# Opinion Dynamics with Noisy Information

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*Abstract*— **This paper considers a social opinion model with noisy information when one agent obtains the opinion of another. Stochastic approximation with bounded confidence is introduced to update the opinions. The asymptotic behavior of the stochastic algorithm is intimately related to a deterministic vector field. We show that the presence of noise can cause a defragmentation of the state space. This in turn can generate more orderly collective behavior, which is very different from noiseless models which have the well known fragmentation property during the evolution of the individual opinions.**

## I. Introduction

The studies on social opinion formation have attracted considerable interest of researchers in different areas including social science, economics, statistical physics, and systems and control [1], [4], [9], [10], [18], [16]. A comprehensive survey up to 2009 is available in [5]. Under the Hegselmann-Krause modeling [10], [16], agents simultaneously update their opinions by using opinions of others which are within a confidence interval. A slightly different rule was proposed in [9] where at each step a pair of agents is randomly selected to perform update with bounded confidence. A well known phenomenon in these bounded confidence based models is the so-called fragmentation effect. When the agents have random initial states, very often the agents form different clusters where members in the same cluster converge to the same limit and agents of different clusters will remain dissent. This implies after some time, agents in different clusters cease to have effective opinion exchange. Some lower bound estimates of the inter-cluster distances in the steady state are developed in [4]. The work [19] used shrinking confidence intervals and analyzed the formation and detection of communities.

In the past research, some attention has been given to noisy opinion dynamics. The early work [11] applied additive noise to the learning rule [11]. The role of noise for more realistic modeling is also discussed in [5, pp. 610-611]. By introducing free will, an agent has positive probabilities to have a jump in its opinion according to a certain distribution or perform an opinion learning rule using information from others [3], [20].

This paper introduces a new stochastic noisy modeling of social opinion dynamics. The natural motivation is to consider the introduction of inaccuracies when the communication of opinions takes place. Several scenarios may contribute to noisy information. The first is due to indirect communication. When agent $i$ obtains the opinion of agent $j$ by a third party such as another agent, a TV news report, or a newspaper article, etc., some inaccuracy or distortion may occur when the opinion of agent $j$ is conveyed. The second scenario involves consciously introducing ambiguity by the agent who is providing opinion. For example, when a sensitive issue is discussed, a person may do so just to avoid controversy or to leave some room for future clarification. Another scenario is related to biased modification. An agent may say something different from his (or her) genuine thought to some extent, and has the tendency of showing a milder position. For example, when asked to publicly speak on a sensitive issue, a candidate of an electoral campaign having an opinion of strong support (or objection) may choose to express a softer version of his opinion.

Our modeling framework is significantly different from [3], [11], [20] since our focus is on unreliability of the opinion communication among the agents. Owning to this unreliability, each agent needs to adaptively adjust its opinion update rule for the purpose of cautious learning. This distinctive feature makes our approach different from the existing research [3], [11], [20] where the algorithms have a certain time homogeneity. Our algorithm will incorporate bounded confidence into stochastic approximation. It turns out that the algorithm has inherent nonlinearity. Our main contribution is the determination of the structure of the equilibrium set of the associated nonlinear vector field. For the application of stochastic approximation to consensus problems, the reader is referred to [2], [12], [13], [14], [15], [17], [21].

The paper is organized as follows. Section II describes the social opinion model with noisy information acquisition and bounded confidence, which leads to a framework of nonlinear stochastic approximation. The main results are presented in Section III in terms of the equilibrium set of the vector field governing the stochastic approximation algorithm. Section IV sketches the proof of Theorem 4. Simulations are illustrated in Section V. Section VI concludes the paper. The analysis of the equilibrium set of Section IV relies on some key results of graph decomposition which are provided in Appendix A and are interesting in their own right.

## II. The Network Model and Opinion Dynamics

We introduce some standard preliminary on graph modeling of the network topology. A directed graph (digraph) $G=(\mathscr{N},\mathscr{E})$ consists of a set of nodes $\mathscr{N}=\{1,\ldots,n\}$ and a set of directed edges $\mathscr{E}$. A directed edge (simply called an edge) is denoted by an ordered pair $(i,j)\in\mathscr{N}\times\mathscr{N}$, where $i\neq j$. A directed path (from node $i_1$ to node $i_l$) in

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$G$ consists of a sequence of nodes $i_1,\dots,i_l$, $l\geq 2$, such that $(i_k,i_{k+1})\in\mathscr{E}$. The digraph $G$ is strongly connected if from any node to any other node, there exists a directed path. A directed tree is a digraph where each node $i$, except the root, has exactly one parent node $j$ so that $(j,i)\in G$. The digraph $G$ is said to contain a spanning tree if there exists a directed tree $G_{\rm tr}=(\mathscr{N},\mathscr{E}_{\rm tr})$ such that $\mathscr{E}_{\rm tr}\subset\mathscr{E}$. For two disjoint subsets $S_1$ and $S_2$ of $\mathscr{N}$, if there exist $i_1\in S_1$ and $i_2\in S_2$ such that $(i_1,i_2)\in\mathscr{E}$, we say $S_2$ is reachable from $S_1$ by one hop. We call $i_1$ and $i_2$ the exit and entry nodes, respectively. A node without incoming edges is called a source. A node without out-going edges is called a sink.

### A. Opinion Update with Bounded Confidence

The social opinion network is modeled by $G$, where each agent is identified as a node in $G$. The two names agent and node will be used interchangeably. The digraph $G$ determines the communication of information among the agents. If $(j,i)\in\mathscr{E}$, agent $i$ receives information from agent $j$ which is called a neighbor of agent $i$. The neighbor set of agent $i$ is denoted by $\mathscr{N}_i=\{j|(j,i)\in\mathscr{E}\}$.

The opinion of agent $i$ at time $t$ is represented by a real number $x_t^i$, and is also called its state. Each agent knows its own state exactly. The opinion exchange between two agents is noisy and modeled by

$$y_t^{ij}=x_t^j+w_t^{ij},\qquad j\in\mathscr{N}_i, \tag{1}$$

which is received by agent $i$ from agent $j$.

For agent $i$, let $r_i>0$ be a fixed number to be called its confidence threshold. For its opinion update, agent $i$ needs to deal with two cases.

Case 1): $|y_t^{ij}-x_t^i|\leq r_i$. We say $y_t^{ij}$ is within the confidence range of agent $i$, and so it is accepted. In this case, agent $j$ is called a valid neighbor of agent $i$.

Case 2): $|y_t^{ij}-x_t^i|>r_i$. Then $y_t^{ij}$ is not trustworthy and so ignored by agent $i$.

Define the valid neighbor set of agent $i$ by

$$\mathscr{N}_{it}=\{j|j\in\mathscr{N}_i,|y_t^{ij}-x_t^i|\leq r_i\},$$

which depends on $x_t^i$ and noisy opinions $y_t^{ij}$, $j\in\mathscr{N}_i$, and the threshold parameter $r_i$.

The opinions evolve according to the following heuristic rules. Each agent takes information based on the valid neighbor set. Next, it performs cautious learning since the obtained information is noise corrupted. We propose the state update rule

$$x_{t+1}^i=\left(1-a_t\sum_{j\in\mathscr{N}_{it}}b_{ij}\right)x_t^i+a_t\sum_{j\in\mathscr{N}_{it}}b_{ij}y_t^{ij}, \tag{2}$$

where $a_t$ is the step size at time $t$, and $b_{ij}$ is a positive number to indicate the relative importance of information from agent $j$. The step size will decrease to zero. If an agent does not have any neighbor, its opinion remains a constant and it is called a stubborn agent [1].

The learning rule differs from most existing algorithms [4], [10], [16], [19] on social opinion dynamics by the cautious learning behavior of the agents which is reflected by the decreasing step size. This algorithm shares some similarity to consensus algorithms with measurement noise [12], [13], [14], which have linear dynamics. The early works [7], [6] used decreasing step sizes in noiseless consensus problems to model hardening positions.

We introduce the following assumptions.

(A1) $a_t>0$ for all $t$, $\sum_{t=0}^\infty a_t=\infty$, $\sum_{t=0}^\infty a_t^2<\infty$. □

(A2) For each $(i,j)\in\mathscr{E}$, the noises $\{w^{ij},w_t^{ij},t\geq 0\}$ are i.i.d., and have a probability density function (p.d.f.) $f_{w^{ij}}(z)$, which has support equal to $\mathbb{R}$, i.e., $f_{w^{ij}}(z)>0$ for any $z$. □

For later notational convenience, we introduce $w^{ij}$ as a fictitious random variable. We use (A1) only in the simulations.

### B. A Perspective of Nonlinear Stochastic Approximation

The main objective of this paper is to study the dynamic properties of the opinion update algorithm and examine the impact of the noise. We write (2) in the equivalent form

$$x_{t+1}^i=x_t^i+a_t\sum_{j\in\mathscr{N}_{it}}b_{ij}(x_t^j+w_t^{ij}-x_t^i).$$

Denote $Y_t^i=\sum_{j\in\mathscr{N}_{it}}b_{ij}(x_t^j+w_t^{ij}-x_t^i)$, which is the correcting term in the adjustment of $x_t^i$. Denote the vector of the $n$ individual states

$$x_t=(x_t^1,\dots,x_t^n)^T.$$

For $w^{ij}$, denote the truncated moment:

$$M_{ij}(z,r_i)=\int_{-r_i}^{r_i}uf_{w^{ij}}(u-z)du.$$

To obtain information on the tendency of the state adjustment, we define the drift function of agent $i$ as

$$\begin{aligned}F_i(x^1,\dots x^n)&=E\left[Y_t^i\middle|x_t=x\right]\\&=\sum_{j\in\mathscr{N}_i}b_{ij}\int_{-r_i}^{r_i}uf_{w^{ij}}(u-(x^j-x^i))du\\&=\sum_{j\in\mathscr{N}_i}b_{ij}M_{ij}(x^j-x^i,r_i).\end{aligned}$$

*Example 1:* If $w^{ij}$ has the normal distribution $N(0,\sigma^2)$, $\sigma>0$, we have

$$M_{ij}(x^j-x^i,r_i)=\frac{1}{\sqrt{2\pi}\sigma}\int_{-r_i}^{r_i}u\exp\left\{-\frac{[u-(x^j-x^i)]^2}{2\sigma^2}\right\}du.$$

□

Define

$$W_{t+1}^i=Y_t^i-E\left[Y_t^i|x_t\right],\quad W_t=(W_t^1,\dots,W_t^n)^T,$$
$$F=(F_1,\dots,F_n)^T,$$

where $F$ determines a vector field in $\mathbb{R}^n$. We have the relation

$$x_{t+1}^i=x_t^i+a_tF_i(x_t)+a_tW_{t+1}^i, \tag{3}$$

which has the vector form

$$x_{t+1}=x_t+a_tF(x_t)+a_tW_{t+1},$$

where $W_{t+1}$ acts as an additive noise vector with zero mean. The study of the original opinion update algorithm

(2) reduces to the investigation of the nonlinear stochastic approximation algorithm. The properties of the function $F$ will play a central role. If the vector field behaves sufficiently well, the algorithm is expected to converge to a point which is an equilibrium of $F$ (i.e., $F$ equals zero at that point). The focus of this paper is the analysis of the function $F$.

By elementary estimates it can be shown that $M_{ij}(z,r_i)$ is a continuous function of $z$ on $(-\infty,\infty)$. We introduce the following assumption for the noise.

(A3) For each $(i,j)\in\mathscr{E}$, (i) $M_{ij}(z,r_i)>0$ for $z>0$; (ii) $M_{ij}(z,r_i)<0$ for $z<0$; (iii) $M_{ij}(0,r_i)=0$. □

The purpose of introducing (A3) is to enable the opinion adjustment rule to learn in the "right" direction. We give a sufficient condition to ensure (A3).

*Proposition 1:* Suppose (a) the p.d.f. $f_{w^{ij}}(z)$ is strictly increases on $(-\infty,0)$, and strictly decreases on $(0,\infty)$ (b) $f_{w^{ij}}(z)$ is an even function on some interval $(-r_0,r_0)$, $r_0>0$. Then (A3) is satisfied for all $r_i\in(0,r_0]$.

*Proof:* Let $r_i\in(0,r_0]$ be fixed. If $z\in[r_i,\infty)$, clearly

$$\begin{aligned}
M_{ij}(z,r_i) &= \int_{-r_i-z}^{r_i-z}(u+z)f_{w^{ij}}(u)du \\
&= \int_{-r_i-z}^{-z}(u+z)f_{w^{ij}}(u)du+\int_{-z}^{r_i-z}(u+z)f_{w^{ij}}(u)du \\
&> \int_{-r_i-z}^{-z}(u+z)f_{w^{ij}}(-z)du+\int_{-z}^{r_i-z}(u+z)f_{w^{ij}}(-z)du=0.
\end{aligned}$$

Now, fix any $z\in(0,r_i)$. We have

$$M_{ij}(z,r_i)=\int_{-r_i}^{0}uf_{w^{ij}}(u-z)du+\int_{0}^{r_i}uf_{w^{ij}}(u-z)du \tag{4}$$

$$=\int_{0}^{r_i}u[f_{w^{ij}}(u-z)-f_{w^{ij}}(-u-z)]du. \tag{5}$$

If $u\in(0,z)$, $f_{w^{ij}}(u-z)-f_{w^{ij}}(-u-z)>0$. If $u\in(z,r_i)$,

$$f_{w^{ij}}(u-z)=f_{w^{ij}}(-u+z)>f_{w^{ij}}(-u-z). \tag{6}$$

Therefore $M_{ij}(z,r_i)>0$ for all $z\in(0,r_i)$. This verifies (i). Condition (ii) is verified similarly. Finally, (iii) follows since $f_{w^{ij}}$ is even on $(-r_0,r_0)$. □

*Remark 1:* (A3) holds for the normal distribution $N(0,\sigma^2)$, $\sigma>0$, and a symmetric exponential distribution. □

## III. Main Results

*Proposition 2:* $\{W_t,t\geq 0\}$ is a sequence of martingale differences with respect to the increasing sequence of $\sigma$-algebras $\mathscr{F}_t=\sigma(x_0,\ldots,x_t)$. □

*Definition 3:* A point $x\in\mathbb{R}^n$ is called an equilibrium of $F$ if $F(x)=0$. The equilibrium set $S(F)$ of $F$ consists of all its equilibrium points. □

We have the following result on the equilibrium set. Its proof is given in Section IV.

*Theorem 4:* If $G$ contains a spanning tree without a leader, then the equilibrium set $S(F)=\text{span}\{\mathbf{1}_n\}$. □

If $G$ contains a spanning tree with a leader $i_L$, then $i_L$ is a stubborn agent. Let its state be fixed as $x_0^{i_L}$.

*Corollary 5:* If $G$ contains a spanning tree with a leader $x_0^{i_L}$, then the equilibrium set is the singleton $S(F)=\{x_0^{i_L}\mathbf{1}_n\}$.

*Proof:* We may adapt the proof of Theorem 4 to prove the corollary. □

## IV. Proof of Theorem 4

We need to make some technical preparation. The next lemma is obvious and we omit the proof.

*Lemma 6:* We have $\text{span}\{\mathbf{1}_n\}\subset S(F)$. □

*Lemma 7:* If $G$ is strongly connected, $S(F)=\text{span}\{\mathbf{1}_n\}$.

*Proof:* See Appendix B. □

For the set of nodes $\mathscr{N}$ in $G$, we decompose it into strongly connected components (SCCs) $C_0,\ldots,C_K$, $K\geq 0$. There may exist an edge pointing from one SCC to another.

Since $G$ contains a spanning tree, there exists a node $i_0$ which can reach any other node by a directed path. Without loss of generality, assume $i_0\in C_0$. Let $G_{\text{mg}}=(\mathscr{N}_{\text{mg}},\mathscr{E}_{\text{mg}})$ denote the meta-graph [8]. It has the set of nodes $\{v_0,\ldots,v_K\}$ corresponding to the SCCs $C_0,\ldots,C_K$ in $G$.

If $K=0$, $G$ is strongly connected.

If $K\geq 1$, we apply Theorem 14 to list the nodes $\{v_0,\ldots,v_K\}$ of the meta-graph in the form

$$\{v_0\},\quad \{v_{1,1},\ldots,v_{1,K_1}\},\quad \cdots,\quad \{v_{l,1},\cdots,v_{l,K_l}\}, \tag{7}$$

where $l$ is the greatest maximal depth in the meta-graph (see Appendix A for its definition). We have $1+K_1+\ldots+K_l=1+K$. Corresponding to (7), we list all associated SCCs of $G$ into the array:

$$\begin{array}{l}
C_0,\\
C_{1,1},\ldots,C_{1,K_1},\\
\quad\cdots\\
C_{l,1},\ldots,C_{l,K_l}.
\end{array}$$

For instance $C_{k,j}$ corresponds to $v_{k,j}$. Summarizing the above, we have the following proposition by Theorem 14.

*Proposition 8:* Suppose $G$ is leaderless and $K\geq 1$. Then

(i) $C_0$ contains at least 2 nodes. None of these nodes have a neighbor outside $C_0$.

(ii) For $i\in C_{k,j}$, $k\geq 1$, each of its neighbors is from $C_{k,j}$ or sets in the upper levels of the array. For each $j$, there exists $i\in C_{k,j}$, which has at least one neighbor in $\cup_{1\leq m\leq K_{k-1}}C_{k-1,m}$ (the union is interpreted as $C_0$ if $k=1$). □

### A. Proof of Theorem 4.

*Proof:* By Lemma 6, it suffices to show $S(F)\subset\text{span}\{\mathbf{1}_n\}$. For the decomposition into SCCs, if $K=0$, $G$ is strongly connected and the theorem reduces to Lemma 7.

Now suppose $K\geq 1$, and $x\in S(F)$.

Step 1. For $G$ without a leader, $C_0$ contains at least 2 nodes. Denote them by $i_1,\ldots,i_{k_1}$. None of them have a neighbor outside $C_0$, so that $(F^{i_1}(x^1,\ldots,x^n),\ldots,F^{i_{k_1}}(x^1,\ldots,x^n))$ involves only the variables $(x^{i_1},\ldots,x^{i_{k_1}})$. For instance,

$$F^{i_1}(x^1,\ldots,x^n)=\sum_{j\in\mathscr{N}_{i_1}}b_{i_1j}\int_{-r_{i_1}}^{r_{i_1}}uf_{w^{i_1,j}}(z-(x^j-x^{i_1}))du,$$

where $\mathscr{N}_{i_1}\subset C_0$. Since $C_0$ is strongly connected, this case reduces to the scenario of Lemma 7 after replacing $G$ by the digraph $(C_0,\mathscr{E}|_{C_0})$, where $\mathscr{E}|_{C_0}$ denotes the set of edges of $G$ which have the initial and terminal nodes in $C_0$. Hence, we conclude

$$x^{i_1}=\ldots=x^{i_{k_1}}=\xi,$$

where $\xi$ denotes the common value of the $k_1$ coordinates.

Step 2. Now we list all elements in $C_{1,1} \cup \ldots \cup C_{1,K_1}$ as $i_{k_1+1}, i_{k_1+2}, \ldots, i_{k_2}$, where $k_2 \geq k_1 + 1$. Denote

$$\bar{x} = \max\{x^{i_{k_1+1}}, \ldots, x^{i_{k_2}}\}, \quad \underline{x} = \min\{x^{i_{k_1+1}}, \ldots, x^{i_{k_2}}\}.$$

We show $\bar{x} = \underline{x} = \xi$ by contradiction. Assume $\bar{x} > \xi$ and $\bar{x} = x^{i_s}$ for some $s$ satisfying $k_1 + 1 \leq s \leq k_2$. So

$$0 = F^{i_s}(x^1, \ldots, x^n) = \sum_{j \in \mathcal{N}_{i_s}} b_{i_s j} \int_{-r_{i_s}}^{r_{i_s}} u f_{w^{i_s j}}(u - (x^j - x^{i_s}))du.$$

By Proposition 8-(ii), $x^j \leq \bar{x}$. By Lemma 15,

$$x^j = \bar{x}, \qquad j \in \mathcal{N}_{i_s}. \tag{8}$$

Let $C(i)$ be the SCC of $G$ containing node $i$. If $C(i_s)$ is not a singleton, we show that $x^{i'_s} = x^{i_s}$ if $i'_s \in C(i_s)$. Note node $i_s$ has at least one neighbor $i_{s,1}$ in $C(i_s)$. By (8), that neighbor should have $x^{i_{s,1}} = \bar{x}$. Combining $i_s$ and $i_{s,1}$ together, we find another node, if there is such one remaining in $C(i_s)$, such that its state is also $\bar{x}$. By induction, we conclude that all nodes in $C(i_s)$ have the same value $\bar{x}$.

We may select node $i'_s \in C(i_s)$ such that there exists an edge pointing to node $i'_s$ from $C_0$ by Proposition 8-(ii). Without introducing additional notation, we assume node $i_s$ already has this property. Suppose $i_k \in C_0$ and there is an edge from $i_k$ to $i_s$. By using (8), we see that $x^{i_k} = \bar{x}$, which contradicts with $\xi < \bar{x}$. Thus, we conclude that $\bar{x} \leq \xi$. Similarly, we may show

$$\underline{x} \geq \xi.$$

Combining the two inequalities yields $\bar{x} = \underline{x} = \xi$.

Step 3. By induction, we conclude

$$x^1 = \ldots = x^n.$$

This completes the proof. □

## V. Simulation Examples

The simulation examples are based on a complete graph of 40 agents with $r_i \equiv 0.25$.

The initial states $x_0^i$, $1 \leq i \leq 40$, are generated as i.i.d. random variables uniformly distributed on $[0, 1]$. Fig. 1 (top) illustrates the Hegselmann-Krause model without noise. The opinions converge into 3 clusters. Fig. 1 (bottom) shows the convergence of the stochastic approximation algorithm where $a_t = 0.5(t+1)^{-0.55}$ and $w^{ij}$ has the normal distribution $N(0, 0.2^2)$.

In the next example of stochastic approximation, the 40 agents are divided into two groups to have poor initial inter-cluster connectivity. Cluster $A$ with $(x^1, \ldots, x^{20})$ and $B$ with $(x^{21}, \ldots, x^{40})$ have their initial opinions distributed in a small neighborhood of 0.6 and 1.5, respectively. The confidence threshold 0.25 is much smaller than the approximate separation distance $1.5 - 0.6 = 0.9$ between the two clusters. We see the convergence in Fig. 2 is extremely slow. However, this is expectable. For example, when $|x_t^1 - x_t^{26}| = 0.9$, the probability for the $26th$ agent to become a valid neighbor of the first agent is at the order of $10^{-7}$. Nevertheless, we still observe convergence.

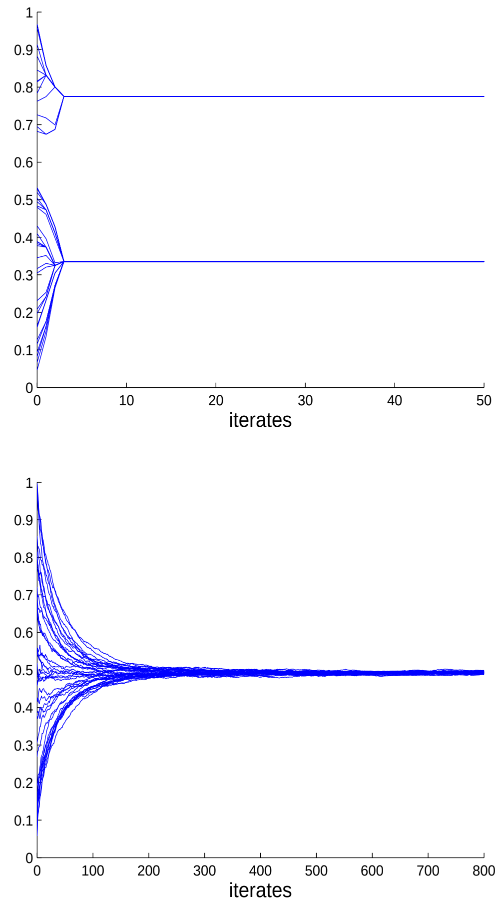


Fig. 1. Top: the Hegselmann-Krause model; bottom: stochastic approximation with bounded confidence.

## VI. Concluding Remarks

This paper addresses noisy information exchange in social opinion systems. We exploit the noise enhanced connectivity between the agents and adopt a framework of nonlinear stochastic approximation for the opinion evolution. The vector field underlying the algorithm is shown to have an equilibrium set where each point is an agreement state. This feature differs from other works where the state space can fragment into several parts due to bounded confidence in the opinion update. For future work, it is of interest to study the sample path convergence of the opinion updating rule.

## Appendix A: Graph Decomposition

Suppose $G = (\mathcal{N}, \mathcal{E})$ is a digraph. Let $\mathcal{N}$ be partitioned as the disjoint union of $S_1, \ldots, S_K$ where each set $S_k$ is a strongly connected component (SCC). The meta-graph of $G$ is defined as a digraph $G_{\rm mg} = (\mathcal{N}_{\rm mg}, \mathcal{E}_{\rm mg})$, where $\mathcal{N}_{\rm mg} = \{1, \ldots, K\}$ and $(i, j) \in \mathcal{E}_{\rm mg}$ if and only if $S_j$ is reachable from $S_i$ by one hop. Therefore, the meta-graph is obtained by collapsing each SCC into a single node.

*Proposition 9:* If $G$ contains a spanning tree, then $G_{\rm mg}$ has the following properties:

(i) it is a directed acyclic graph;

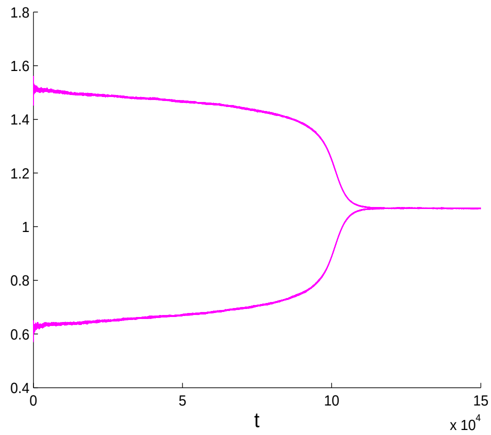


Fig. 2. The initial opinions of the two clusters are around 0.6 and 1.5, respectively.

(ii) it has exactly one source and at least one sink;
(iii) it contains a spanning tree.

*Proof:* By [8, pp. 100-101], $G_{\rm mg}$ is a directed acyclic graph with at least one source and at least one sink. Assume $G_{\rm mg}$ has two different sources $s_1$ and $s_2$. Let $S_1$ and $S_2$ be the corresponding SCCs in $G$ and so neither of them have incoming edges. Since $G$ contains a spanning tree, there exists a node $i_R$ such it can reach any other node by a directed path. We consider two cases: (i) If $i_R \in S_1$, it cannot reach $S_2$ since there are no edges entering $S_2$. (ii) If $i_R \notin S_1$, it cannot reach $S_1$. The two cases lead to a contradiction. So there is exactly one source.

Suppose the SCC $S_{k_0}$ contains $i_R$ and corresponds to node $k_0$ in $G_{\rm mg}$. Since $i_R$ is connected to any other node of $G$ by a directed path, $k_0$ is connected to any other node of $G_{\rm mg}$ by a directed path. Hence $G_{\rm mg}$ contains a spanning tree. In fact in this case $k_0$ is the unique source. □

The length of a directed path is the number of edges (allowed to repeat if cycles appear) lying between the initial and terminal nodes. Below it is always assumed that $G$ contains a spanning tree. We introduce the following definition.

*Definition 10:* For each node $i \neq i_S$ in $G_{\rm mg}$, the maximal depth $Md(i, G_{\rm mg})$ is the maximal length of all directed paths from the source $i_S$ to $i$. □

We make the convention $Md(i_S, G_{\rm mg}) = 0$.

*Proposition 11:* For any node $i \neq i_S$ in $G_{\rm mg}$, $1 \leq Md(i, G_{\rm mg}) \leq |\mathcal{N}_{\rm mg}| - 1$.

*Proof:* Since the digraph $G_{\rm mg}$ contains a spanning tree and is acyclic, there exists a directed path from $i_S$ to $i$ and the total number of such directed paths is finite. Moreover, any directed path from $i_S$ to $i$ has at most $|\mathcal{N}_{\rm mg}| - 1$ edges since otherwise it would contain a cycle. □

*Proposition 12:* Denoting $d_1 = \max_j Md(j, G_{\rm mg})$, each node with its maximal depth equal to $d_1$ is a sink.

*Proof:* Suppose $Md(i, G_{\rm mg}) = d_1$ and $i$ is not a sink. We construct a directed path from $i_S$ to $i$ and extend it until a next node $i'$. This is feasible since $i$ is not a sink. Then $Md(i', G_{\rm mg}) \geq Md(i, G_{\rm mg}) + 1 = d_1 + 1$, which is a contradiction. □

*Remark 2:* $G_{\rm mg}$ may have sinks whose maximal depth is less than $d_1$. □

Below we describe a procedure to obtain a subgraph from $G_{\rm mg}$. To avoid triviality, we assume that $G_{\rm mg}$ contains at least 2 nodes. We remove all nodes of $G_{\rm mg}$ which have their maximal depth equal to $d_1 = \max_j Md(j, G_{\rm mg})$. By Proposition 12, these nodes only have incoming edges. We also remove all these incoming edges. Let the resulting subgraph be denoted by $G^1_{\rm mg}$.

*Proposition 13:* Let $d_2 = \max_j Md(j, G^1_{\rm mg})$. We have the assertions.

(i) If $i$ is in $G^1_{\rm mg}$, then $Md(i, G^1_{\rm mg}) = Md(i, G_{\rm mg})$;
(ii) $d_2 = d_1 - 1$;
(iii) $G^1_{\rm mg}$ is still a digraph having the three properties in Proposition 9.

*Proof:* (i) It is clear that for a node $i$ in $G^1_{\rm mg}$, none of its incoming edges are removed in the procedure when $G^1_{\rm mg}$ is constructed. The set of directed paths from $i_S$ to $i$ is the same no matter it is regarded as a node in $G^1_{\rm mg}$ or $G_{\rm mg}$.

(ii) First, we have $d_2 \leq d_1 - 1$. Assume $Md(i_0, G_{\rm mg}) = d_1$. Then there exists an edge $(i_1, i_0)$ of $G_{\rm mg}$ and there is a directed path of length $d_1$ from $i_S$ to $i_0$ via $i_1$. Then $i_1$ is a node of $G^1_{\rm mg}$ since it must remain after the above removal procedure. It is clear that $Md(i_1, G^1_{\rm mg}) = d_1 - 1$. Therefore, $d_2 = d_1 - 1$.

(iii) First, $G^1_{\rm mg}$ is a directed acyclic graph with $i_S$ being a source. Suppose $i \neq i_S$ is in $G^1_{\rm mg}$. Since $i$ is also in $G_{\rm mg}$, there is a directed path $p_{i_S,i}$ from $i_S$ to $i$. Note that $p_{i_S,i}$ does not have any node which was removed in constructing $G^1_{\rm mg}$. Therefore, $p_{i_S,i}$ is within $G^1_{\rm mg}$. So $G^1_{\rm mg}$ contains a spanning tree. By the proof of Proposition 9, we see $G^1_{\rm mg}$ satisfies Proposition 9(ii). □

By using Propositions 9 and 13 and applying the removal procedure repeatedly, we establish the following decomposition theorem.

*Theorem 14:* For the digraph $G_{\rm mg}$, its set of nodes can be decomposed as a disjoint union of the following subsets:

$$\begin{array}{l} S_0 = \{i_S\}, \\ S_1 = \{i_1, \ldots, i_{k_1}\}, \\ S_2 = \{i_{k_1+1}, \ldots, i_{k_2}\}, \\ \cdots \\ S_{l-1} = \{i_{k_{l-2}+1}, \ldots, i_{k_{l-1}}\}, \\ S_l = \{i_{k_{l-1}+1}, \ldots, i_{k_l}\}, \end{array} \tag{9}$$

where we have $|\mathcal{N}_{\rm mg}| = \sum_{i=0}^{l} |S_i|$ and

(i) all nodes in a subset $S_i$ have the same maximal depth equal to $i$;
(ii) there exists no edge between any two nodes in the same subset $S_i$;
(iii) if $(i_1, i_2)$ is an edge of $G_{\rm mg}$, then there exists $0 \leq k_1 < k_2 \leq l$ such that $i_1 \in S_{k_1}$ and $i_2 \in S_{k_2}$;
(iv) if $i \in S_k$, $1 \leq k \leq l$, there exists a node $i' \in S_{k-1}$ such that $(i', i)$ is an edge of $G_{\rm mg}$.

*Proof:* (i) When $G^1_{\rm mg}$ is constructed, let the set of nodes deleted from $G_{\rm mg}$ be denoted by $S_l$. Similarly, by Proposition

13-(iii), we may repeat this removal procedure by deleting the set $S_{l-1}$ of nodes in $G^1_{\rm mg}$ which have their maximal depth equal to $d_2$. This constructs the digraph $G^2_{\rm mg}$. Repeating this for a finite number of steps, we obtain the sets $S_l, S_{l-1}\ldots, S_0$, and the digraphs $G^1_{\rm mg},\ldots,G^l_{\rm mg}$. Each of $G_{\rm mg}, G^1_{\rm mg},\ldots,G^{l-1}_{\rm mg}$ has the three properties in Proposition 9. It is obvious that all nodes in the same set $S_i$ share the same maximal depth, and along the sequence $S_l, S_{l-1},\ldots,S_1$, the maximal depth decreases by one from one set to the next. Now we only need to show that $Md(i_1, G^l_{\rm mg}) = 1$. It is clear that all nodes in $S_1$ appear as sinks in $G^{l-1}_{\rm mg}$. Since $S_1$ contains a spanning tree, from $i_S$ to each node and in particular to $i_1$, there exists an edge. So $Md(i_1, G^l_{\rm mg}) = 1$.

(ii) For $k = l, l-1,\ldots,2$, each node in $S_k$ is always deleted, together with its incoming edges, as a sink of the current digraph $G^{l-k}_{\rm mg}$ (some sinks may not qualify for deletion), where we denote $G^0_{\rm mg} = G_{\rm mg}$. For the previous steps there is no chance to remove an edge between two nodes in $S_k$. Hence (ii) follows.

(iii) By the above removal procedure, all edges are eventually deleted. Whenever an edge is being deleted, it points to a node with a strictly greater maximal depth than its initial node.

(iv) Consider $i \in S_k$. When $i$ is removed from within $G^{l-k}_{\rm mg}$, it appears as a sink. Its neighbor set within $G^{l-k}_{\rm mg}$ contains at least one node $i'$ with maximal depth equal to $k-1$. Hence $i' \in S_{k-1}$. □

*Remark 3:* (iii) implies there is no back edge pointing to a set $S_k$ which was listed earlier. (iv) means there is always a node from the immediate upper level in (9) connecting to the given node; it is possible to have edges originating from other upper levels. □

## APPENDIX B

*Lemma 15:* Suppose $\lambda_j > 0$ and for some $\alpha$,

$$\beta_j \geq \alpha \ (\text{resp., } \beta_j \leq \alpha), \quad j = 1,\ldots,k. \tag{10}$$

Then

$$\sum_{j=1}^{k} \lambda_j \int_{-r}^{r} u f_{w^{ij}}(u - (\beta_j - \alpha))du \geq 0 \ (\text{resp., } \leq 0),$$

where the equality holds only if (10) becomes $k$ equalities. □

**Proof of Lemma 7.** Let $(x^1,\ldots,x^n)$ be an equilibrium point. Then $F^i(x^1,\ldots,x^n) = 0$ for $i = 1,\ldots,n$. Suppose

$$x^{i_1} \leq x^{i_2} \leq \ldots \leq x^{i_n},$$

where $(i_1,\ldots,i_n)$ is a permutation of $(1,\ldots,n)$. We have $F^{i_1}(x^1,\ldots,x^n) = 0$ and so

$$0 = \sum_{j\in\mathcal{N}_{i_1}} b_{i_1 j} \int_{-r_{i_1}}^{r_{i_1}} u f_{w^{i_1 j}}(u - (x^j - x^{i_1}))du,$$

which implies for each $j \in \mathcal{N}_{i_1}$, $x^j = x^{i_1}$ by Lemma 15. Fix $i_l \in \mathcal{N}_{i_1}$, $l \geq 2$. Therefore we have a sequence of equal values

$$x^{i_1} = x^{i_2} = \ldots = x^{i_l}. \tag{11}$$

By the strong connectivity, there exists $i_k \in \{i_1, i_2,\ldots,i_l\}$ which has a neighbor in $\mathcal{N}\backslash\{i_1, i_2,\ldots,i_l\}$ whenever $\{i_1, i_2,\ldots,i_l\} \neq \mathcal{N}$. By the previous step, we obtain a sequence

$$x^{i_1} = x^{i_2} = \ldots = x^{i_l} = x^{i_{l+1}} = \ldots$$

which is longer than (11) by at least one. Repeating this procedure, we conclude

$$x^{i_1} = x^{i_2} = \ldots = x^{i_n}.$$

Recalling Lemma 6, the lemma follows. □

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