

Risk-Sensitive Linear-Quadratic Mean-Field Games: Asymptotic Solvability and Decentralized $O(1/N)$ -Nash Equilibria*

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In honour of the 80th birthday of Professor Peter Caines

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Abstract This paper considers risk-sensitive linear-quadratic mean-field games. By the so-called direct approach via dynamic programming, the authors determine the feedback Nash equilibrium in an N -player game. Subsequently, the authors design a set of decentralized strategies by passing to the mean-field limit. The authors prove that the set of decentralized strategies constitutes an $O(1/N)$ -Nash equilibrium when applied by the N players, and hence obtain so far the tightest equilibrium error bounds for this class of models.

Keywords Asymptotic Nash equilibria, decentralized strategies, linear-quadratic, mean-field games, risk-sensitive costs.

1 Introduction

Since its inception in the first decade of this century^[1, 2], mean field game theory has developed into a major research area, providing a powerful methodology to tackle complex decision problems involving a large number of noncooperative players. In addition to the intense mathematical and theoretical investigations (see [1–5]) and references therein), the application of mean field games has spread to a vast range of domains: Finance^[6, 7], wireless communication networks^[8], load management in power systems^[9], electrical vehicle charging control^[10], cryptocurrency mining^[11], opinion dynamics^[12], collective motion^[13, 14], large-scale Kuramoto oscillator models^[15], theoretical physics^[16], among others.

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The vast literature of mean field game theory and applications has been largely built upon two fundamental methodologies: The direct approach and the fixed point approach. The direct approach starts by formally solving an N -player game to obtain a large coupled solution equation system. The next step is to derive a limit for the solution by taking $N \rightarrow \infty$ ^[2]. By contrast, the fixed point approach is to solve an optimal control problem of a single agent based on consistent mean field approximations and formalize a fixed point problem to determine the mean field; this approach is also called the Nash certainty equivalence approach^[1]. The solution of the fixed point problem may be used to design decentralized strategies in the original large but finite population model to achieve an ε -Nash equilibrium^[17]. A more detailed account of the two methodologies and their relation is available in [18].

In the existing literature, linear-quadratic (LQ) mean field games^[17] have been of significant interest, with the very appealing feature of closed-form solutions while very few mean field game models beyond the LQ cases allow explicit solutions. This class of mean field games has been intensively studied by many researchers, addressing LQ Markov jump models^[19], ergodic costs^[20], an adjoint equation based fixed point approach^[21], agent dynamics modelled by backward stochastic differential equations^[22], control input constraints^[23], $O(1/N)$ -Nash equilibrium results with controlled diffusion and indefinite weights^[24], partial information with forward-backward stochastic dynamics^[25]. The LQ mean field game can be generalized to include a major player^[26, 27], which has applications in competitive trading in financial markets^[6].

For risk-sensitive mean-field games, one can refer to [28–34]. In this paper, we consider a class of risk-sensitive LQ mean-field games and aim to analyze the individual player's performance loss due to the use of decentralized information. While there has existed extensive work examining performance of risk sensitive LQ mean field games under different specific modeling, researchers have encountered major difficulty to get rigorous and satisfactory performance estimates. A key reason is that when risk-sensitive costs (as exponentiated costs) are used in the game, the mean square error estimates for mean field approximations over the whole time horizon, which have been widely used in mean field games with risk-neutral costs, are no longer adequate. As a result, various restrictions on the model are needed in the analysis while the performance loss bounds are generally loose, at most achieving an error bound of $O(1/\sqrt{N})$.

We adopt the direct approach developed for LQ mean field games in [18] and extend the re-scaling technique there to analyze the asymptotic behavior of high order Riccati ordinary differential equations (ODEs). By this approach, we are able to overcome the difficulty confronting different researchers. Our contributions are outlined as the following. i) Using the direct approach via dynamic programming, we obtain the centralized strategies and the resulting decentralized strategies by taking the former's mean field limit; ii) by the Feynman-Kac formula, we derive the corresponding costs for the centralized and decentralized strategies; iii) we prove that the set of decentralized strategies is an $O(1/N)$ -Nash equilibrium of the risk-sensitive N -player game. Compared with^[29, 31–33], we obtain the tightest equilibrium error bound.

The rest of the paper is organized as follows. We formulate the risk-sensitive LQ mean-field game in Section 2. In Section 3, we analyze feedback Nash equilibria in N -player games and

study their asymptotic solvability. In Section 4, we construct the limit decentralized strategies and compute the corresponding individual costs. In Section 5, we prove that the set of decentralized strategies is an $O(1/N)$ -Nash equilibrium for the risk-sensitive game of N players. Section 6 provides some numerical examples. Section 7 concludes the paper.

2 The N -Player Model

2.1 Problem Formulation

For the risk-sensitive LQ mean-field game, we start by introducing a population of N players $\mathcal{A}_i$, $1 \leq i \leq N$. The dynamics of player $\mathcal{A}_i$ satisfy the following stochastic differential equation (SDE):

$$dX_i(t) = \left(AX_i(t) + Bu_i(t) + GX^{(N)}(t) \right) dt + DdW_i(t), \quad (1)$$

where we have the state $X_i(t) \in \mathbb{R}^n$, the control $u_i(t) \in \mathbb{R}^{n_1}$, the population average state $X^{(N)}(t) = \frac{1}{N} \sum_{i=1}^N X_i(t)$, and $\{W_i, 1 \leq i \leq N\}$ are N independent $\mathbb{R}^{n_2}$ -valued Brownian motions. The initial states are independent, and also independent of the N Brownian motions. The coefficients A, B, G and D are constant matrices with appropriate dimensions.

For a real matrix $Z \in \mathbb{R}^{m \times n}$ and a positive semi-definite matrix $Q \in \mathbb{R}^{m \times m}$, we denote $|Z|_Q^2 = Z^T Q Z$. Let $u_{-i} = (u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_N)$ be the set of strategies except $\mathcal{A}_i$. The cost of player $\mathcal{A}_i$ is defined as

$$J_i^{(N)}(u_i, u_{-i}) = \frac{1}{\alpha} \ln \mathbb{E} \left[e^{\alpha L_i(u_i, u_{-i})} \right], \quad 1 \leq i \leq N, \quad (2)$$

where $\alpha > 0$ is a risk-sensitive parameter and

$$L_i(u_i, u_{-i}) = \int_0^T \left(|X_i(t) - \Gamma X^{(N)}(t)|_Q^2 + |u_i(t)|_R^2 \right) dt + |X_i(T) - \Gamma_f X^{(N)}(T)|_{Q_f}^2.$$

Here the constant matrices $Q \geq 0$, $R > 0$, $Q_f \geq 0$, Γ, Γ_f have compatible dimensions.

Remark 2.1 For notational simplicity, we consider constant parameters $(A, B, G, D, Q, R, \Gamma)$ in (1) and (2). When these parameters are replaced by time dependence functions (such as $A(t), \dots, \Gamma(t)$), the analysis is similar.

Let the functions $\phi(t, x_1, \dots, x_N) : [0, T] \times \mathbb{R}^n \times \dots \times \mathbb{R}^n \rightarrow \mathbb{R}^{n_1}$ and $\varphi(t, x) : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n_1}$ be continuous, and Lipschitz continuous in $(x_1, \dots, x_N)$ and x , respectively. For player $\mathcal{A}_i$, $1 \leq i \leq N$, we introduce

$$\mathcal{U}_i^c = \{u_i \mid u_i = \phi(t, x_1, \dots, x_N)\}, \quad \mathcal{U}_i^d = \{u_i \mid u_i = \varphi(t, x_i)\}$$

as sets for centralized (state) feedback strategies and decentralized (state) feedback strategies, respectively.

Consider the following two scenarios of initial states:

- (I) $X_i(0), 1 \leq i \leq N$, are $\mathbb{R}^n$ -valued independent random variables which have mean μ_i and satisfy $P(X_i(0) \in S_X) = 1$ for each i and for the same compact set $S_X \subset \mathbb{R}^n$. Moreover, the average mean $\mu^{(N)} = \frac{1}{N} \sum_{i=1}^N \mu_i$ converges to μ as $N \rightarrow \infty$.

- (II) $X_i(0), 1 \leq i \leq N$, are $\mathbb{R}^n$ -valued i.i.d. normal random variables with mean μ and covariance matrix $\Sigma > 0$.

In later performance evaluation, the independent normal initial states in Scenario (II) will allow us to compute the asymptotic individual costs (as $N \rightarrow \infty$) in an explicit form. Let us introduce the following definitions and problems.

Definition 2.2 The set of centralized strategies $u_i^*(\cdot) \in \mathcal{U}_i^c, 1 \leq i \leq N$, is a Nash equilibrium if for each i ,

$$J_i^{(N)}(u_i^*, u_{-i}^*) \leq J_i^{(N)}(u_i, u_{-i}^*), \quad \forall u_i \in \mathcal{U}_i^c.$$

Problem (CFNE) For the risk-sensitive LQ mean-field game model with dynamics (1) and individual costs (2), find a set of centralized feedback Nash equilibrium strategies $(u_1^*, \dots, u_N^*)$.

The Nash equilibrium for Problem (CFNE) is a set of centralized strategies based on closed-loop perfect state (CLPS) information^[35], where each player requires the state information of all players. This informational requirement makes the implementation difficult when N is large. To address this issue, we are interested in finding decentralized strategies.

Definition 2.3 The set of decentralized strategies $\hat{u}_i(\cdot) \in \mathcal{U}_i^d, 1 \leq i \leq N$, is an ε_N -Nash equilibrium if for each i ,

$$J_i^{(N)}(\hat{u}_i, \hat{u}_{-i}) \leq J_i^{(N)}(u_i, \hat{u}_{-i}) + \varepsilon_N, \quad \forall u_i \in \mathcal{U}_i^c.$$

Problem (DF ε -NE) For the model with dynamics (1) and costs (2), find a set of decentralized feedback strategies that possesses an ε -Nash equilibrium property.

Remark 2.4 Existing works heavily rely on the fixed-point approach. By estimating the error in mean-field approximations, they can only obtain a bound of $O(1/\sqrt{N})$ for the cost gap between the centralized strategies and decentralized strategies. We employ the direct approach and use the Feynman-Kac formula to directly estimate the cost difference, leading to a bound of $O(1/N)$. Compared with the existing works, our approach needs weaker conditions and yields tighter estimates.

2.2 Notations

Let $\otimes$ stand for the Kronecker product, $\mathbf{1}_{m \times n}$ for an $m \times n$ matrix with all entries equal to 1 (in particular, $\mathbf{1}_m = \mathbf{1}_{m \times 1}$), and I_n for an $n \times n$ identity matrix. We denote by $\mathbb{S}^n$ the set of $n \times n$ symmetric matrices, and by column vectors $\{e_1^n, \dots, e_n^n\}$ the canonical basis of $\mathbb{R}^n$. For an $l \times m$ real matrix $Z = (z_{sk})_{1 \leq s \leq l, 1 \leq k \leq m}$, $|Z|$ stands for its Frobenius norm, $\|Z\|_{l_1} = \sum_{s,k} |z_{sk}|$ for its l_1 -norm. Define $\det(M)$ as the determinant of matrix $M \in \mathbb{R}^{n \times n}$. For a differentiable function $f(t, x) : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$, where $x = (x^1, \dots, x^n)^T$, define $\partial_{x^i} f$ and $\partial_{x^i x^j}^2 f$ as the first and second order partial derivatives with $1 \leq i, j \leq n$, $\partial_x f$ and $\partial_{xx}^2 f$ as the gradient (as a column vector) and the Hessian matrix with respect to x , respectively.

Hereinafter, define J_{ij} , $1 \leq i \neq j \leq N$, as the matrix resulting from exchanging the i -th and the j -th rows of submatrices in I_{Nn} . It is easy to show $J_{ij}^T = J_{ij} = J_{ij}^{-1}$, $1 \leq i \neq j \leq N$ (see [18]).

3 Nash Equilibria with N Players

Define

$$\begin{aligned}
 X &= (X_1^T, \dots, X_N^T)^T, \quad W = (W_1^T, \dots, W_N^T)^T, \quad \mathbf{x}_0 = (X_1(0)^T, \dots, X_N(0)^T)^T, \\
 \mathbf{A} &= \text{diag}[A, \dots, A] + \mathbf{1}_{N \times N} \otimes \frac{G}{N}, \quad \mathbf{B}_i = e_i^N \otimes B, \quad \mathbf{D} = \text{diag}[D, \dots, D], \\
 \mathbf{Q}_1 &= \left(I - \frac{1}{N} \Gamma, -\frac{1}{N} \Gamma, \dots, -\frac{1}{N} \Gamma \right)^T Q \left(I - \frac{1}{N} \Gamma, -\frac{1}{N} \Gamma, \dots, -\frac{1}{N} \Gamma \right), \\
 \mathbf{Q}_{1f} &= \left(I - \frac{1}{N} \Gamma_f, -\frac{1}{N} \Gamma_f, \dots, -\frac{1}{N} \Gamma_f \right)^T Q_f \left(I - \frac{1}{N} \Gamma_f, -\frac{1}{N} \Gamma_f, \dots, -\frac{1}{N} \Gamma_f \right), \\
 \mathbf{Q}_j &= J_{1j}^T \mathbf{Q}_1 J_{1j}, \quad \mathbf{Q}_{jf} = J_{1j}^T \mathbf{Q}_{1f} J_{1j}, \quad 2 \leq j \leq N, \\
 M_0 &= BR^{-1}B^T, \quad N_0 = 2\alpha DD^T, \quad M = M_0 - N_0.
 \end{aligned} \tag{3}$$

The system dynamics (1) and costs (2) can be rewritten as

$$\begin{cases} dX(t) = \left(\mathbf{A}X(t) + \sum_{i=1}^N \mathbf{B}_i u_i(t) \right) dt + \mathbf{D}dW(t), \\ X(0) = \mathbf{x}_0, \end{cases} \tag{4}$$

and

$$J_i^{(N)}(u_i, u_{-i}) = \frac{1}{\alpha} \ln \mathbb{E} \left[e^{\alpha \left[\int_0^T (|X(t)|_{\mathbf{Q}_i}^2 + |u_i(t)|_R^2) dt + |X(T)|_{\mathbf{Q}_{if}}^2 \right]} \right], \quad 1 \leq i \leq N, \tag{5}$$

respectively.

Given $t \in [0, T]$, we set the deterministic initial condition $(X_1(t)^T, \dots, X_N(t)^T)^T = \mathbf{x} \in \mathbb{R}^{Nn}$ and define

$$J_i^{(N)}(t, \mathbf{x}; u_i, u_{-i}) = \frac{1}{\alpha} \ln \mathbb{E} \left[e^{\alpha \left[\int_t^T (|X(s)|_{\mathbf{Q}_i}^2 + |u_i(s)|_R^2) ds + |X(T)|_{\mathbf{Q}_{if}}^2 \right]} \right], \quad 1 \leq i \leq N.$$

Let the value function of player $\mathcal{A}_i$ be

$$W_i(t, \mathbf{x}) = e^{\alpha J_i^{(N)}(t, \mathbf{x}; u_i^*, u_{-i}^*)} = \mathbb{E} \left[e^{\alpha \left[\int_t^T (|X(s)|_{\mathbf{Q}_i}^2 + |u_i^*(s)|_R^2) ds + |X(T)|_{\mathbf{Q}_{if}}^2 \right]} \right], \quad 1 \leq i \leq N,$$

where (u_i^*, u_{-i}^*) is a set of centralized feedback Nash equilibrium strategies (if it exists). Taking $W_i(t, \mathbf{x})$ in this exponential form will enable us to apply dynamic programming.

By dynamic programming^[36], we have

$$\begin{aligned}
 -\partial_t W_i(t, \mathbf{x}) &= \inf_{u_i \in \mathbb{R}^{n_1}} \left[\partial_{\mathbf{x}}^T W_i \left(\mathbf{A}\mathbf{x} + \mathbf{B}_i u_i + \sum_{j \neq i}^N \mathbf{B}_j u_j^* \right) + \alpha W_i(t, \mathbf{x}) (|\mathbf{x}|_{\mathbf{Q}_i}^2 + |u_i|_R^2) \right] \\
 &\quad + \frac{1}{2} \text{Tr} (\partial_{\mathbf{x}\mathbf{x}}^2 W_i \mathbf{D} \mathbf{D}^T), \quad 1 \leq i \leq N.
 \end{aligned} \tag{6}$$

Denoting $V_i(t, \mathbf{x}) = \frac{1}{\alpha} \ln W_i(t, \mathbf{x})$, we use (6) to derive

$$\begin{cases} -\partial_t V_i(t, \mathbf{x}) = \partial_{\mathbf{x}}^T V_i \left(\mathbf{A}\mathbf{x} - \frac{1}{2} \sum_{j=1}^N \mathbf{B}_j R^{-1} \mathbf{B}_j^T \partial_{\mathbf{x}} V_j \right) + \frac{1}{2} \text{Tr} (\partial_{\mathbf{x}\mathbf{x}}^2 V_i \mathbf{D} \mathbf{D}^T) \\ \quad + \frac{\alpha}{2} |\mathbf{D}^T \partial_{\mathbf{x}} V_i|^2 + |\mathbf{x}|_{\mathbf{Q}_i}^2 + \frac{1}{4} \partial_{\mathbf{x}}^T V_i \mathbf{B}_i R^{-1} \mathbf{B}_i^T \partial_{\mathbf{x}} V_i, \\ V_i(T, \mathbf{x}) = |\mathbf{x}|_{\mathbf{Q}_{if}}^2, \quad 1 \leq i \leq N. \end{cases} \quad (7)$$

And the equilibrium strategy of player $\mathcal{A}_i$ is given by

$$u_i^* = -\frac{1}{2} R^{-1} \mathbf{B}_i^T \partial_{\mathbf{x}} V_i, \quad 1 \leq i \leq N. \quad (8)$$

For $1 \leq i \leq N$, taking the ansatz $V_i(t, \mathbf{x}) = \mathbf{x}^T \mathbf{P}_i \mathbf{x} + \mathbf{r}_i$, and plugging it into (7), we have the ordinary differential equations (ODEs) for $\mathbf{P}_i(t)$ and $\mathbf{r}_i(t)$:

$$\begin{cases} \dot{\mathbf{P}}_i + \mathbf{A}^T \mathbf{P}_i + \mathbf{P}_i \mathbf{A} + \mathbf{P}_i (\mathbf{B}_i R^{-1} \mathbf{B}_i^T + 2\alpha \mathbf{D} \mathbf{D}^T) \mathbf{P}_i + \mathbf{Q}_i \\ \quad - \mathbf{P}_i \sum_{j=1}^N \mathbf{B}_j R^{-1} \mathbf{B}_j^T \mathbf{P}_j - \sum_{j=1}^N \mathbf{P}_j \mathbf{B}_j R^{-1} \mathbf{B}_j^T \mathbf{P}_i = 0, \\ \mathbf{P}_i(T) = \mathbf{Q}_{if}, \end{cases} \quad (9)$$

$$\begin{cases} \dot{\mathbf{r}}_i + \text{Tr} (\mathbf{P}_i \mathbf{D} \mathbf{D}^T) = 0, \\ \mathbf{r}_i(T) = 0. \end{cases} \quad (10)$$

We have the following theorem.

Theorem 3.1 *If the coupled Riccati equation system (9) has a solution on $[0, T]$, the set of centralized feedback Nash equilibrium strategies $(u_1^*, \dots, u_N^*)$ of Problem (CFNE) exists and is given by*

$$u_i^*(t) = -R^{-1} \mathbf{B}_i^T \mathbf{P}_i(t) X(t), \quad 1 \leq i \leq N, \quad (11)$$

where the state $X(t)$ satisfies

$$\begin{cases} dX(t) = \left(\mathbf{A}X(t) - \sum_{i=1}^N \mathbf{B}_i R^{-1} \mathbf{B}_i^T \mathbf{P}_i(t) X(t) \right) dt + \mathbf{D} dW(t), \\ X(0) = \mathbf{x}_0. \end{cases} \quad (12)$$

Here the initial condition $\mathbf{x}_0 \in \mathbb{R}^{Nn}$ is specified by scenario (I), or by scenario (II) provided that $\mathbb{E}[e^{\alpha V_1(t, \mathbf{x})}] < \infty$.

Notice that the centralized feedback Nash strategies (11) are characterized by the coupled Riccati equation system (9). If (9) has a solution $(\mathbf{P}_1, \dots, \mathbf{P}_N)$ on $[0, T]$, such a solution is unique due to local Lipschitz continuity of the vector field in (9). Subsequently, we obtain a unique solution for the equation (10).

Now we focus on analyzing Riccati equation system (9). Following the route in [18], we introduce the following definition.

Definition 3.2 The sequence of risk-sensitive LQ Nash games (1)–(2) with CLPS information has asymptotic solvability if there exists N^* such that for all $N \geq N^*$, $(P_1, \dots, P_N)$ in (9) has a solution on $[0, T]$ and

$$\sup_{N \geq N^*} \sup_{1 \leq i \leq N, 0 \leq t \leq T} \|P_i(t)\|_{l_1} < \infty.$$

To study the asymptotic solvability of risk-sensitive LQ Nash games specified by (1) and (2), we introduce the following Riccati ODE system and assumptions.

$$\begin{cases} \dot{A}_1 + A_1 A + A^T A_1 - A_1 M A_1 + Q = 0, \\ A_1(T) = Q_f, \end{cases} \quad (13)$$

$$\begin{cases} \dot{A}_2 + A_1 G + A^T A_2 + A_2 (A + G) - A_2 M_0 A_1 \\ - A_1 M_0 A_2 + A_1 N_0 A_2 - A_2 M_0 A_2 - Q_f \Gamma = 0, \\ A_2(T) = -Q_f \Gamma_f, \end{cases} \quad (14)$$

$$\begin{cases} \dot{A}_3 + A_2^T G + G^T A_2 + A_3 (A + G - M_0 A_1 - M_0 A_2) \\ + (A + G - M_0 A_1 - M_0 A_2)^T A_3 - A_2^T M A_2 + \Gamma^T Q \Gamma = 0, \\ A_3(T) = \Gamma_f^T Q_f \Gamma_f. \end{cases} \quad (15)$$

Assumption 3.3 For the given α , $BR^{-1}B^T - 2\alpha DD^T \geq 0$.

Assumption 3.4 Riccati equation (14) has a solution on $[0, T]$ (which is then unique).

Remark 3.5 Under Assumption 3.3, the equation (13) is a standard Riccati equation and so has a unique solution on $[0, T]$. After determining A_1 and A_2 under Assumptions 3.3–3.4, (15) becomes a linear ODE, giving a unique solution.

Lemma 3.6 Under Assumptions 3.3–3.4, we have the following assertions.

(i) If the Riccati equation system (9) has a solution on $[0, T]$, we have the representation of P_i , $1 \leq i \leq N$:

$$P_1 = \begin{pmatrix} \Pi_1^N & \Pi_2^N & \cdots & \Pi_2^N \\ \Pi_2^{NT} & \Pi_3^N & \cdots & \Pi_3^N \\ \vdots & \vdots & \ddots & \vdots \\ \Pi_2^{NT} & \Pi_3^N & \cdots & \Pi_3^N \end{pmatrix}, \quad P_j = J_{1j}^T P_1 J_{1j}, \quad 2 \leq j \leq N, \quad (16)$$

where each submatrix in the partition of P_1 is $\mathbb{R}^{n \times n}$ -valued.

(ii) For the solutions of (9) and of (13)–(15), we have

$$\sup_{0 \leq t \leq T} (|\Pi_1^N - A_1| + |N\Pi_2^N - A_2| + |N^2\Pi_3^N - A_3|) = O\left(\frac{1}{N}\right). \quad (17)$$

(iii) The sequence of risk-sensitive LQ Nash games in (1)–(2) has asymptotic solvability.

Proof (i) Using the proving method for Theorem 3 in [18] and Lemma 4.1 in [24], we obtain (16).

(ii) Let $\Lambda_1^N = \Pi_1^N$, $\Lambda_2^N = N\Pi_2^N$ and $\Lambda_3^N = N^2\Pi_3^N$. Combining (9) with (16), we have the following ODE system:

$$\begin{cases} \dot{\Lambda}_1^N + \Lambda_1^N A + A^T \Lambda_1^N - \Lambda_1^N M \Lambda_1^N + Q + g_1 = 0, \\ \Lambda_1^N(T) = \left| I - \frac{1}{N} \Gamma_f \right|_{Q_f}^2, \\ g_1 = \frac{1}{N} (G^T \Lambda_1^N + \Lambda_1^N G) + \left| I - \frac{1}{N} \Gamma \right|_Q^2 - Q + \frac{N-1}{N^2} (G^T \Lambda_2^{N^T} + \Lambda_2^N G \\ \quad + \Lambda_2^N N_0 \Lambda_2^{N^T} - \Lambda_2^N M_0 \Lambda_2^N - \Lambda_2^{N^T} M_0 \Lambda_2^{N^T}), \\ \dot{\Lambda}_2^N + \Lambda_1^N G + A^T \Lambda_2^N + \Lambda_2^N (A + G) - \Lambda_2^N M_0 \Lambda_1^N - \Lambda_1^N M_0 \Lambda_2^N \\ \quad + \Lambda_1^N N_0 \Lambda_2^N - \Lambda_2^N M_0 \Lambda_2^N - Q \Gamma + g_2 = 0, \\ \Lambda_2^N(T) = -Q_f \Gamma_f + \frac{1}{N} \Gamma_f^T Q_f \Gamma_f, \\ g_2 = \frac{1}{N} (G^T \Lambda_2^N - \Lambda_2^N G) + \frac{N-1}{N^2} G^T \Lambda_3^N + \frac{2}{N} \Lambda_2^N M_0 \Lambda_2^N \\ \quad - \frac{N-1}{N^2} (\Lambda_2^{N^T} M_0 - \Lambda_2^N N_0) \Lambda_3^N + \frac{1}{N} \Gamma^T Q \Gamma, \\ \dot{\Lambda}_3^N + \Lambda_2^{N^T} G + G^T \Lambda_2^N + \Lambda_3^N (A + G - M_0 \Lambda_1^N - M_0 \Lambda_2^N) - \Lambda_2^{N^T} M \Lambda_2^N \\ \quad + (A + G - M_0 \Lambda_1^N - M_0 \Lambda_2^N)^T \Lambda_3^N + \Gamma^T Q \Gamma + g_3 = 0, \\ \Lambda_3^N(T) = \Gamma_f^T Q_f \Gamma_f, \\ g_3 = \frac{1}{N} (2\Lambda_3^N M_0 \Lambda_2^N + 2\Lambda_2^{N^T} M_0 \Lambda_3^N - \Lambda_3^N G - G^T \Lambda_3^N) + \frac{N-1}{N^2} \Lambda_3^N N_0 \Lambda_3^N, \end{cases} \quad (18)$$

where g_1, g_2 and g_3 each have coefficients with order of magnitude $O(\frac{1}{N})$. Note that equations (13)–(15) are the limit of the ODE system (18) as $N \rightarrow \infty$. According to Theorem 4 in [18], there exists an N^* such that for all $N \geq N^*$, the ODE system (18) has a solution on $[0, T]$ and

$$\sup_{0 \leq t \leq T} (|\Lambda_1^N - A_1| + |\Lambda_2^N - A_2| + |\Lambda_3^N - A_3|) = O\left(\frac{1}{N}\right), \quad (19)$$

which implies (17).

(iii) In view of (17), we have

$$\sup_{N \geq N^*} \sup_{0 \leq t \leq T} (|\Pi_1^N| + |N\Pi_2^N| + |N^2\Pi_3^N|) < \infty,$$

which implies that the sequence of risk-sensitive LQ Nash games (1)–(2) has asymptotic solvability. ■

Remark 3.7 Lemma 3.6 (iii) implies that under Assumptions 3.3–3.4, (9) has a unique solution on $[0, T]$ for sufficiently large N . This fact facilitates our study of the asymptotic behaviour for the closed-loop system with N players as $N \rightarrow \infty$.

4 Decentralized Strategies and Their Costs

We construct the (limit) decentralized strategies based on the centralized feedback Nash equilibrium strategies (11) and Lemma 3.6. Taking $N \rightarrow \infty$, the limit decentralized strategy of player $\mathcal{A}_i$ is given by

$$\hat{u}_i(t) = -R^{-1}B^T \left(A_1(t)\hat{X}_i(t) + A_2(t)\bar{X}(t) \right), \quad 1 \leq i \leq N, \quad (20)$$

where $A_1(t)$ and $A_2(t)$ satisfy (13)–(14), and $\bar{X}(t)$ is regarded as the mean-field limit of the population average state $\hat{X}^{(N)}(t)$. Under the set of strategies (20), the N players have the closed-loop dynamics

$$\begin{cases} d\hat{X}_i(t) = \left[(A - M_0A_1(t))\hat{X}_i + G\hat{X}^{(N)}(t) - M_0A_2(t)\bar{X}(t) \right] dt \\ \quad + DdW_i(t), \quad 1 \leq i \leq N, \\ d\bar{X}(t) = (A + G - M_0A_1(t) - M_0A_2(t))\bar{X}(t)dt, \\ \hat{X}_i(0) = X_i(0), \quad \bar{X}(0) = \mu. \end{cases} \quad (21)$$

In order to analyze the asymptotic property of costs associated with the decentralized strategies (20), we introduce the following ODE system:

$$\begin{cases} \dot{\hat{A}}_1 + \hat{A}_1(A - M_0A_1) + (A - M_0A_1)^T\hat{A}_1 + A_1M_0A_1 + \hat{A}_1N_0\hat{A}_1 + Q = 0, \\ \hat{A}_1(T) = Q_f, \end{cases} \quad (22)$$

$$\begin{cases} \dot{\hat{A}}_2 + \hat{A}_1G + \hat{A}_2(A + G - M_0A_1) + (A - M_0A_1)^T\hat{A}_2 + \hat{A}_1N_0\hat{A}_2 - Q\Gamma = 0, \\ \hat{A}_2(T) = -Q_f\Gamma_f, \end{cases} \quad (23)$$

$$\begin{cases} \dot{\hat{A}}_3 + \hat{A}_2^TG + G^T\hat{A}_2 + \hat{A}_3(A + G - M_0A_1) \\ \quad + (A + G - M_0A_1)^T\hat{A}_3 + \hat{A}_2^TN_0\hat{A}_2 + \Gamma^TQ\Gamma = 0, \\ \hat{A}_3(T) = \Gamma_f^TQ_f\Gamma_f, \end{cases} \quad (24)$$

$$\begin{cases} \dot{\hat{A}}_4 + \hat{A}_4(A + G - M_0A_1 - M_0A_2) + (A - M_0A_1)^T\hat{A}_4 \\ \quad - (\hat{A}_1 + \hat{A}_2)M_0A_2 + A_1M_0A_2 + \hat{A}_1N_0\hat{A}_4 = 0, \\ \hat{A}_4(T) = 0, \end{cases} \quad (25)$$

$$\begin{cases} \dot{\hat{A}}_5 + \hat{A}_5(A + G - M_0A_1 - M_0A_2) + G^T\hat{A}_4 - \hat{A}_2^TM_0A_2 \\ \quad - \hat{A}_3M_0A_2 + (A + G - M_0A_1)^T\hat{A}_5 + \hat{A}_2^TN_0\hat{A}_4 = 0, \\ \hat{A}_5(T) = 0, \end{cases} \quad (26)$$

$$\begin{cases} \dot{\hat{A}}_6 + \hat{A}_6(A + G - M_0A_1 - M_0A_2) - A_2^TM_0(\hat{A}_4 + \hat{A}_5) - (\hat{A}_4 + \hat{A}_5)^TM_0A_2 \\ \quad + (A + G - M_0A_1 - M_0A_2)^T\hat{A}_6 + A_2^TM_0A_2 + \hat{A}_4^TN_0\hat{A}_4 = 0, \\ \hat{A}_6(T) = 0, \end{cases} \quad (27)$$

$$\begin{cases} \dot{\hat{r}} + \text{Tr}(\hat{A}_1DD^T) = 0, \\ \hat{r}(T) = 0. \end{cases} \quad (28)$$

Remark 4.1 Under Assumptions 3.3–3.4, ODEs (13)–(15) have a unique solution on $[0, T]$. We can verify that Λ_1 obtained in (13) is a solution of (22). The uniqueness of the solution $\hat{\Lambda}_1$ in (22) is obvious. Hence we can solve linear ODEs (23)–(28) following the same order.

Let us introduce an additional assumption.

Assumption 4.2 For given α and Σ , we have $\alpha\hat{\Lambda}_1(0) - \frac{1}{2}\Sigma^{-1} < 0$.

Theorem 4.3 Suppose Assumptions 3.3–3.4 hold.

(i) If the initial states satisfy scenario (I), the corresponding cost functional is given by

$$J_i^{(N)}(\hat{u}_i, \hat{u}_{-i}) = \frac{1}{\alpha} \ln \mathbb{E} \left[e^{\alpha L(X_i(0), \mu)} \right] + O\left(\frac{1}{N}\right) + O(|\mu^{(N)} - \mu|), \quad 1 \leq i \leq N, \quad (29)$$

where $L(X_i(0), \mu)$ takes the form:

$$\begin{aligned} L(X_i(0), \mu) = & X_i(0)^T \hat{\Lambda}_1(0) X_i(0) + 2X_i(0)^T \left(\hat{\Lambda}_2(0) + \hat{\Lambda}_4(0) \right) \mu \\ & + \mu^T \left(\hat{\Lambda}_3(0) + \hat{\Lambda}_5(0)^T + \hat{\Lambda}_5(0) + \hat{\Lambda}_6(0) \right) \mu + \hat{r}(0), \end{aligned}$$

while $\hat{\Lambda}_k, 1 \leq k \leq 6$ and $\hat{r}$ are solutions of (22)–(27) and (28), respectively.

(ii) If the initial states satisfy scenario (II), and moreover, Assumption 4.2 holds, then we have

$$\begin{aligned} J_i^{(N)}(\hat{u}_i, \hat{u}_{-i}) = & -\frac{1}{2\alpha} \ln \left[\det \left(I_n - 2\alpha \Sigma \hat{\Lambda}_1(0) \right) \right] + \mu^T \left(\hat{\Lambda}_3(0) + \hat{\Lambda}_5(0)^T + \hat{\Lambda}_5(0) + \hat{\Lambda}_6(0) \right) \mu \\ & - \frac{\mu^T \Sigma^{-1} \mu}{2\alpha} - \frac{1}{2\alpha} \mu^T \left| \Sigma^{-1} + 2\alpha \left(\hat{\Lambda}_2(0) + \hat{\Lambda}_4(0) \right) \right|_{(2\alpha \hat{\Lambda}_1(0) - \Sigma^{-1})^{-1}}^2 \mu \\ & + \hat{r}(0) + O\left(\frac{1}{N}\right), \quad 1 \leq i \leq N. \end{aligned} \quad (30)$$

Proof Denote

$$\begin{aligned} \Theta &= R^{-1} B^T A_1, \quad \Theta_0 = R^{-1} B^T A_2, \quad \hat{\Theta}_i = \Theta \otimes e_i^{N^T}, \quad \mathbf{B} = \text{diag}[B, \dots, B], \\ \hat{\Theta}_0 &= \Theta_0 \otimes \mathbf{1}_N, \quad \hat{\Theta} = (\hat{\Theta}_1^T, \dots, \hat{\Theta}_N^T)^T, \quad \hat{X} = (\hat{X}_1^T, \dots, \hat{X}_N^T)^T. \end{aligned} \quad (31)$$

With the notation in (3) and (31), the limit decentralized strategies (20) and the corresponding dynamics (21) are rewritten as

$$\hat{u}_i(t) = -\hat{\Theta}_i \hat{X}(t) - \Theta_0 \bar{X}(t), \quad 1 \leq i \leq N \quad (32)$$

and

$$\begin{cases} d\hat{X}(t) = \left(\mathbf{A} \hat{X}(t) - \mathbf{B} \hat{\Theta} \hat{X}(t) - \mathbf{B} \hat{\Theta}_0 \bar{X}(t) \right) dt + \mathbf{D} dW(t), \\ d\bar{X}(t) = (A + G - B\Theta - B\Theta_0) \bar{X}(t) dt, \\ \hat{X}(0) = \mathbf{x}_0, \quad \bar{X}(0) = \mu, \end{cases} \quad (33)$$

respectively.

For given initial time $t \in [0, T]$, we consider (33) on $[t, T]$ and set the initial conditions $(\hat{X}_1^T(t), \dots, \hat{X}_N^T(t))^T = \hat{\mathbf{x}} \in \mathbb{R}^{Nn}$ and $\bar{X}(t) = \bar{x} \in \mathbb{R}^n$, and define

$$\widehat{W}_i(t, \hat{\mathbf{x}}, \bar{x}) = \mathbb{E} \left[e^{\alpha \left[\int_t^T (|\hat{X}(s)|_{Q_i}^2 + |\hat{u}_i(s)|_R^2) ds + |\hat{X}(T)|_{Q_{if}}^2 \right]} \right], \quad 1 \leq i \leq N. \quad (34)$$

According to Feynman-Kac formula, we have

$$\begin{aligned} -\partial_t \widehat{W}_i(t, \hat{\mathbf{x}}, \bar{x}) &= \partial_{\hat{\mathbf{x}}}^T \widehat{W}_i \left[\left(\mathbf{A} - \mathbf{B}\hat{\Theta} \right) \hat{\mathbf{x}} - \mathbf{B}\hat{\Theta}_0 \bar{x} \right] + \partial_{\bar{x}}^T \widehat{W}_i \left(\mathbf{A} + \mathbf{G} - \mathbf{B}\hat{\Theta} - \mathbf{B}\hat{\Theta}_0 \right) \bar{x} \\ &\quad + \frac{1}{2} \text{Tr} \left(\partial_{\hat{\mathbf{x}}\hat{\mathbf{x}}}^2 \widehat{W}_i \mathbf{D}\mathbf{D}^T \right) + \alpha \widehat{W}_i \left(|\hat{\mathbf{x}}|_{Q_i}^2 + |\hat{\Theta}_i \hat{\mathbf{x}} + \hat{\Theta}_0 \bar{x}|_R^2 \right), \quad 1 \leq i \leq N. \end{aligned} \quad (35)$$

For $1 \leq i \leq N$, setting $\widehat{V}_i(t, \hat{\mathbf{x}}, \bar{x}) = \frac{1}{\alpha} \ln \widehat{W}_i(t, \hat{\mathbf{x}}, \bar{x})$ and taking the ansatz $\widehat{V}_i(t, \hat{\mathbf{x}}, \bar{x}) = \hat{\mathbf{x}}^T \hat{\mathbf{P}}_{11}^i \hat{\mathbf{x}} + 2\hat{\mathbf{x}}^T \hat{\mathbf{P}}_{12}^i \bar{x} + \bar{x}^T \hat{\mathbf{P}}_{22}^i \bar{x} + \hat{\mathbf{r}}_i$, we use (35) to derive

$$\begin{cases} \dot{\hat{\mathbf{P}}}_{11}^i + \hat{\mathbf{P}}_{11}^i (\mathbf{A} - \mathbf{B}\hat{\Theta}) + (\mathbf{A} - \mathbf{B}\hat{\Theta})^T \hat{\mathbf{P}}_{11}^i + 2\alpha \hat{\mathbf{P}}_{11}^i \mathbf{D}\mathbf{D}^T \hat{\mathbf{P}}_{11}^i + \hat{\Theta}_i^T R \hat{\Theta}_i + \mathbf{Q}_i = 0, \\ \hat{\mathbf{P}}_{11}^i(T) = \mathbf{Q}_{if}, \end{cases} \quad (36)$$

$$\begin{cases} \dot{\hat{\mathbf{P}}}_{12}^i + \hat{\mathbf{P}}_{12}^i (\mathbf{A} + \mathbf{G} - \mathbf{B}\hat{\Theta} - \mathbf{B}\hat{\Theta}_0) + (\mathbf{A} - \mathbf{B}\hat{\Theta})^T \hat{\mathbf{P}}_{12}^i \\ \quad - \hat{\mathbf{P}}_{11}^i \mathbf{B}\hat{\Theta}_0 + 2\alpha \hat{\mathbf{P}}_{11}^i \mathbf{D}\mathbf{D}^T \hat{\mathbf{P}}_{12}^i + \hat{\Theta}_i^T R \hat{\Theta}_0 = 0, \\ \hat{\mathbf{P}}_{12}^i(T) = 0, \end{cases} \quad (37)$$

$$\begin{cases} \dot{\hat{\mathbf{P}}}_{22}^i + \hat{\mathbf{P}}_{22}^i (\mathbf{A} + \mathbf{G} - \mathbf{B}\hat{\Theta} - \mathbf{B}\hat{\Theta}_0) + (\mathbf{A} + \mathbf{G} - \mathbf{B}\hat{\Theta} - \mathbf{B}\hat{\Theta}_0)^T \hat{\mathbf{P}}_{22}^i \\ \quad - \hat{\mathbf{P}}_{12}^{iT} \mathbf{B}\hat{\Theta}_0 - \hat{\Theta}_0^T \mathbf{B}^T \hat{\mathbf{P}}_{12}^i + 2\alpha \hat{\mathbf{P}}_{12}^{iT} \mathbf{D}\mathbf{D}^T \hat{\mathbf{P}}_{12}^i + \hat{\Theta}_0^T R \hat{\Theta}_0 = 0, \\ \hat{\mathbf{P}}_{22}^i(T) = 0, \end{cases} \quad (38)$$

$$\begin{cases} \dot{\hat{\mathbf{r}}}_i + \text{Tr} \left(\hat{\mathbf{P}}_{11}^i \mathbf{D}\mathbf{D}^T \right) = 0, \\ \hat{\mathbf{r}}_i(T) = 0. \end{cases} \quad (39)$$

In analogue to Lemma 3.6, and according to Remark 3.7 and Lemma A.1, we establish the following assertions under Assumptions 3.3–3.4.

(a) For all sufficiently large N , the ODEs (36)–(39) have a unique solution on $[0, T]$, with the representation

$$\begin{aligned} \hat{\mathbf{P}}_{11}^1 &= \begin{pmatrix} \hat{\Pi}_1^N & \hat{\Pi}_2^N & \cdots & \hat{\Pi}_2^N \\ \hat{\Pi}_2^{NT} & \hat{\Pi}_3^N & \cdots & \hat{\Pi}_3^N \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\Pi}_2^{NT} & \hat{\Pi}_3^N & \cdots & \hat{\Pi}_3^N \end{pmatrix}, \quad \hat{\mathbf{P}}_{12}^1 = \begin{pmatrix} \hat{\Pi}_4^N \\ \hat{\Pi}_5^N \\ \vdots \\ \hat{\Pi}_5^N \end{pmatrix}, \quad \hat{\mathbf{P}}_{22}^1 = \hat{\Pi}_6^N, \\ \hat{\mathbf{P}}_{11}^j &= J_{1j}^T \hat{\mathbf{P}}_{11}^1 J_{1j}, \quad \hat{\mathbf{P}}_{12}^j = J_{1j}^T \hat{\mathbf{P}}_{12}^1, \quad \hat{\mathbf{P}}_{22}^j = \hat{\Pi}_6^N, \quad 2 \leq j \leq N. \end{aligned} \quad (40)$$

Setting $\hat{\mathbf{r}}_i = \hat{\mathbf{r}}$ for all $1 \leq i \leq N$, the equation (39) can be rewritten as

$$\begin{cases} \dot{\hat{\mathbf{r}}} + \text{Tr} \left(\hat{\Pi}_1^N \mathbf{D}\mathbf{D}^T \right) + (N-1) \text{Tr} \left(\hat{\Pi}_3^N \mathbf{D}\mathbf{D}^T \right) = 0, \\ \hat{\mathbf{r}}(T) = 0. \end{cases} \quad (41)$$

(b) For the solutions of (22)–(28), (36)–(38) and (41), we have

$$\sup_{0 \leq t \leq T} \left(|\widehat{\Pi}_1^N - \widehat{\Lambda}_1| + |N\widehat{\Pi}_2^N - \widehat{\Lambda}_2| + |N^2\widehat{\Pi}_3^N - \widehat{\Lambda}_3| \right. \\ \left. + |\widehat{\Pi}_4^N - \widehat{\Lambda}_4| + |N\widehat{\Pi}_5^N - \widehat{\Lambda}_5| + |\widehat{\Pi}_6^N - \widehat{\Lambda}_6| + |\widehat{r} - \widehat{r}| \right) = O\left(\frac{1}{N}\right). \quad (42)$$

If the initial states satisfy scenario (I), the corresponding cost of player $\mathcal{A}_i$ is given by

$$J_i^{(N)}(\widehat{u}_i, \widehat{u}_{-i}) = \frac{1}{\alpha} \ln \mathbb{E} \left[e^{\alpha(\mathbf{x}_0^T \widehat{\mathbf{P}}_{11}^i(0) \mathbf{x}_0 + 2\mathbf{x}_0^T \widehat{\mathbf{P}}_{12}^i(0) \mu + \mu^T \widehat{\mathbf{P}}_{22}^i(0) \mu + \widehat{r}(0))} \right], \quad 1 \leq i \leq N. \quad (43)$$

Denote

$$f(X_i(0), X_{-i}^{(N)}(0)) = e^{\alpha(X_i(0)^T \widehat{\Lambda}_1(0) X_i(0) + 2X_i(0)^T \widehat{\Lambda}_2(0) X_{-i}^{(N)}(0) + X_{-i}^{(N)}(0)^T \widehat{\Lambda}_3(0) X_{-i}^{(N)}(0))} \\ \times e^{\alpha(2X_i(0)^T \widehat{\Lambda}_4(0) \mu + 2X_{-i}^{(N)}(0)^T \widehat{\Lambda}_5(0) \mu + \mu^T \widehat{\Lambda}_6(0) \mu + \widehat{r}(0))},$$

where we have $X_{-i}^{(N)}(0) = \frac{1}{N} \sum_{j \neq i}^N X_j(0)$. In view of (40), (42) and compactness of the set S_X , there exists a deterministic constant C such that

$$|e^{\alpha(\mathbf{x}_0^T \widehat{\mathbf{P}}_{11}^i(0) \mathbf{x}_0 + 2\mathbf{x}_0^T \widehat{\mathbf{P}}_{12}^i(0) \mu + \mu^T \widehat{\mathbf{P}}_{22}^i(0) \mu + \widehat{r}(0))} - f(X_i(0), X_{-i}^{(N)}(0))| \leq \frac{C}{N}, \quad (44)$$

which implies that the cost (43) of player $\mathcal{A}_i$ reduces to

$$J_i^{(N)}(\widehat{u}_i, \widehat{u}_{-i}) = \frac{1}{\alpha} \ln \mathbb{E} \left[f(X_i(0), X_{-i}^{(N)}(0)) \right] + O\left(\frac{1}{N}\right), \quad 1 \leq i \leq N. \quad (45)$$

According to Lemma B.1, we have

$$|\mathbb{E}[f(X_i(0), X_{-i}^{(N)}(0))] - \mathbb{E}[f(X_i(0), \mu)]| = O\left(\frac{1}{N}\right) + O(|\mu^{(N)} - \mu|), \quad (46)$$

Combining (45) with (46), we obtain (29).

If the initial states satisfy scenario (II), we have

$$J_i^{(N)}(\widehat{u}_i, \widehat{u}_{-i}) = \frac{1}{\alpha} \ln \mathbb{E} \left[e^{\alpha L^{(N)}(X_i(0), X_{-i}^{(N)}(0))} \right], \quad 1 \leq i \leq N, \quad (47)$$

where $L^{(N)}(X_i(0), X_{-i}^{(N)}(0))$ is given by

$$L^{(N)}(X_i(0), X_{-i}^{(N)}(0)) = X_i(0)^T \widehat{\Pi}_1^N(0) X_i(0) + 2X_i(0)^T N\widehat{\Pi}_2^N(0) X_{-i}^{(N)}(0) \\ + X_{-i}^{(N)}(0)^T N^2 \widehat{\Pi}_3^N(0) X_{-i}^{(N)}(0) + 2X_i(0)^T \widehat{\Pi}_4^N(0) \mu \\ + 2X_{-i}^{(N)}(0)^T N\widehat{\Pi}_5^N(0) \mu + \mu^T \widehat{\Pi}_6^N(0) \mu + \widehat{r}(0).$$

Denoting

$$\mathcal{M}_1 = \alpha \begin{pmatrix} \widehat{\Pi}_1^N(0) & N\widehat{\Pi}_2^N(0) \\ N\widehat{\Pi}_2^N(0)^T & N^2 \widehat{\Pi}_3^N(0) \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \Sigma^{-1} & 0 \\ 0 & \frac{N^2}{N-1} \Sigma^{-1} \end{pmatrix}, \\ \mathcal{M}_2 = \alpha \begin{pmatrix} \widehat{\Pi}_4^N(0) \\ N\widehat{\Pi}_5^N(0) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \Sigma^{-1} \\ N\Sigma^{-1} \end{pmatrix}, \quad X_{i,N} = \begin{pmatrix} X_i(0) \\ X_{-i}^{(N)}(0) \end{pmatrix},$$

we obtain

$$e^{\alpha J_i^{(N)}(\hat{u}_i, \hat{u}_{-i})} = \frac{1}{(\sqrt{2\pi})^{2n} \det(\Sigma) \sqrt{\frac{N-1}{N^2}}} \left(\int_{\mathbb{R}^{2n}} e^{X_{i,N}^T \mathcal{M}_1 X_{i,N} + 2X_{i,N}^T \mathcal{M}_2 \mu} dX_{i,N} \right) e^{\mu^T (\alpha \hat{\Pi}_6^N(0) - \frac{N}{2} \Sigma^{-1}) \mu + \alpha \hat{r}(0)}. \quad (48)$$

Under Assumption 4.2, we have $\mathcal{M}_1 < 0$ for all large N . Combining (42) with (48), we have

$$\begin{aligned} e^{\alpha J_i^{(N)}(\hat{u}_i, \hat{u}_{-i})} &= \frac{1}{\sqrt{\det \left(I_n - 2\alpha \Sigma \hat{\Lambda}_1(0) \right) + O\left(\frac{1}{N}\right)}} \times e^{\alpha \mu^T (\hat{\Lambda}_3(0) + \hat{\Lambda}_5(0) + \hat{\Lambda}_5(0)^T + \hat{\Lambda}_6(0)) \mu} \times e^{\frac{-\mu^T \Sigma^{-1} \mu}{2}} \\ &\quad \times e^{-\frac{1}{2} \mu^T \left[\Sigma^{-1} + 2\alpha (\hat{\Lambda}_2(0) + \hat{\Lambda}_4(0)) \right]_{(2\alpha \hat{\Lambda}_1(0) - \Sigma^{-1})^{-1}}^2} \mu \times e^{\alpha \hat{r}(0) + O\left(\frac{1}{N}\right)}, \end{aligned} \quad (49)$$

which implies (30). ■

5 Decentralized ε -Nash Equilibrium

To show that the limit decentralized strategies (20) possess an ε -Nash equilibrium property, we introduce the risk-sensitive LQ control problem for the unilateral improvement of a single player. Due to the symmetry of the system dynamics and the costs, we only need to consider the risk-sensitive LQ optimal control problem of player $\mathcal{A}_1$. The optimal control problems of other players can be transformed into the optimal control problem of player $\mathcal{A}_1$ by rearrangement of individual states.

5.1 The Risk-Sensitive Optimal Control Problem of Player $\mathcal{A}_1$

Notice that the set of decentralized feedback strategies $(\hat{u}_1, \dots, \hat{u}_N)$ satisfies (32). For player $\mathcal{A}_1$, we fix the decentralized strategies $\hat{u}_j, j \geq 2$, of all other players, and consider the following dynamics

$$\begin{cases} d\tilde{X}(t) = \left(\mathbf{A}\tilde{X}(t) + \mathbf{B}_1 u_1(t) - \sum_{j=2}^N \mathbf{B}_j \hat{\Theta}_j \tilde{X}(t) - \sum_{j=2}^N \mathbf{B}_j \Theta_0 \bar{X}(t) \right) dt + \mathbf{D} dW(t), \\ d\bar{X}(t) = (\mathbf{A} + \mathbf{G} - \mathbf{B}\Theta - \mathbf{B}\Theta_0) \bar{X}(t) dt, \\ \tilde{X}(0) = \mathbf{x}_0, \quad \bar{X}(0) = \mu, \end{cases} \quad (50)$$

where the state $\tilde{X} = (\tilde{X}_1^T, \dots, \tilde{X}_N^T)^T \in \mathbb{R}^{Nn}$, the coefficients of (50) follow the notation in (3) and (31).

Problem (RSC₁) For the dynamics (50), find an optimal feedback control $\tilde{u}_1 \in \mathcal{U}_1^c$ to minimize

$$J_1^{(N)}(u_1, \hat{u}_{-1}) = \frac{1}{\alpha} \ln \mathbb{E} \left[e^{\alpha \left[\int_0^T (|\tilde{X}(t)|_{\tilde{\mathbf{Q}}_1}^2 + |u_1(t)|_R^2) dt + |\tilde{X}(T)|_{\tilde{\mathbf{Q}}_1^f}^2 \right]} \right]. \quad (51)$$

For given time $t \in [0, T]$, we set the initial conditions $\tilde{X}(t) = \tilde{x} \in \mathbb{R}^{Nn}$ and $\bar{X}(t) = \bar{x}$, and define

$$\widetilde{W}(t, \tilde{x}, \bar{x}) = \inf_{u_1 \in \mathcal{U}_1^c} \mathbb{E} \left[e^{\alpha \left[\int_t^T (|\tilde{X}(s)|_{\mathbf{Q}_1}^2 + |u_1(s)|_R^2) ds + |\tilde{X}(T)|_{\mathbf{Q}_{1f}}^2 \right]} \right]. \quad (52)$$

Again using dynamic programming^[36], we obtain

$$\begin{aligned} -\partial_t \widetilde{W}(t, \tilde{x}, \bar{x}) = & \inf_{u_1 \in \mathbb{R}^{n_1}} \left[\partial_{\tilde{x}}^T \widetilde{W} \left(\mathbf{A}\tilde{x} + \mathbf{B}_1 u_1 - \sum_{j=2}^N \mathbf{B}_j \hat{\Theta}_j \tilde{x} - \sum_{j=2}^N \mathbf{B}_j \Theta_0 \bar{x} \right) + \frac{1}{2} \text{Tr} \left(\partial_{\tilde{x}\tilde{x}}^2 \widetilde{W} \mathbf{D} \mathbf{D}^T \right) \right. \\ & \left. + \partial_{\bar{x}}^T \widetilde{W} (A + G - B\Theta - B\Theta_0) \bar{x} + \alpha \widetilde{W} (|\tilde{x}|_{\mathbf{Q}_1}^2 + |u_1|_R^2) \right]. \end{aligned} \quad (53)$$

By the transformation $\tilde{V}(t, \tilde{x}, \bar{x}) = \frac{1}{\alpha} \ln \widetilde{W}(t, \tilde{x}, \bar{x})$, we use (53) to derive

$$\left\{ \begin{aligned} -\partial_t \tilde{V}(t, \tilde{x}, \bar{x}) = & \inf_{u_1 \in \mathbb{R}^{n_1}} \left[\partial_{\tilde{x}}^T \tilde{V} \left(\mathbf{A}\tilde{x} + \mathbf{B}_1 u_1 - \sum_{j=2}^N \mathbf{B}_j \hat{\Theta}_j \tilde{x} - \sum_{j=2}^N \mathbf{B}_j \Theta_0 \bar{x} \right) \right. \\ & \left. + \partial_{\tilde{x}}^T \tilde{V} (A + G - B\Theta - B\Theta_0) \bar{x} + \frac{1}{2} \text{Tr} \left(\partial_{\tilde{x}\tilde{x}}^2 \tilde{V} \mathbf{D} \mathbf{D}^T \right) \right. \\ & \left. + \frac{\alpha}{2} |\mathbf{D}^T \partial_{\tilde{x}} \tilde{V}|^2 + |\tilde{x}|_{\mathbf{Q}_1}^2 + |u_1|_R^2 \right], \\ \tilde{V}(T, \tilde{x}, \bar{x}) = & |\tilde{x}|_{\mathbf{Q}_{1f}}^2. \end{aligned} \right. \quad (54)$$

Since $R > 0$, the optimal control of player $\mathcal{A}_1$ is given by

$$\tilde{u}_1 = -\frac{1}{2} R^{-1} \mathbf{B}_1^T \partial_{\tilde{x}} \tilde{V}. \quad (55)$$

Take the ansatz

$$\tilde{V}(t, \tilde{x}, \bar{x}) = \tilde{x}^T \tilde{\mathbf{P}}_{11}(t) \tilde{x} + 2\tilde{x}^T \tilde{\mathbf{P}}_{12}(t) \bar{x} + \bar{x}^T \tilde{\mathbf{P}}_{22}(t) \bar{x} + \tilde{r}(t), \quad (56)$$

where $\tilde{\mathbf{P}}_{11}(t)$, $\tilde{\mathbf{P}}_{12}(t)$, $\tilde{\mathbf{P}}_{22}(t)$ and $\tilde{r}(t)$ are deterministic functions to be determined.

Substituting (55) and (56) into (54), we obtain

$$\left\{ \begin{aligned} & \dot{\tilde{\mathbf{P}}}_{11} + \tilde{\mathbf{P}}_{11} \left(\mathbf{A} - \sum_{j=2}^N \mathbf{B}_j \hat{\Theta}_j \right) + \left(\mathbf{A} - \sum_{j=2}^N \mathbf{B}_j \hat{\Theta}_j \right)^T \tilde{\mathbf{P}}_{11} \\ & - \tilde{\mathbf{P}}_{11} (\mathbf{B}_1 R^{-1} \mathbf{B}_1^T - 2\alpha \mathbf{D} \mathbf{D}^T) \tilde{\mathbf{P}}_{11} + \mathbf{Q}_1 = 0, \\ & \tilde{\mathbf{P}}_{11}(T) = \mathbf{Q}_{1f}, \end{aligned} \right. \quad (57)$$

$$\left\{ \begin{aligned} & \dot{\tilde{\mathbf{P}}}_{12} + \tilde{\mathbf{P}}_{12} (A + G - B\Theta - B\Theta_0) - \tilde{\mathbf{P}}_{11} \left(\sum_{j=2}^N \mathbf{B}_j \Theta_0 \right) \\ & + \left(\mathbf{A} - \sum_{j=2}^N \mathbf{B}_j \hat{\Theta}_j \right)^T \tilde{\mathbf{P}}_{12} - \tilde{\mathbf{P}}_{11} (\mathbf{B}_1 R^{-1} \mathbf{B}_1^T - 2\alpha \mathbf{D} \mathbf{D}^T) \tilde{\mathbf{P}}_{12} = 0, \\ & \tilde{\mathbf{P}}_{12}(T) = 0, \end{aligned} \right. \quad (58)$$

$$\begin{cases} \dot{\tilde{\mathbf{P}}}_{22} + \tilde{\mathbf{P}}_{22}(A + G - B\Theta - B\Theta_0) - \tilde{\mathbf{P}}_{12}^T \left(\sum_{j=2}^N \mathbf{B}_j \Theta_0 \right) - \left(\sum_{j=2}^N \mathbf{B}_j \Theta_0 \right)^T \tilde{\mathbf{P}}_{12} \\ + (A + G - B\Theta - B\Theta_0)^T \tilde{\mathbf{P}}_{22} - \tilde{\mathbf{P}}_{12}^T (\mathbf{B}_1 R^{-1} \mathbf{B}_1^T - 2\alpha \mathbf{D} \mathbf{D}^T) \tilde{\mathbf{P}}_{12} = 0, \\ \tilde{\mathbf{P}}_{22}(T) = 0, \end{cases} \quad (59)$$

$$\begin{cases} \dot{\tilde{\mathbf{r}}} + \text{Tr}(\tilde{\mathbf{P}}_{11} \mathbf{D} \mathbf{D}^T) = 0, \\ \tilde{\mathbf{r}}(T) = 0. \end{cases} \quad (60)$$

The solvability of (57)–(60) will be given in Lemma 5.2. Based on the above process, the corresponding cost of player $\mathcal{A}_1$ can be rewritten as

$$J_1^{(N)}(\tilde{u}_1, \hat{u}_{-1}) = \frac{1}{\alpha} \ln \mathbb{E} \left[e^{\alpha(\mathbf{x}_0^T \tilde{\mathbf{P}}_{11}(0) \mathbf{x}_0 + 2\mathbf{x}_0^T \tilde{\mathbf{P}}_{12}(0) \mu + \mu^T \tilde{\mathbf{P}}_{22}(0) \mu + \tilde{\mathbf{r}}(0))} \right]. \quad (61)$$

5.2 Asymptotic Analysis

In order to characterize the asymptotic behaviour (as $N \rightarrow \infty$) of $\tilde{\mathbf{P}}_{11}$, $\tilde{\mathbf{P}}_{12}$ and $\tilde{\mathbf{P}}_{22}$, we introduce the following ODE system:

$$\begin{cases} \dot{\tilde{\Lambda}}_1 + \tilde{\Lambda}_1 A + A^T \tilde{\Lambda}_1 - \tilde{\Lambda}_1 M \tilde{\Lambda}_1 + Q = 0, \\ \tilde{\Lambda}_1(T) = Q_f, \end{cases} \quad (62)$$

$$\begin{cases} \dot{\tilde{\Lambda}}_2 + \tilde{\Lambda}_1 G + \tilde{\Lambda}_2(A + G - M_0 \Lambda_1) + A^T \tilde{\Lambda}_2 - \tilde{\Lambda}_1 M \tilde{\Lambda}_2 - Q \Gamma = 0, \\ \tilde{\Lambda}_2(T) = -Q_f \Gamma_f, \end{cases} \quad (63)$$

$$\begin{cases} \dot{\tilde{\Lambda}}_3 + \tilde{\Lambda}_2^T G + G^T \tilde{\Lambda}_2 + \tilde{\Lambda}_3(A + G - M_0 \Lambda_1) + \Gamma^T Q \Gamma \\ + (A + G - M_0 \Lambda_1)^T \tilde{\Lambda}_3 - \tilde{\Lambda}_2^T M \tilde{\Lambda}_2 = 0, \\ \tilde{\Lambda}_3(T) = \Gamma_f^T Q_f \Gamma_f, \end{cases} \quad (64)$$

$$\begin{cases} \dot{\tilde{\Lambda}}_4 + \tilde{\Lambda}_4(A + G - M_0 \Lambda_1 - M_0 \Lambda_2) + A^T \tilde{\Lambda}_4 - \tilde{\Lambda}_2 M_0 \Lambda_2 - \tilde{\Lambda}_1 M \tilde{\Lambda}_4 = 0, \\ \tilde{\Lambda}_4(T) = 0, \end{cases} \quad (65)$$

$$\begin{cases} \dot{\tilde{\Lambda}}_5 + \tilde{\Lambda}_5(A + G - M_0 \Lambda_1 - M_0 \Lambda_2) + G^T \tilde{\Lambda}_4 - \tilde{\Lambda}_3 M_0 \Lambda_2 \\ + (A + G - M_0 \Lambda_1)^T \tilde{\Lambda}_5 - \tilde{\Lambda}_2^T M \tilde{\Lambda}_4 = 0, \\ \tilde{\Lambda}_5(T) = 0, \end{cases} \quad (66)$$

$$\begin{cases} \dot{\tilde{\Lambda}}_6 + \tilde{\Lambda}_6(A + G - M_0 \Lambda_1 - M_0 \Lambda_2) - \tilde{\Lambda}_2^T M_0 \tilde{\Lambda}_5 - \tilde{\Lambda}_5^T M_0 \Lambda_2 \\ + (A + G - M_0 \Lambda_1 - M_0 \Lambda_2)^T \tilde{\Lambda}_6 - \tilde{\Lambda}_4^T M \tilde{\Lambda}_4 = 0, \\ \tilde{\Lambda}_6(T) = 0, \end{cases} \quad (67)$$

$$\begin{cases} \dot{\tilde{\mathbf{r}}} + \text{Tr}(\tilde{\Lambda}_1 \mathbf{D} \mathbf{D}^T) = 0, \\ \tilde{\mathbf{r}}(T) = 0. \end{cases} \quad (68)$$

Remark 5.1 Under Assumption 3.3, the equation (62) is a standard Riccati equation and so has a unique solution on $[0, T]$. In fact, we obtain $\tilde{\Lambda}_1 = \Lambda_1$, where the latter is given

by (13). If Assumption 3.4 holds, we can solve the ODEs (63)–(68) following the same order as linear equations.

In analogue to Lemma 3.6, and according to Lemma A.1, we obtain the following lemma.

Lemma 5.2 *Under Assumptions 3.3–3.4, we have the following assertions.*

(i) *For all sufficiently large N , the ODEs (57)–(60) have a solution on $[0, T]$. Moreover, $\tilde{P}_{11}$, $\tilde{P}_{12}$ and $\tilde{P}_{22}$ have the representation*

$$\tilde{P}_{11} = \begin{pmatrix} \tilde{\Pi}_1^N & \tilde{\Pi}_2^N & \cdots & \tilde{\Pi}_2^N \\ \tilde{\Pi}_2^{NT} & \tilde{\Pi}_3^N & \cdots & \tilde{\Pi}_3^N \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\Pi}_2^{NT} & \tilde{\Pi}_3^N & \cdots & \tilde{\Pi}_3^N \end{pmatrix}, \quad \tilde{P}_{12} = \begin{pmatrix} \tilde{\Pi}_4^N \\ \tilde{\Pi}_5^N \\ \vdots \\ \tilde{\Pi}_5^N \end{pmatrix}, \quad \tilde{P}_{22} = \tilde{\Pi}_6^N. \quad (69)$$

(ii) *For the solutions of (57)–(60) and of (62)–(68), we have*

$$\sup_{0 \leq t \leq T} \left(|\tilde{\Pi}_1^N - \tilde{\Lambda}_1| + |N\tilde{\Pi}_2^N - \tilde{\Lambda}_2| + |N^2\tilde{\Pi}_3^N - \tilde{\Lambda}_3| \right. \\ \left. + |\tilde{\Pi}_4^N - \tilde{\Lambda}_4| + |N\tilde{\Pi}_5^N - \tilde{\Lambda}_5| + |\tilde{\Pi}_6^N - \tilde{\Lambda}_6| + |\tilde{r} - \tilde{r}| \right) = O\left(\frac{1}{N}\right).$$

5.3 Optimal Cost of Player $\mathcal{A}_1$

Lemma 5.2 (i) implies that Problem (RSC₁) has a solution for all sufficiently large N . Here we consider such large N , and in view of (55) and (56), determine the optimal control law of player $\mathcal{A}_1$ as

$$\tilde{u}_1(t) = -R^{-1}B^T \left(\tilde{\Pi}_1^N(t)\tilde{X}_1(t) + \tilde{\Pi}_2^N(t) \sum_{j=2}^N \tilde{X}_j(t) + \tilde{\Pi}_4^N(t)\bar{X}(t) \right), \quad (70)$$

where $\tilde{X}_i(t)$, $1 \leq i \leq N$, and $\bar{X}(t)$ satisfy

$$\begin{cases} d\tilde{X}_1(t) = \left[\left(A - M_0\tilde{\Pi}_1^N(t) \right) \tilde{X}_1(t) + G\tilde{X}^{(N)}(t) - M_0\tilde{\Pi}_2^N(t) \sum_{j=2}^N \tilde{X}_j(t) \right. \\ \quad \left. - M_0\tilde{\Pi}_4^N(t)\bar{X}(t) \right] dt + DdW_1(t), \\ d\tilde{X}_j(t) = \left[\left(A - M_0A_1(t) \right) \tilde{X}_j(t) + G\tilde{X}^{(N)}(t) - M_0A_2(t)\bar{X}(t) \right] dt \\ \quad + DdW_j(t), \quad 2 \leq j \leq N, \\ d\bar{X}(t) = \left(A + G - M_0A_1(t) - M_0A_2(t) \right) \bar{X}(t) dt, \\ \tilde{X}_i(0) = X_i(0), \quad \bar{X}(0) = \mu. \end{cases} \quad (71)$$

Here the population average state $\tilde{X}^{(N)}(t) = \frac{1}{N} \sum_{i=1}^N \tilde{X}_i(t)$, and $A_1(t)$, $A_2(t)$, $\tilde{\Pi}_1^N(t)$, $\tilde{\Pi}_2^N(t)$ and $\tilde{\Pi}_4^N(t)$ satisfy the equations (13)–(14), (57)–(58) and (69), respectively.

Theorem 5.3 *Suppose Assumptions 3.3–3.4 hold.*

(i) *For initial states of scenario (I), the corresponding cost of player $\mathcal{A}_1$ is given by*

$$J_1^{(N)}(\tilde{u}_1, \hat{u}_{-1}) = \frac{1}{\alpha} \ln \mathbb{E} \left[e^{\alpha \tilde{L}(X_1(0), \mu)} \right] + O\left(\frac{1}{N}\right) + O(|\mu^{(N)} - \mu|), \quad (72)$$

where $\tilde{L}(X_1(0), \mu)$ is in the form of

$$\begin{aligned} \tilde{L}(X_1(0), \mu) = & X_1(0)^T \tilde{\Lambda}_1(0) X_1(0) + 2X_1(0)^T \left(\tilde{\Lambda}_2(0) + \tilde{\Lambda}_4(0) \right) \mu \\ & + \mu^T \left(\tilde{\Lambda}_3(0) + \tilde{\Lambda}_5(0)^T + \tilde{\Lambda}_5(0) + \tilde{\Lambda}_6(0) \right) \mu + \tilde{r}(0), \end{aligned}$$

while $\tilde{\Lambda}_j, 1 \leq j \leq 6$ and $\tilde{r}$ satisfy (62)–(67) and (68), respectively.

(ii) For initial states of scenario (II) while additionally, Assumption 4.2 holds, we have

$$\begin{aligned} & J_1^{(N)}(\tilde{u}_1, \hat{u}_{-1}) \\ = & -\frac{1}{2\alpha} \ln \left[\det \left(I_n - 2\alpha \Sigma \tilde{\Lambda}_1(0) \right) \right] + \mu^T \left(\tilde{\Lambda}_3(0) + \tilde{\Lambda}_5(0)^T + \tilde{\Lambda}_5(0) + \tilde{\Lambda}_6(0) \right) \mu - \frac{\mu^T \Sigma^{-1} \mu}{2\alpha} \\ & - \frac{1}{2\alpha} \mu^T \left| \Sigma^{-1} + 2\alpha \left(\tilde{\Lambda}_2(0) + \tilde{\Lambda}_4(0) \right) \right|_{(2\alpha \tilde{\Lambda}_1(0) - \Sigma^{-1})^{-1}}^2 \mu + \tilde{r}(0) + O\left(\frac{1}{N}\right). \end{aligned} \quad (73)$$

Proof The proof follows an approach similar to that of Theorem 4.3. ■

5.4 ε -Nash Equilibrium

We have the main theorem regarding the performance of the obtained decentralized strategies.

Theorem 5.4 Suppose Assumptions 3.3–3.4 hold.

(i) For initial states of scenario (I), the set of decentralized strategies $\hat{u}_i, 1 \leq i \leq N$, in (20) satisfies

$$|J_i^{(N)}(\hat{u}_i, \hat{u}_{-i}) - \inf_{u_i \in \mathcal{U}_i^c} J_i^{(N)}(u_i, \hat{u}_{-i})| = O\left(\frac{1}{N}\right) + O(|\mu^{(N)} - \mu|), \quad (74)$$

where $\mu^{(N)} = \frac{1}{N} \sum_{i=1}^N \mu_i \rightarrow \mu$, as $N \rightarrow \infty$.

(ii) For initial states of scenario (II) with the additional Assumption 4.2, the set of decentralized strategies $\hat{u}_i, 1 \leq i \leq N$, in (20) satisfies

$$|J_i^{(N)}(\hat{u}_i, \hat{u}_{-i}) - \inf_{u_i \in \mathcal{U}_i^c} J_i^{(N)}(u_i, \hat{u}_{-i})| = O\left(\frac{1}{N}\right). \quad (75)$$

The estimates in (74) and (75) imply the set of decentralized strategies $(\hat{u}_1, \dots, \hat{u}_N)$ is an asymptotic Nash equilibrium of Problem (DF ε -NE).

Proof According to the equation (14), equations (22)–(28) and equations (62)–(68), we can show

$$\begin{aligned} \hat{\Lambda}_1(t) &= \tilde{\Lambda}_1(t), \quad \hat{\Lambda}_2(t) = \tilde{\Lambda}_2(t), \quad \hat{\Lambda}_4(t) = \tilde{\Lambda}_4(t), \quad \hat{r}(t) = \tilde{r}(t), \\ \hat{\Lambda}_3(t) + \hat{\Lambda}_5(t) + \hat{\Lambda}_5(t)^T + \hat{\Lambda}_6(t) &= \tilde{\Lambda}_3(t) + \tilde{\Lambda}_5(t) + \tilde{\Lambda}_5(t)^T + \tilde{\Lambda}_6(t). \end{aligned} \quad (76)$$

If the initial states satisfy scenario (I), plugging (76) into (29) and (72), we have

$$|J_i^{(N)}(\widehat{u}_i, \widehat{u}_{-i}) - J_i^{(N)}(\widetilde{u}_i, \widehat{u}_{-i})| = O\left(\frac{1}{N}\right) + O(|\mu^{(N)} - \mu|),$$

which implies (74).

If the initial states satisfy scenario (II), the difference between the costs (30) and (73) is

$$\begin{aligned} & J_i^{(N)}(\widehat{u}_i, \widehat{u}_{-i}) - J_i^{(N)}(\widetilde{u}_i, \widehat{u}_{-i}) \\ &= \frac{-1}{2\alpha} \ln \frac{\det(I_n - 2\alpha \Sigma \widehat{\Lambda}_1(0))}{\det(I_n - 2\alpha \Sigma \widetilde{\Lambda}_1(0))} + \mu^T \left(\widehat{\Lambda}_3(0) + \widehat{\Lambda}_5(0)^T + \widehat{\Lambda}_5(0) + \widehat{\Lambda}_6(0) - \widetilde{\Lambda}_3(0) \right. \\ & \quad \left. - \widetilde{\Lambda}_5(0)^T - \widetilde{\Lambda}_5(0) - \widetilde{\Lambda}_6(0) \right) \mu - \frac{1}{2\alpha} \mu^T \left[\left| \Sigma^{-1} + 2\alpha \left(\widehat{\Lambda}_2(0) + \widehat{\Lambda}_4(0) \right) \right|_{(2\alpha \widehat{\Lambda}_1(0) - \Sigma^{-1})^{-1}}^2 \right. \\ & \quad \left. - \left| \Sigma^{-1} + 2\alpha \left(\widetilde{\Lambda}_2(0) + \widetilde{\Lambda}_4(0) \right) \right|_{(2\alpha \widetilde{\Lambda}_1(0) - \Sigma^{-1})^{-1}}^2 \right] \mu + \widehat{r}(0) - \widetilde{r}(0) + O\left(\frac{1}{N}\right). \end{aligned} \quad (77)$$

Substituting (76) into (77), we derive (75). ■

Corollary 5.5 *Suppose Assumptions 3.3–3.4 hold. Then for initial states of scenario (I) with $\mu_i = \mu$ for all i , and initial states of scenario (II) further subject to Assumption 4.2, we have*

$$|J_i^{(N)}(\widehat{u}_i, \widehat{u}_{-i}) - \inf_{u_i \in \mathcal{U}_i^c} J_i^{(N)}(u_i, \widehat{u}_{-i})| = O\left(\frac{1}{N}\right),$$

which implies that the set of decentralized strategies $(\widehat{u}_1, \dots, \widehat{u}_N)$ is an $O(1/N)$ -Nash equilibrium of Problem (DF ε -NE).

6 Numerical Examples

Note that Assumption 3.4 plays an important role in obtaining the asymptotic Nash equilibrium. We give two numerical examples to illustrate that (14) has a solution on $[0, T]$ under some conditions, but does not always have a solution.

In what follows, we analyze a scalar case of the Riccati ODEs (13)–(14), i.e., $n = 1$. Denote the constant $\Lambda_{1\infty} = A + \sqrt{A^2 + Q} > 0$ as the stabilizing solution of the algebraic Riccati equation

$$\Lambda_{1\infty}^2 - 2A\Lambda_{1\infty} - Q = 0.$$

Example 6.1 Take the parameters $B = G = D = Q = \Gamma = 1$, $\alpha = A = R = \frac{1}{2}$, $Q_f = \Lambda_{1\infty} = \frac{1}{2} + \frac{\sqrt{5}}{2}$ and $\Gamma_f = 0$. We have $\Lambda_1(t) = \Lambda_{1\infty}$ as the unique solution of (13) on $[0, T]$ for any $T > 0$. Then (14) can be rewritten as

$$\begin{cases} \dot{\Lambda}_2(t) + (2 - 3\Lambda_{1\infty})\Lambda_2(t) - 2\Lambda_2^2(t) + \Lambda_{1\infty} - 1 = 0, \\ \Lambda_2(T) = 0. \end{cases} \quad (78)$$

Note that $\Lambda_{1\infty} - 1 = \frac{\sqrt{5}-1}{2} > 0$, so (78) is a standard Riccati equation and has a unique solution.

Example 6.2 Take $T = \frac{\sqrt{2}\pi}{4}$, $A = 1$, $G = 1 + 3\sqrt{2}$, $\Gamma = 8 + 4\sqrt{2}$ and $Q_f = A_{1\infty} = 1 + \sqrt{2}$. All other parameters are the same as in Example 6.1. We also have $A_1(t) = A_{1\infty}$ as the unique solution of (13) on $[0, T]$, and now (14) reduces to

$$\begin{cases} \dot{A}_2(t) - 2A_2^2(t) - 1 = 0, \\ A_2(T) = 0. \end{cases} \quad (79)$$

Note that $A_2(t) = \frac{\sqrt{2}}{2} \tan(\sqrt{2}t - \frac{\pi}{2})$ is a solution of (79) on $(0, T]$, but (79) does not have a global solution on $[0, T]$.

Example 6.3 Take $T = 3/2$, $\Gamma = 3/2$ and $Q_f = 0$. All other parameters are the same as in Example 6.1. Equations (13) and (14) reduce to

$$\begin{cases} \dot{A}_1(t) + A_1(t) - A_1^2(t) + 1 = 0, \\ A_1(T) = 0, \end{cases} \quad (80)$$

$$\begin{cases} \dot{A}_2(t) + (2 - 3A_1(t))A_2(t) - 2A_2^2(t) + A_1(t) - \frac{3}{2} = 0, \\ A_2(T) = 0. \end{cases} \quad (81)$$

Note that (80) is a standard Riccati equation and so has a unique solution on $[0, T]$.

We plot the numerical solution of $A_1(t)$ and $A_2(t)$ in Figure 1. Figure 1 (a) displays the solution curve of $A_1(t)$ on $[0, T]$. Figure 1 (b) shows that $A_2(t)$ tends to negative infinity as t approaches around 0.58, which implies that $A_2(t)$ does not have a global solution on $[0, T]$.

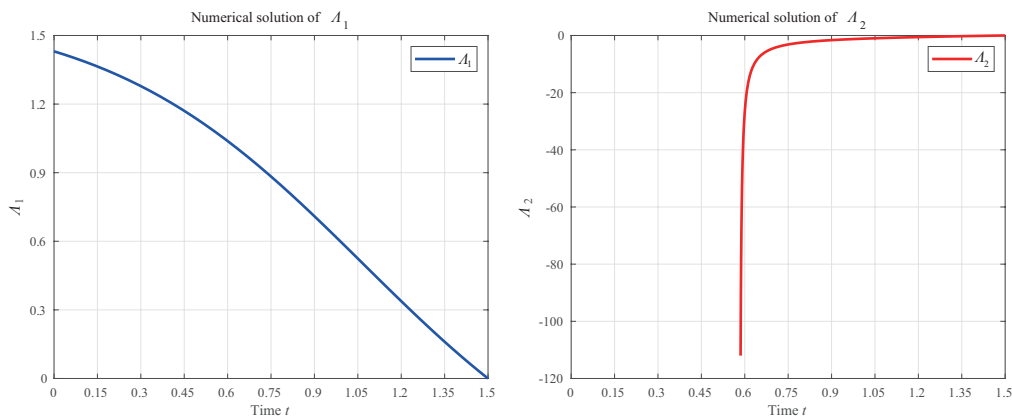


Figure 1 Numerical solutions of A_1 and A_2 in Example 6.3

7 Conclusion

This paper studies risk-sensitive LQ mean-field games. We derive the centralized strategies and subsequently construct decentralized strategies by applying the direct approach in mean-field game theory and taking the mean-field limit. Using the value function method, we obtain tight estimates of the costs and establish an $O(1/N)$ -Nash equilibrium for the decentralized strategies.

Conflict of Interest

HUANG Minyi is an editorial board member for Journal of Systems Science & Complexity and was not involved in the editorial review or the decision to publish this article. All authors declare that there are no competing interests.

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Appendix A

We give the following lemma, which plays an important role in proving Theorem 4.3 and Lemma 5.2. The proof is similar to that of Theorem 3 in [18].

Lemma A.1 For $1 \leq i \leq N$ and $t \in [0, T]$, let $(\mathbf{K}_i, \mathbf{L}_i)$ be given by following ODEs:

$$\begin{cases} \dot{\mathbf{K}}_i(t) + \mathbf{A}_i(t)^T \mathbf{K}_i(t) + \mathbf{K}_i(t) \mathbf{A}_i(t) - \mathbf{K}_i(t) \mathbf{M}_i(t) \mathbf{K}_i(t) + \mathbf{H}_i(t) = 0, \\ \mathbf{K}_i(T) = \mathbf{H}_{if}, \end{cases} \quad (\text{A.1})$$

$$\begin{cases} \dot{\mathbf{L}}_i(t) + \mathbf{L}_i(t) \mathbf{C}(t) + \mathbf{F}_i(t) \mathbf{L}_i(t) + \mathbf{S}_i(t) = 0, \\ \mathbf{L}_i(T) = \mathbf{S}_{if}, \end{cases} \quad (\text{A.2})$$

where the deterministic bounded functions $\mathbf{A}_i(t) \in \mathbb{R}^{Nn \times Nn}$, $\mathbf{F}_i(t) \in \mathbb{R}^{Nn \times Nn}$, $\mathbf{S}_i(t) \in \mathbb{R}^{Nn}$, $\mathbf{C}(t) \in \mathbb{R}$, the deterministic bounded symmetric functions $\mathbf{M}_i(t) \in \mathbb{S}^{Nn}$, $\mathbf{H}_i(t) \in \mathbb{S}^{Nn}$, and the constant matrices $\mathbf{H}_{if} \in \mathbb{S}^{Nn}$ and $\mathbf{S}_{if} \in \mathbb{R}^{Nn}$ take the form (to save space, we use lower case letters to denote submatrices and subvectors in the partition below):

$$\begin{aligned} \mathbf{A}_1 &= \begin{pmatrix} a_1 & a_2 & a_2 & \cdots & a_2 \\ a_5 & a_3 & a_4 & \cdots & a_4 \\ a_5 & a_4 & a_3 & \cdots & a_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_5 & a_4 & a_4 & \cdots & a_3 \end{pmatrix}, \quad \mathbf{M}_1 = \begin{pmatrix} m_1 & m_2 & m_2 & \cdots & m_2 \\ m_2^T & m_3 & m_4 & \cdots & m_4 \\ m_2^T & m_4 & m_3 & \cdots & m_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_2^T & m_4 & m_4 & \cdots & m_3 \end{pmatrix}, \\ \mathbf{H}_1 &= \begin{pmatrix} h_1 & h_2 & h_2 & \cdots & h_2 \\ h_2^T & h_3 & h_3 & \cdots & h_3 \\ h_2^T & h_3 & h_3 & \cdots & h_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_2^T & h_3 & h_3 & \cdots & h_3 \end{pmatrix}, \quad \mathbf{F}_1 = \begin{pmatrix} f_1 & f_2 & f_2 & \cdots & f_2 \\ f_5 & f_3 & f_4 & \cdots & f_4 \\ f_5 & f_4 & f_3 & \cdots & f_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_5 & f_4 & f_4 & \cdots & f_3 \end{pmatrix}, \\ \mathbf{H}_{1f} &= \begin{pmatrix} h_{1f} & h_{2f} & h_{2f} & \cdots & h_{2f} \\ h_{2f}^T & h_{3f} & h_{3f} & \cdots & h_{3f} \\ h_{2f}^T & h_{3f} & h_{3f} & \cdots & h_{3f} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{2f}^T & h_{3f} & h_{3f} & \cdots & h_{3f} \end{pmatrix}, \quad \mathbf{S}_1 = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_2 \end{pmatrix}, \quad \mathbf{S}_{1f} = \begin{pmatrix} s_{1f} \\ s_{2f} \\ \vdots \\ s_{2f} \end{pmatrix}, \\ \Phi_j &= J_{1j}^T \Phi_1 J_{1j}, \quad \mathbf{S}_j = J_{1j}^T \mathbf{S}_1, \quad \mathbf{S}_{jf} = J_{1j}^T \mathbf{S}_{1f}, \\ \Phi_j &= \mathbf{A}_j, \mathbf{M}_j, \mathbf{H}_j, \mathbf{H}_{jf}, \mathbf{F}_j, \quad 2 \leq j \leq N. \end{aligned} \quad (\text{A.3})$$

(Note that m_4 within $\mathbf{M}_1$ is symmetric.)

If Riccati equation (A.1) has a solution on $[0, T]$ for all i , then $\mathbf{K}_i(t) \in \mathbb{S}^{Nn}$ and $\mathbf{L}_i(t) \in \mathbb{R}^{Nn}$ have the representation:

$$\mathbf{K}_1(t) = \begin{pmatrix} k_1(t) & k_2(t) & \cdots & k_2(t) \\ k_2(t)^T & k_3(t) & \cdots & k_3(t) \\ \vdots & \vdots & \ddots & \vdots \\ k_2(t)^T & k_3(t) & \cdots & k_3(t) \end{pmatrix}, \quad \mathbf{L}_1(t) = \begin{pmatrix} l_1(t) \\ l_2(t) \\ \vdots \\ l_2(t) \end{pmatrix},$$

$$\mathbf{K}_j(t) = J_{1j}^T \mathbf{K}_1(t) J_{1j}, \quad \mathbf{L}_j(t) = J_{1j}^T \mathbf{L}_1(t), \quad 2 \leq j \leq N. \quad (\text{A.4})$$

Proof Note that the coefficients of the ODE systems (A.1)–(A.2) are deterministic bounded functions, so the linear ODE (A.2) has a unique solution on $[0, T]$. If the equation (A.1) has a solution on $[0, T]$, such a solution is unique. Taking transpose on both sides of (A.1) and comparing the resulting ODE with (A.1), we get $\mathbf{K}_i(t) = \mathbf{K}_i(t)^T$, $1 \leq i \leq N$.

For $i = 1$ and $2 \leq j \leq N$, left-multiplying both sides of equation (A.1) by J_{1j}^T , right-multiplying by J_{1j} , we have

$$\begin{cases} J_{1j}^T \dot{\mathbf{K}}_1(t) J_{1j} + J_{1j}^T \mathbf{K}_1(t) J_{1j} J_{1j}^T \mathbf{A}_1(t) J_{1j} + J_{1j}^T \mathbf{A}_1(t)^T J_{1j} J_{1j}^T \mathbf{K}_1(t) J_{1j} \\ - J_{1j}^T \mathbf{K}_1(t) J_{1j} J_{1j}^T \mathbf{M}_1(t) J_{1j} J_{1j}^T \mathbf{K}_1(t) J_{1j} + J_{1j}^T \mathbf{H}_1(t) J_{1j} = 0, \\ J_{1j}^T \mathbf{K}_1(T) = J_{1j}^T \mathbf{H}_{1f} J_{1j}. \end{cases} \quad (\text{A.5})$$

Plugging (A.3) into (A.5) and comparing the coefficients of the equations (A.1) and (A.5), we have

$$\mathbf{K}_j(t) = J_{1j}^T \mathbf{K}_1(t) J_{1j}, \quad 2 \leq j \leq N. \quad (\text{A.6})$$

Next we focus on analyzing the structure of $\mathbf{K}_1(t) \in \mathbb{S}^{Nn}$. It is straightforward to show

$$J_{sk}^T \Phi_1 J_{sk} = \Phi_1, \quad J_{sk}^T \mathbf{S}_1 = \mathbf{S}_1, \quad J_{sk}^T \mathbf{S}_{1f} = \mathbf{S}_{1f},$$

for $\Phi_1 = \mathbf{A}_1, \mathbf{M}_1, \mathbf{H}_1, \mathbf{H}_{1f}, \mathbf{F}_1$, $2 \leq s \neq k \leq N$.

For $i = 1$, left-multiplying both sides of the equation (A.1) by J_{sk}^T , right-multiplying by J_{sk} , and defining $J_{sk}^T \mathbf{K}_1(t) J_{sk} = \mathbf{K}_{sk}^\dagger(t)$, we have

$$\begin{cases} \dot{\mathbf{K}}_{sk}^\dagger(t) + \mathbf{K}_{sk}^\dagger(t) \mathbf{A}_1(t) + \mathbf{A}_1(t)^T \mathbf{K}_{sk}^\dagger(t) - \mathbf{K}_{sk}^\dagger(t) \mathbf{M}_1(t) \mathbf{K}_{sk}^\dagger(t) + \mathbf{H}_1(t) = 0, \\ \mathbf{K}_{sk}^\dagger(T) = \mathbf{H}_{1f}, \quad 2 \leq s \neq k \leq N. \end{cases} \quad (\text{A.7})$$

Comparing (A.1) with (A.7), we have $\mathbf{K}_1(t) = J_{sk}^T \mathbf{K}_1(t) J_{sk}$, $2 \leq s \neq k \leq N$. Combining it with the symmetry of $\mathbf{K}_1(t)$, we see that the representation of $\mathbf{K}_1(t)$ is given by

$$\mathbf{K}_1(t) = \begin{pmatrix} k_1^o(t) & k_2^o(t) & k_2^o(t) & \cdots & k_2^o(t) \\ k_2^o(t)^T & k_3^o(t) & k_4^o(t) & \cdots & k_4^o(t) \\ k_2^o(t)^T & k_4^o(t) & k_3^o(t) & \cdots & k_4^o(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_2^o(t)^T & k_4^o(t) & k_4^o(t) & \cdots & k_3^o(t) \end{pmatrix}.$$

Setting $k_{34}^o(t) = k_3^o(t) - k_4^o(t)$, we have

$$\begin{cases} \dot{k}_{34}^o(t) + k_{34}^o(t)(a_3(t) - a_4(t)) + (a_3(t) - a_4(t))^T k_{34}^o(t) - k_{34}^o(t)(m_3(t) - m_4(t)) k_{34}^o(t) = 0, \\ k_{34}^o(T) = 0, \end{cases}$$

which implies $k_3^o(t) = k_4^o(t)$. Then the representation of $\mathbf{K}_1(t)$ is in the form of (A.4) by defining $k_1^o(t) = k_1(t)$, $k_2^o(t) = k_2(t)$, $k_3^o(t) = k_4^o(t) = k_3(t)$. The representation of $\mathbf{L}_i(t)$, $1 \leq i \leq N$, can be derived by a similar procedure. $\blacksquare$

Remark A.2 Since a_2, a_4 and a_5 are not the same, we can not obtain the representation of K_i , $1 \leq i \leq N$ directly by using Theorem 3 in [4] or Lemma 4.1 in [10]. Compared to Theorem 3 in [4] or Lemma 4.1 in [10], we introduce an additional inhomogeneous term (A.2).

Appendix B

Lemma B.1 Suppose that the function $f(x, y) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ has up to the second order partial derivatives with respect to y , which are continuous in (x, y) . If the initial states $X_i(0) \in \mathbb{R}^n$, $1 \leq i \leq N$, satisfy Scenario (I), we have

$$|\mathbb{E}[f(X_i(0), X_{-i}^{(N)}(0))] - \mathbb{E}[f(X_i(0), \mu)]| = O(|\mu^{(N)} - \mu|) + O\left(\frac{1}{N}\right), \quad 1 \leq i \leq N, \quad (\text{B.1})$$

where the random variable $X_{-i}^{(N)}(0) = \frac{1}{N} \sum_{j \neq i}^N X_j(0)$, $\mathbb{E}[X_i(0)] = \mu_i$ for all i , and the average mean $\mu^{(N)} = \frac{1}{N} \sum_{i=1}^N \mu_i \rightarrow \mu$ as $N \rightarrow \infty$.

Proof When there is no ambiguity, we omit the time variable t , writing $X_i = X_i(0)$ for simplicity.

Since X_i , $1 \leq i \leq N$, have a compact support, the first and the second order derivatives of $f(X_i, X_{-i}^{(N)})$ with respect to $X_{-i}^{(N)}$ are bounded.

By the first order Taylor expansion, we have

$$\begin{aligned} \left| \mathbb{E} \left[f(X_i, \mu_{-i}^{(N)}) - f(X_i, \mu) \right] \right| &\leq \left| \mathbb{E} \left[\left(\partial_y f(X_i, \xi_{\mu_{-i}^{(N)}}) \right)^T \right] \right| |\mu_{-i}^{(N)} - \mu| \\ &= O\left(\frac{1}{N}\right) + O(|\mu^{(N)} - \mu|), \end{aligned} \quad (\text{B.2})$$

where we have $\mu_{-i}^{(N)} = \frac{1}{N} \sum_{j \neq i}^N \mu_j$ and $\xi_{\mu_{-i}^{(N)}}$ depends on $\mu_{-i}^{(N)}$.

Again using the second order Taylor expansion, we have

$$\begin{aligned} &\left| \mathbb{E} \left[f(X_i, X_{-i}^{(N)}) - f(X_i, \mu_{-i}^{(N)}) \right] \right| \\ &= \left| \mathbb{E} \left[\left(\partial_y f(X_i, \mu_{-i}^{(N)}) \right)^T (X_{-i}^{(N)} - \mu_{-i}^{(N)}) + \frac{1}{2} (X_{-i}^{(N)} - \mu_{-i}^{(N)})^T \partial_{yy}^2 f(X_i, \xi_{X_{-i}^{(N)}}) (X_{-i}^{(N)} - \mu_{-i}^{(N)}) \right] \right| \\ &= \frac{1}{2N^2} \sum_{j \neq i}^N \left| \mathbb{E} \left[(X_j - \mu_j)^T \partial_{yy}^2 f(X_i, \xi_{X_{-i}^{(N)}}) (X_j - \mu_j) \right] \right| = O\left(\frac{1}{N}\right). \end{aligned} \quad (\text{B.3})$$

Here the last two equalities hold since X_i , $1 \leq i \leq N$ are independent, and the partial derivatives $\partial_y f(X_i, \mu_{-i}^{(N)})$ and $\partial_{yy}^2 f(X_i, \xi_{X_{-i}^{(N)}})$ are bounded.

Combining (B.2) with (B.3), we have

$$\begin{aligned} \left| \mathbb{E} \left[f(X_i, X_{-i}^{(N)}) - f(X_i, \mu) \right] \right| &\leq \left| \mathbb{E} \left[f(X_i, X_{-i}^{(N)}) - f(X_i, \mu_{-i}^{(N)}) \right] \right| + \left| \mathbb{E} \left[f(X_i, \mu_{-i}^{(N)}) - f(X_i, \mu) \right] \right| \\ &= O\left(\frac{1}{N}\right) + O(|\mu^{(N)} - \mu|). \end{aligned} \quad (\text{B.4})$$

The proof is completed. $\blacksquare$