mu(dy) \Delta^{j,1} \mu(dz) \right| \\ &\leq \frac{C}{(N-1)^2} (1 + |x^1| + |x^j| + \langle |y| \rangle_{\mu^{-1}}). \end{aligned}$$

We use Assumption 3.1 to obtain

$$(4.24) \quad \begin{aligned} |\partial_y \delta_{\mu} \bar{U}(t, \mu; y)| &\leq |\partial_y V(t, y, \mu)| + |\langle \mu(dw), \partial_y \delta_{\mu} V(t, w, \mu; y) \rangle| \\ &\leq C(1 + |y| + \langle |y| \rangle_{\mu}). \end{aligned}$$

By (4.22), (4.23), and (4.24), we obtain the bound of $|\xi^b|$ in (4.15). Finally, we follow the method in (4.22) to obtain (4.17) and (4.18). $\square$

LEMMA 4.9. *There exists a constant C_{K_0} depending on K_0 such that for (3.4)–(3.5), we have*

$$(4.25) \quad \sup_N \sup_{u^1 \in \mathcal{U}_{\mathcal{F}^N}^{K_0}} \max_{1 \leq i \leq N, 0 \leq t \leq T} \mathbb{E} |X_t^i|^2 \leq C_{K_0}.$$

Proof. By Itô's formula, we have

$$\mathbb{E} |X_t^1|^2 = \mathbb{E} |X_0^1|^2 + \mathbb{E} \int_0^t \left[X_s^{1\top} f(X_s^1, u_s^1, \mu_s^{-1}) + \text{Tr}(\Sigma + \Sigma_0) \right] ds.$$

We have

$$\begin{aligned} |X_s^{1\top} f(X_s^1, u_s^1, \mu_s^{-1})| &\leq C|X_s^1| \cdot \left(1 + |X_s^1| + |u_s^1| + \frac{1}{N-1} \sum_{k=2}^N |X_s^k|\right) \\ &\leq C \left(1 + |X_s^1|^2 + |u_s^1|^2 + \frac{1}{N-1} \sum_{k=2}^N |X_s^k|^2\right). \end{aligned}$$

Then it follows that

$$(4.26) \quad \mathbb{E}|X_t^1|^2 \leq \mathbb{E}|X_0^1|^2 + C \int_0^t \left(1 + \mathbb{E}|u_s^1|^2 + \max_{1 \leq k \leq N} \mathbb{E}|X_s^k|^2\right) ds.$$

Similarly, for $k \geq 2$, we use assumption (A1) and the linear growth of ϕ in Definition 2.2 to obtain

$$(4.27) \quad \mathbb{E}|X_t^k|^2 \leq \mathbb{E}|X_0^k|^2 + C \int_0^t \left(1 + \max_{1 \leq k \leq N} \mathbb{E}|X_s^k|^2\right) ds.$$

Next, we follow the proof of Proposition 4.3 to use Grönwall's lemma to obtain (4.25). $\square$

Proof of Theorem 4.5. By Theorem 4.7 and Lemmas 4.8 and 4.9, we obtain

$$\begin{aligned} J_{\text{soc}}^{(N)}(\hat{u}^1, \hat{\mathbf{u}}^{-1}) &= \mathbb{E}[U(0, X_0^1, \mu_0^{-1}) + (N-1)\bar{U}(0, \mu_0^{-1})] + O(1/N), \\ J_{\text{soc}}^{(N)}(u^1, \hat{\mathbf{u}}^{-1}) &\geq \mathbb{E}[U(0, X_0^1, \mu_0^{-1}) + (N-1)\bar{U}(0, \mu_0^{-1})] - O(1/N) \end{aligned}$$

for all $u^1(\cdot) \in \mathcal{U}_{\mathcal{F}^N}^{K_0}$. Then Theorem 4.5 follows readily.

5. Explicit solutions in LQ models. This section uses LQ mean field social optimization models to illustrate explicit solutions of the master equations of V , M , and U . The individual agent has the dynamics

$$(5.1) \quad dX_t^i = (AX_t^i + Bu_t^i + GX_t^{(-i)})dt + dW_t^i + D_0 dW_t^0, \quad 1 \leq i \leq N,$$

where the initial states are independent with $\mathbb{E}|X_0^i|^2 < \infty$. The individual cost is given by

$$(5.2) \quad J_i = \mathbb{E} \int_0^T (|X_t^i - \Gamma X_t^{(-i)} - \eta|_Q^2 + u_t^{i\top} R u_t^i) dt + \mathbb{E}|X_T^i - \Gamma_f X_T^{(-i)} - \eta_f|_{Q_f}^2,$$

where $X_t^{(-i)} := \frac{1}{N-1} \sum_{j=1, j \neq i}^N X_t^j$ and $|x|_Q^2 := x^\top Q x$. We have symmetric matrices $Q \geq 0$ and $R > 0$. Denote the social cost $J_{\text{soc}}^{(N)} = \sum_{i=1}^N J_i$.

5.1. The master equation of the value function. Denote $\bar{x} := \langle \mu(dy), y \rangle \in \mathbb{R}^n$ as a function of μ . For the model (5.1)–(5.2), corresponding to (2.6) we determine

$$\begin{aligned} \Phi &= V_x(t, x, \mu)(Ax + Bu^i + G\bar{x}) + |x - \Gamma\bar{x} - \eta|_Q^2 + u^{i\top} R u^i \\ &\quad + \langle \mu(dy), \partial_x \delta_\mu V(t, y, \mu; x)(Ax + Bu^i + G\bar{x}) \rangle. \end{aligned}$$

The minimizer of Φ takes the form

$$\hat{u}^i = \phi(t, x, \mu) = -\frac{1}{2} R^{-1} B^\top [V_x^\top(t, x, \mu) + \langle \mu(dy), \partial_x \delta_\mu V(t, y, \mu; x) \rangle].$$

The master equation (2.4) becomes

$$\begin{aligned}
 & -\partial_t V(t, x, \mu) \\
 & = V_x(t, x, \mu)(Ax + G\bar{x}) + \frac{1}{2}\text{Tr}[V_{xx}(t, x, \mu)(DD^\top + D_0D_0^\top)] \\
 & \quad + |x - \Gamma\bar{x} - \eta|_Q^2 - \frac{1}{4}V_xBR^{-1}B^\top V_x^\top \\
 & \quad + \frac{1}{4}\langle \mu(dy), \partial_x \delta_\mu V(t, y, \mu; x) \rangle \cdot BR^{-1}B^\top \cdot \langle \mu(dy), \partial_x^\top \delta_\mu V(t, y, \mu; x) \rangle \\
 & \quad + \langle \mu(dy), \partial_y \delta_\mu V(t, x, \mu; y) [Ay + B\phi(t, y, \mu) + G\bar{x}] \rangle \\
 & \quad + \frac{1}{2}\langle \mu(dy), \text{Tr}[\partial_y^2 \delta_\mu V(t, x, \mu; y)(DD^\top + D_0D_0^\top)] \rangle \\
 & \quad + \langle \mu(dy), \text{Tr}[\partial_{xy} \delta_\mu V(t, x, \mu; y)D_0D_0^\top] \rangle \\
 & \quad + \frac{1}{2}\langle \mu^{\otimes 2}(dydz), \text{Tr}[\partial_{yz} \delta_{\mu\mu} V(t, x, \mu; y, z)D_0D_0^\top] \rangle,
 \end{aligned}$$

where $V(T, x, \mu) = |x - \Gamma_f \bar{x} - \eta_f|_{Q_f}^2$.

We take the ansatz

$$\begin{aligned}
 (5.3) \quad V(t, x, \mu) &= x^\top P(t)x + \bar{x}^\top \Lambda(t)\bar{x} + 2x^\top H(t)\bar{x} \\
 &\quad + 2x^\top S(t) + 2\bar{x}^\top \theta(t) + r(t),
 \end{aligned}$$

where the functions $P(t)$, $\Lambda(t)$, $\dots$, $r(t)$ are to be determined, with $P(t)$ and $\Lambda(t)$ being symmetric. We calculate

$$\begin{aligned}
 \delta_\mu V(t, x, \mu; y) &= 2\bar{x}^\top \Lambda y + 2x^\top H y + 2y^\top \theta + \chi(t, x, \mu), \\
 \delta_{\mu\mu} V(t, x, \mu; y, z) &= 2z^\top \Lambda y + \delta_\mu \chi(t, x, \mu; z) + \chi_1(t, x, y, \mu)
 \end{aligned}$$

and derive the ordinary differential equation (ODE) system

$$(5.4) \quad \dot{P} = -A^\top P - PA + PBR^{-1}B^\top P - Q,$$

$$\begin{aligned}
 (5.5) \quad \dot{\Lambda} &= -(\Lambda + H)BR^{-1}B^\top(\Lambda + H^\top) \\
 &\quad - \Lambda[A + G - BR^{-1}B^\top(P + \Lambda + H + H^\top)] \\
 &\quad - [A + G - BR^{-1}B^\top(P + \Lambda + H + H^\top)]^\top \Lambda \\
 &\quad + H^\top BR^{-1}B^\top H - H^\top G - G^\top H - \Gamma^\top Q\Gamma,
 \end{aligned}$$

$$\begin{aligned}
 (5.6) \quad \dot{H} &= -A^\top H + PBR^{-1}B^\top H \\
 &\quad - H[A + G - BR^{-1}B^\top(P + \Lambda + H + H^\top)] - PG + Q\Gamma,
 \end{aligned}$$

$$(5.7) \quad \dot{S} = -A^\top S + PBR^{-1}B^\top S + HBR^{-1}B^\top(S + \theta) + Q\eta,$$

$$\begin{aligned}
 (5.8) \quad \dot{\theta} &= H^\top BR^{-1}B^\top S + \Lambda BR^{-1}B^\top(S + \theta) \\
 &\quad - [A + G - BR^{-1}B^\top(P + \Lambda + H + H^\top)]^\top \theta \\
 &\quad - (\Lambda + H)BR^{-1}B^\top \theta - G^\top S - \Gamma^\top Q\eta,
 \end{aligned}$$

$$\begin{aligned}
 (5.9) \quad \dot{r} &= S^\top BR^{-1}B^\top S + \theta^\top BR^{-1}B^\top(2S + \theta) \\
 &\quad - \text{Tr}[(P + \Lambda + H + H^\top)(D_0D_0^\top)] - \text{Tr}(PDD^\top) - \eta^\top Q\eta,
 \end{aligned}$$

where the terminal conditions are

$$\begin{aligned}
 P(T) &= Q_f, \quad \Lambda(T) = \Gamma_f^\top Q_f \Gamma_f, \quad H(T) = -Q_f \Gamma_f, \\
 S(T) &= -Q_f \eta_f, \quad \theta(T) = \Gamma_f Q_f \eta_f, \quad r(T) = \eta_f^\top Q_f \eta_f.
 \end{aligned}$$

We determine the control law

$$(5.10) \quad \phi(t, x, \mu) = -R^{-1}B^\top [Px + (\Lambda + H + H^\top)\bar{x} + S + \theta].$$

THEOREM 5.1. *The ODE system (5.4)–(5.9) has a unique solution $(P, \Lambda, \dots, r)$ on $[0, T]$.*

Proof. We first obtain a unique solution P on $[0, T]$. Denote $Z := P + \Lambda + H + H^\top$. By the ODEs of (P, Λ, H) , we can show that Z satisfies the following equation:

$$(5.11) \quad \dot{Z} = -(A + G)^\top Z - Z(A + G) + ZBR^{-1}B^\top Z - (I - \Gamma)^\top Q(I - \Gamma),$$

where $Z(T) = (I - \Gamma_f^\top)Q_f(I - \Gamma_f)$. We start by solving the standard Riccati equation (5.11) to get a unique solution $Z(t)$ on $[0, T]$. Setting $P + \Lambda + H + H^\top$ in (5.6) as Z , we determine H using a linear equation and further obtain $\Lambda = Z - P - H - H^\top$. Subsequently, we solve two linear equations to obtain (S, θ) . Next, r is determined from a linear equation.

The above calculation gives a solution of (5.4)–(5.9) on $[0, T]$, which is clearly unique by the local Lipschitz continuity of the vector field of the ODE system. $\square$

5.2. The equation of M . Given the control law ϕ in (5.10), we determine

$$f^*(t, x, \mu) = (A - BR^{-1}B^\top P)x + \hat{G}\bar{x} - BR^{-1}B^\top(S + \theta),$$

where

$$(5.12) \quad \hat{G}(t) := G - BR^{-1}B^\top(\Lambda(t) + H(t) + H^\top(t)).$$

Now master equation (3.12) reduces to the following form:

$$(5.13) \quad \begin{aligned} 0 = & \partial_t M(t, \mu) + \langle \mu(dy), [\partial_y \delta_\mu M(t, \mu; y)] f^*(t, y, \mu) \rangle \\ & + \frac{1}{2} \langle \mu(dy), \text{Tr}[\partial_y^2 \delta_\mu M(t, \mu; y)(DD^\top + D_0 D_0^\top)] \rangle \\ & + \frac{1}{2} \langle \mu^{\otimes 2}(dydz), \text{Tr}[\partial_{yz} \delta_{\mu\mu} M(t, \mu; y, z) D_0 D_0^\top] \rangle \\ & - \langle \mu^{\otimes 2}(dydz), [\partial_y \delta_\mu V(t, z, \mu; y)] \cdot [\delta_\mu f^*(t, y, \mu; y) - \delta_\mu f^*(t, y, \mu; z)] \rangle \\ & + \text{Tr}(\Lambda DD^\top), \end{aligned}$$

$$(5.14) \quad M(T, \mu) = 0, \quad (t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^n).$$

We take the ansatz

$$M(t, \mu) = \langle \mu, y^\top \Pi_1^\circ(t)y \rangle + \bar{x}^\top \Pi_2^\circ(t)\bar{x} + 2\bar{x}^\top \theta^\circ(t) + r^\circ(t),$$

where $\Pi_1^\circ(t)$ and $\Pi_2^\circ(t)$ are symmetric. We derive the following linear ODE system:

$$\begin{aligned} \dot{\Pi}_1^\circ &= -\Pi_1^\circ(A - BR^{-1}B^\top P) - (A - BR^{-1}B^\top P)^\top \Pi_1^\circ - (H\hat{G} + \hat{G}^\top H^\top), \\ \dot{\Pi}_2^\circ &= -\Pi_2^\circ(A + \hat{G} - BR^{-1}B^\top P) - (A + \hat{G} - BR^{-1}B^\top P)^\top \Pi_2^\circ \\ &\quad - \Pi_1^\circ \hat{G} - \hat{G}^\top \Pi_1^\circ + H\hat{G} + \hat{G}^\top H^\top, \\ \dot{\theta}^\circ &= -(A + \hat{G} - BR^{-1}B^\top P)^\top \theta^\circ + (\Pi_1^\circ + \Pi_2^\circ)BR^{-1}B^\top(S + \theta), \\ \dot{r}^\circ &= 2\theta^{\circ\top}BR^{-1}B^\top(S + \theta) - \text{Tr}[(\Pi_1^\circ + \Lambda)DD^\top + (\Pi_1^\circ + \Pi_2^\circ)(D_0 D_0^\top)], \end{aligned}$$

with the terminal conditions $\Pi_1^\circ(T) = \Pi_2^\circ(T) = 0$, $\theta^\circ(T) = 0$, and $r^\circ(T) = 0$, where the functions $(P, \Lambda, H, \hat{G}, S, \theta)$ have been determined from Theorem 5.1 and (5.12). The next proposition is obvious.

PROPOSITION 5.2. *The ODE system of $(\Pi_1^o, \Pi_2^o, \theta^o, r^o)$ has a unique solution on $[0, T]$.*

Remark 5.3. In view of Theorem 5.1 and Proposition 5.2, for the model (5.1)–(5.2), we may verify Assumptions 3.1, 3.2, 3.3, and 4.1.

5.3. The solution of U . We use (3.21) to construct a solution of U in the linear quadratic case and obtain

$$(5.15) \quad U(t, x, \mu) = x^\top P x + \langle \mu, y^\top (\Pi_1^o - H - H^\top) y \rangle + \bar{x}^\top (\Pi_2^o - \Lambda) \bar{x} \\ + 2x^\top (\Lambda + H + H^\top) \bar{x} + 2x^\top (S + \theta) + 2\bar{x}^\top \theta^o + r + r^o,$$

where $\bar{x} := \langle y \rangle_\mu$. On the other hand, we may also directly solve (3.10) by looking for quadratic solutions of the following form:

$$(5.16) \quad U(t, x, \mu) = x^\top \Pi_1^\diamond(t) x + \langle \mu, y^\top \Pi_2^\diamond(t) y \rangle + \bar{x}^\top \Pi_3^\diamond(t) \bar{x} \\ + 2x^\top \Pi_4^\diamond(t) \bar{x} + 2x^\top S^\diamond(t) + 2\bar{x}^\top \theta^\diamond(t) + r^\diamond(t),$$

where $\Pi_1^\diamond(t)$, $\Pi_2^\diamond(t)$, and $\Pi_3^\diamond(t)$ are symmetric matrix functions of t .

THEOREM 5.4. *For the model (5.1)–(5.2), the master equation (3.10) has a unique solution U within the class of quadratic solutions, which coincides with the representation (5.15).*

Proof. By use of (5.16), we derive an ODE system for $(\Pi_1^\diamond, \Pi_2^\diamond, \dots, r^\diamond)$. We further show solvability of the ODEs one by one in the following order to take $\Pi_1^\diamond = P$, $\Pi_2^\diamond = \Pi_1^\diamond - H - H^\top$, $\Pi_4^\diamond = \Lambda + H + H^\top$, $\Pi_3^\diamond = \Pi_2^\diamond - \Lambda$, $S^\diamond = S + \theta$, $\theta^\diamond = \theta^o$, and $r^\diamond = r + r^o$. The solution is clearly unique by the local Lipschitz continuity property of the vector field of the ODE system of $(\Pi_1^\diamond, \Pi_2^\diamond, \dots, r^\diamond)$. $\square$

5.4. An example from systemic risk. In the system of interbank lending and borrowing, the state processes of N banks, as the log-capitalization defined in [13], have dynamics

$$dX_t^i = u_t^i dt + \sigma(\sqrt{1 - \rho^2} dW_t^i + \rho dW_t^0), \quad 0 \leq \rho \leq 1,$$

with the initial value X_0^i , where $W_t^0, W_t^1, \dots, W_t^N$ are independent standard Brownian motions. The N banks obtain the social optimal strategy through minimizing

$$J_{\text{soc}}^{(N)}(\mathbf{u}(\cdot)) = \sum_{i=1}^N J_i(\mathbf{u}(\cdot)), \quad \mathbf{u} := (u^1, \dots, u^N),$$

where

$$J_i(\mathbf{u}(\cdot)) = \mathbb{E} \int_0^T L(X_t^i, u_t^i, X_t^{(-i)}) dt + \mathbb{E} g(X_T^i, X_T^{(-i)})$$

with $X_t^{(-i)} := \frac{1}{N-1} \sum_{j \neq i} X_t^j$, running cost

$$L(x^i, u^i, z) = (u^i)^2 + 2qu^i(x^i - z) + \epsilon_0(x^i - z)^2, \quad q^2 \leq \epsilon_0,$$

and terminal cost $g(x^i, z) = c(x^i - z)^2$. The parameters q , ϵ_0 , and c are positive with $q^2 \leq \epsilon_0$ to ensure convexity of the running cost $L(x^i, u^i, z)$. See [13] for an interpretation of q in terms of incentive for lending/borrowing.

5.4.1. The direct solution. Denote $\mathbf{x} = (x^1, \dots, x^N)^\top$. The value function is

$$U_{\text{soc}}^N(t, \mathbf{x}) = \inf_{\mathbf{u}(\cdot)} \sum_{i=1}^N \mathbb{E} \left[\int_t^T L(X_s^i, u_s^i, X_s^{(-i)}) ds + g(X_T^i, X_T^{(-i)}) \right],$$

where $(t, \mathbf{x}) \in [0, T] \times \mathbb{R}^N$. By solving the HJB equation of U_{soc}^N , we obtain $U_{\text{soc}}^N(t, \mathbf{x}) = \mathbf{x}^\top \mathbf{P}(t) \mathbf{x} + r(t)$, where $\mathbf{P}(t)$ has N diagonal entries π_1 and off-diagonal entries all equal to π_2 , with

$$\begin{aligned} 0 &= \dot{\pi}_1 - (\pi_1 + q)^2 - \left(\pi_2 - \frac{q}{N-1} \right)^2 (N-1) + \frac{\epsilon_0 N}{N-1}, \\ 0 &= \dot{\pi}_2 - 2(\pi_1 + q) \left(\pi_2 - \frac{q}{N-1} \right) - \left(\pi_2 - \frac{q}{N-1} \right)^2 (N-2) - \frac{\epsilon_0 N}{(N-1)^2}, \end{aligned}$$

where $\pi_1(T) = cN/(N-1)$ and $\pi_2(T) = -cN/(N-1)^2$. The optimal control law of the i th bank is

$$(5.17) \quad \hat{u}^{N,i}(t, \mathbf{x}) = -(\pi_1 + q)x^i - \left(\pi_2 - \frac{q}{N-1} \right) \sum_{k \neq i, k=1}^N x^k.$$

Denote the ODE

$$(5.18) \quad 0 = \dot{P}_d - (P_d + q)^2 + \epsilon_0, \quad P_d(T) = c,$$

which has a unique solution on $[0, T]$. We can further show $\sup_{0 \leq t \leq T} |\pi_1(t) - P_d(t)| = O(1/N)$. By checking the ODE of $(N-1)\pi_2$, we obtain $\sup_{0 \leq t \leq T} |(N-1)\pi_2(t) + P_d(t)| = O(1/N)$. As $N \rightarrow \infty$, the optimal control law (5.17) takes the limiting form

$$(5.19) \quad \hat{u}^i(t, x^i, \mu) = [P_d(t) + q](\bar{x} - x^i),$$

where $\bar{x} := \langle \mu, y \rangle$. Comparing (5.19) with the open-loop and closed-loop Nash equilibria proposed in [13], the social optimum has the same solution as the mean field game has in the limit with $N \rightarrow \infty$.

5.4.2. The master equation-based control. By (2.6), we obtain the minimizer

$$(5.20) \quad \hat{u}^i = -\frac{1}{2} V_x(t, x, \mu) - q(x - \bar{x}) - \frac{1}{2} \langle \mu(dy), \partial_x \delta_\mu V(t, y, \mu; x) \rangle,$$

where $\bar{x} := \langle \mu, y \rangle$. We take the ansatz $V(t, x, \mu) = P(t)x^2 + \Lambda(t)\bar{x}^2 + 2H(t)x\bar{x} + r(t)$ and derive

$$\begin{aligned} 0 &= \dot{P} - (P + q)^2 + \epsilon_0, \\ 0 &= \dot{\Lambda} + (\Lambda + H)^2 - (H - q)^2 - 2\Lambda(P + \Lambda + 2H) + \epsilon_0, \\ 0 &= \dot{H} - (P + q)(H - q) - H(P + \Lambda + 2H) - \epsilon_0, \\ 0 &= \dot{r} + \sigma^2 P + \sigma^2 \rho^2 (\Lambda + 2H), \end{aligned}$$

where $P(T) = c$, $\Lambda(T) = c$, $H(T) = -c$, $r(T) = 0$. Clearly, $P = P_d$ on $[0, T]$. Denote $Z_1 := \Lambda + 2H$, and use the equations of $\dot{\Lambda}$ and $\dot{H}$ to write the following ODE:

$$(5.21) \quad 0 = \dot{Z}_1 + (Z_1 - q)^2 - 2(P + Z_1)(Z_1 - q) - \epsilon_0, \quad Z_1(T) = -c,$$

for which we can show that $Z_1 = -P$ is a solution. By uniqueness of the solution of (5.21), it follows that $\Lambda + 2H = -P$ on $[0, T]$. Subsequently, we determine H from a linear equation using the equation of $\dot{H}$ after setting $\Lambda + 2H$ as $-P$; this in turn determines Λ . The above solution (P, Λ, H) is unique. The control law (5.20) now becomes

$$\begin{aligned}\dot{u}^i(t, x, \mu) &= -\{[P(t) + q]x + [\Lambda(t) + 2H(t) - q]\bar{x}\} \\ &= [P(t) + q](\bar{x} - x),\end{aligned}$$

which agrees with (5.19).

6. Concluding remarks. Our performance estimate via multiscale analysis has been based on existence of sufficiently well behaved solutions of two master equations for V and M . For future work, it will be of interest to develop existence results following such methods as in [9, 16] for more specific models.

Appendix A: Preliminary lemmas and semi-symmetry property.

LEMMA A.1. Suppose the function $\psi(x, y)$ from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ has continuous partial derivatives $\partial_{xy}\psi(x, y)$ and $\partial_{yx}\psi(x, y)$, and denote $h(x, y) = \partial_{xy}\psi(x, y)$. Then

$$(A.1) \quad \partial_{yx}\psi(x, y) = (h(x, y))^\top,$$

$$(A.2) \quad \partial_{xy}\psi(y, x) = (h(y, x))^\top.$$

Proof. The (i, j) th entry in $\partial_{xy}\psi(x, y)$ is $\partial_{x_i y_j}\psi(x, y)$, and the (j, i) th entry in $\partial_{yx}\psi(x, y)$ is $\partial_{y_j x_i}\psi(x, y)$. Now (A.1) follows from Schwarz's theorem [23]. Next, (A.1) yields (A.2) by switching x and y . $\square$

LEMMA A.2. Let $\bar{\Sigma} \in \mathbb{R}^{n \times n}$ be symmetric. Suppose $\partial_{zy}\delta_\mu V(t, y, \mu; z)$ and $\partial_{yz}\delta_\mu V(t, y, \mu; z)$ are continuous in (y, z) . Then we have

$$(A.3) \quad \text{Tr}[\partial_{zy}\delta_\mu V(t, y, \mu; z)\bar{\Sigma}] = \text{Tr}[\partial_{yz}\delta_\mu V(t, y, \mu; z)\bar{\Sigma}],$$

$$(A.4) \quad \text{Tr}[\partial_{yz}\delta_\mu V(t, y, \mu; z)\bar{\Sigma}]|_{z=y} = \text{Tr}[\partial_{yz}\delta_\mu V(t, z, \mu; y)\bar{\Sigma}]|_{z=y}.$$

Proof. If D and $\bar{\Sigma}$ are $n \times n$ matrices and $\bar{\Sigma}$ is symmetric, then we have

$$(A.5) \quad \text{Tr}(D\bar{\Sigma}) = \text{Tr}(\bar{\Sigma}D) = \text{Tr}[(\bar{\Sigma}D)^\top] = \text{Tr}(D^\top \bar{\Sigma}),$$

which together with Lemma A.1 yields (A.3). Denote $h(y, z) := \partial_{yz}\delta_\mu V(t, y, \mu; z)$. By Lemma A.1, $\partial_{yz}\delta_\mu V(t, z, \mu; y) = (h(z, y))^\top$, which combined with (A.5) implies (A.4). $\square$

Consider a continuous function $\psi(t, s): R_c = [0, T] \times [0, c) \rightarrow \mathbb{R}$, where $c > 0$. Define the partial derivatives $\partial_t\psi(t, s)$, $\partial_s\psi(t, s)$, and $\partial_t\partial_s\psi(t, s)$ on R_c , where each partial derivative is interpreted as a one-sided derivative when we take ∂_t (∂_s , resp.) at $t = 0$ or $t = T$ (at $s = 0$, resp.). For instance, for $0 \leq t \leq T$, we have $\partial_s\psi(t, 0) := \lim_{\epsilon \rightarrow 0^+} \frac{\psi(t, \epsilon) - \psi(t, 0)}{\epsilon}$.

The next lemma extends the symmetry property of second-order partial derivatives in Schwarz's theorem [23] to the case of boundary points of a region. The proof uses essentially the same argument as in [23, pp. 317] and is omitted here.

LEMMA A.3. Suppose ψ satisfies the following conditions:

- (i) The partial derivatives $\partial_t\psi(t, s)$, $\partial_s\psi(t, s)$, and $\partial_t\partial_s\psi(t, s)$ exist on R_c .
- (ii) $\partial_t\partial_s\psi(t, s)$ is continuous at the point $(t_0, 0)$ for $t_0 \in [0, T]$.

Then $\partial_s\partial_t\psi(t_0, 0)$ exists and $\partial_s\partial_t\psi(t_0, 0) = \partial_t\partial_s\psi(t_0, 0)$.

LEMMA A.4. Suppose $\hat{V}(t, x, \mu)$ is a function from $[0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)$ to $\mathbb{R}$ satisfying the following conditions:

- (i) $\partial_t \hat{V}(t, x, \mu)$, $\delta_\mu \hat{V}(t, x, \mu; y)$, and $\partial_t \delta_\mu \hat{V}(t, x, \mu; y)$ exist and are jointly continuous in $(t, x, \mu) \in [0, T] \times \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)$ and $(t, x, y, \mu) \in [0, T] \times \mathbb{R}^{2n} \times \mathcal{P}_2(\mathbb{R}^n)$, respectively;
- (ii) for each constant $K > 0$,

$$|\delta_\mu \hat{V}(t, x, \mu; y)|, |\partial_t \delta_\mu \hat{V}(t, x, \mu; y)| \leq C_K(1 + |y|^2)$$

holds for all $t \in [0, T]$, $x, y \in \mathbb{R}^n$, $\mu \in \mathcal{P}_2(\mathbb{R}^n)$ whenever $|x| \leq K$ and $W_2(\mu, \delta_0) \leq K$ (equivalently $\langle \mu, |y|^2 \rangle \leq K^2$), where C_K is a constant depending on K .

Then the derivative $\delta_\mu(\partial_t \hat{V}(t, x, \mu))(y)$ exists and, moreover,

$$(A.6) \quad \delta_\mu(\partial_t \hat{V}(t, x, \mu))(y) = \partial_t \delta_\mu \hat{V}(t, x, \mu; y)$$

for all $t \in [0, T]$, $x, y \in \mathbb{R}^n$, $\mu \in \mathcal{P}_2(\mathbb{R}^n)$.

Proof. Step 1. We show the normalization property of $\partial_t \delta_\mu \hat{V}(t, x, \mu; y)$. By the normalization property of $\delta_\mu \hat{V}$, we have $\langle \mu(dy), \delta_\mu \hat{V}(t, x, \mu; y) \rangle = 0$, which implies

$$0 = \partial_t \langle \mu(dy), \delta_\mu \hat{V}(t, x, \mu; y) \rangle = \langle \mu(dy), \partial_t \delta_\mu \hat{V}(t, x, \mu; y) \rangle,$$

due to condition (ii) and an application of the dominated convergence theorem.

Step 2. Let μ_1 and μ_2 both from $\mathcal{P}_2(\mathbb{R}^n)$ be fixed. For $t \in [0, T]$ and $s \in [0, 1]$, set $\psi(t, s) := \hat{V}(t, x, \mu_1 + s\nu)$, where $\nu = \mu_2 - \mu_1$. We have

$$(A.7) \quad \partial_t \psi(t, s) = \partial_t \hat{V}(t, x, \mu_1 + s\nu)$$

for $(t, s) \in [0, T] \times [0, 1]$. We proceed to check $\partial_s \psi(t, s)$. Suppose $s \in (0, 1)$. Then, by the definition of the derivative, $\delta_\mu \hat{V}$ evaluated at $\mu_1 + s\nu$ gives

$$(A.8) \quad \lim_{\epsilon \downarrow 0} \frac{\psi(t, s + \epsilon) - \psi(t, s)}{\epsilon} = \langle \nu(dy), \delta_\mu \hat{V}(t, x, \mu_1 + s\nu; y) \rangle.$$

Next, we check $\lim_{\epsilon \downarrow 0} \frac{\psi(t, s - \epsilon) - \psi(t, s)}{-\epsilon}$ and get the same limit as in (A.8) (see [30] for details). We similarly obtain the one-sided derivative of $\psi(t, \cdot)$ as $\langle \nu(dy), \delta_\mu \hat{V}(t, x, \mu_1 + s\nu; y) \rangle$ at $s = 0$ and $s = T$. Therefore, for all $s \in [0, T]$, $\partial_s \psi(t, s)$ exists and

$$(A.9) \quad \partial_s \psi(t, s) = \langle \nu(dy), \delta_\mu \hat{V}(t, x, \mu_1 + s\nu; y) \rangle.$$

Subsequently, by (A.9), for each fixed $t \in (0, T)$, we have

$$(A.10) \quad \begin{aligned} \partial_t \partial_s \psi(t, s) &= \partial_t \langle \nu(dy), \delta_\mu \hat{V}(t, x, \mu_1 + s\nu; y) \rangle \\ &= \langle \nu(dy), \partial_t \delta_\mu \hat{V}(t, x, \mu_1 + s\nu; y) \rangle, \quad s \in [0, T], \end{aligned}$$

where the second equality results from condition (ii). We similarly obtain (A.10) for $t = 0$ and $t = T$.

Step 3. As $(t', s') \rightarrow (t, s) \in [0, T] \times [0, 1]$, we may show $W_2(\mu_1 + s'\nu, \mu_1 + s\nu) \rightarrow 0$ using [40, Thm. 6.9], and therefore $\partial_t \delta_\mu \hat{V}(t', x, \mu_1 + s'\nu; y) \rightarrow \partial_t \delta_\mu \hat{V}(t, x, \mu_1 + s\nu; y)$ by condition (i). Subsequently, $\partial_t \partial_s \psi(t', s') \rightarrow \partial_t \partial_s \psi(t, s)$ by (A.10) and the dominated convergence theorem under condition (ii). Hence, $\partial_t \partial_s \psi(t, s)$ is continuous in $(t, s) \in [0, T] \times [0, 1]$. Therefore, by Lemma A.3, $\partial_s \partial_t \psi(t, s)$ exists at each point $(t, 0)$, $t \in [0, T]$, and, moreover,

$$(A.11) \quad \partial_s \partial_t \psi(t, 0) = \partial_t \partial_s \psi(t, 0) = \langle \nu(dy), \partial_t \delta_\mu \hat{V}(t, x, \mu_1; y) \rangle,$$

where the second equality follows from (A.10). By (A.7) and (A.11), we have

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} [\partial_t \hat{V}(t, x, \mu_1 + \epsilon \nu) - \partial_t \hat{V}(t, x, \mu_1)] &= \partial_s \partial_t \psi(t, 0) \\ &= \langle \nu(dy), \partial_t \delta_\mu \hat{V}(t, x, \mu_1; y) \rangle \end{aligned}$$

for all $\mu_1 \in \mathcal{P}_2(\mathbb{R}^n)$. So $\delta_\mu(\partial_t \hat{V}(t, x, \mu_1))(y)$ exists. Recalling Step 1, we obtain (A.6). $\square$

PROPOSITION A.5 (see [9]). *Suppose that both $\delta_\mu \psi(\mu; y)$ and $\psi_{\mu\mu}(\mu; y, z)$ exist and are jointly continuous in their arguments and that for each $K > 0$, $|\delta_\mu \psi(\mu; y)|$, $|\delta_{\mu\mu} \psi(\mu; y, z)| \leq C_K(1 + |y|^2 + |z|^2)$ holds for all $y, z \in \mathbb{R}^n$ whenever μ satisfies $W_2(\mu, \delta_0) \leq K$. Then*

$$(A.12) \quad \delta_{\mu\mu} \psi(\mu; y, z) = \delta_{\mu\mu} \psi(\mu; z, y) + \delta_\mu \psi(\mu; y) - \delta_\mu \psi(\mu; z).$$

The equality (A.12) will be called the semi-symmetry property of $\delta_{\mu\mu} \psi$ and has been proved in [9, pp. 33–34] for bounded derivatives. But the proof there can be easily adapted to the quadratic growth case. Note that the proof in [9] has required the normalization property of the derivative $\delta_\mu \psi$. Therefore, we adopt this normalization convention in section 1.2.

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