Introduction to Mean Field Game Theory Part I

Minyi Huang

School of Mathematics and Statistics
Carleton University
Ottawa, Canada

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Outline of talk

- Background and motivation for mean field game (MFG) theory
- Illustrative modeling examples
- Fundamental methodologies
- Applications and references

A mean field game is a situation of stochastic (dynamic) decision making where

- each agent interacts with the **aggregate effect** of all other agents;
- agents are **non-cooperative**.

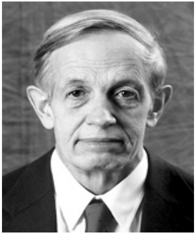
Example: Hybrid electric vehicle recharging control (interacting through aggregate load/price)

Noncooperative games: historical background

1. Cournot duopoly   Cournot equilibrium, 1838
2. 2 person zero-sum   von Neumann's minimax theorem, 1928
3. N-person nonzero-sum   Nash equilibrium, 1950

Antoine Cournot
(1801-1877)

John von Neumann
(1903-1957)



John Nash
(1928-2015)

Two directions for generalizations:

- Include **dynamics**

  Examples: L. Shapley (1953), MDP model; Rufus Isaacs (1965), *Differential games*, Wiley (work of 1950s)

- Consider **large populations** of players

Technical challenges arise ...

- the **Curse of Dimensionality** with a large number of players
- Example: 50 players, each having 2 states, 2 actions. Need a system configuration: $4^{50} = 1.27 \times 10^{30}$

Early efforts to deal with large populations

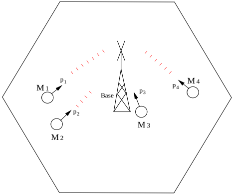
- J. W. Milnor and L.S. Shapley (1978); R. J. Aumann (1975). This is for cooperative games
- E. J. Green (1984). Non-cooperative games
- M. Ali Khan and Y. Sun (2002). Survey on "Non-cooperative games with many players". Static models
- B. Jovanovic and R. W. Rosenthal (1988). Anonymous sequential games. It considers distributional strategies for an infinite population. However, individual behaviour is not addressed
- ...

The early motivation for mean field games

– A wireless network power control problem

- Rate based power control (Huang, Caines, Malhamé'03)
  $dp^i = u^i dt, \quad |u_i| \le u_{\max}, \quad 1 \le i \le N$
- Log-normal channel gain (in dB)
  $dx^i = -a(x^i + b)dt + \sigma dW^i$ (Charalambous et al'99)
  W^i: white noise (Brownian motion)
- Cost function of user i

$$J_i = E\int_0^T \left\{\left[p^i e^{x^i} - \gamma\left(\frac{\alpha}{N}\sum_{j\neq i}^N p^j e^{x^j} + \eta\right)\right]^2 + r(u^i)^2\right\}dt$$



$\frac{\alpha}{N}\sum_{j\neq i}^N p^j e^{x^j} + \eta$: total interference due to other users and thermal noise at receiver

Cocktail party model: each speaker deals with bkgrd noise; one against the mass

Control objective: keep signal to interference ratio around γ

Figure : A cell of N users

Computational load with $N = 2$ (Huang, Caines, Malhamé, IFAC'02, IEEE TAC'04):

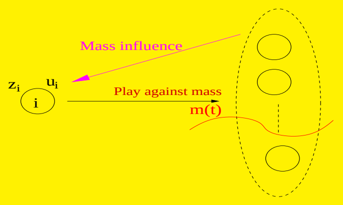
- Grid size for the 4 spatial variables in $\mathbb{R}^4$: 90^4
- Discretize time interval $[0, 1]$ with step size 0.1
- For the computation, memory allocation of 70 million sizeof(float) in C
- Very heavy computation via iterations!!

For large N, such iteration is no longer feasible

This motivated the development of a different approach to tackle dimensionality difficulty

See Aziz and Caines (2017) for more recent numerical analysis using MFG

Conceptual Modeling of mean field games



- Model: Each player interacts with the empirical state distribution $\delta_z^{(N)}$ of N players via dynamics and/or costs (see examples)
  - Individual state and control (z_i, u_i), $\delta_{z(t)}^{(N)} = \frac{1}{N}\sum_{i=1}^{N} \delta_{z_i(t)}$
- Objective: Overcome dimensionality difficulty

Illustrative examples for MFG modeling

- Discrete time linear quadratic (LQ) model
- Continuous time LQ model
- MDP model
- Stochastic growth with relative performance

Example 1: Discrete time linear quadratic (LQ) games

$$X_{t+1}^i = AX_t^i + Bu_t^i + GX_t^{(N)} + DW_{t+1}^i, \quad 1 \le i \le N$$

$$J_i(u^i, u^{-i}) = E \sum_{t=0}^{\infty} \rho^t \left\{ |X_t^i - (\Gamma X_t^{(N)} + \eta)|_Q^2 + (u_t^i)^T R u_t^i \right\}$$

- X_t^i: state; u_t^i: control; u^{-i}: control of all other agents; W_t^i: noise; $Q \ge 0$, $R > 0$, $\rho \in (0,1)$; $|z|_Q = (z^T Q z)^{\frac{1}{2}}$
- The **coupling term** $X_t^{(N)} = \frac{1}{N} \sum_{k=1}^N X_t^k$
- It is possible to consider different parameters such as A_i instead of A, etc.

Example 1': Continuous time linear quadratic (LQ)

$$dX_t^i = (A_{\theta_i} X_t^i + Bu_t^i + GX_t^{(N)})dt + DdW_t^i, \quad 1 \le i \le N$$

$$J_i(u^i, u^{-i}) = E\int_0^\infty e^{-\rho t}\left\{|X_t^i - (\Gamma X_t^{(N)} + \eta)|_Q^2 + (u_t^i)^T R u_t^i\right\} dt$$

- X_t^i: state; u_t^i: control; W_t^i: white noise as Brownian motion; $\rho > 0$; $|z|_Q = (z^T Qz)^{\frac{1}{2}}$
- The **coupling term** $X_t^{(N)} = \frac{1}{N}\sum_{k=1}^N X_t^k$
- θ_i: dynamic parameter to indicate heterogeneity

Note: This is the continuous time counterpart of Example 1.

Example 2: Discrete time Markov decision processes (MDPs)

- N MDPs
$$(x_t^i, a_t^i), \quad 1 \le i \le N,$$
where state x_t^i has transitions affected by action a_t^i but not by a_t^j, $j \neq i$.
- Player i has cost
$$J_i = E \sum_{t=0}^{T} \rho^t l(x_t^i, x_t^{(N)}, a_t^i), \qquad \rho \in (0, 1].$$

Example 3: Growth with relative performance

Dynamics of N agents:

$$dX_t^i = \left[A(X_t^i)^\alpha - \delta X_t^i - C_t^i\right] dt - \sigma X_t^i dW_t^i, \quad 1 \le i \le N$$

- X_t^i: capital stock, $X_0^i > 0$, $EX_0^i < \infty$, C_t^i: consumption rate
- Ax^α, $\alpha \in (0,1)$: Cobb-Douglas production function, $0 < \alpha < 1$, $A > 0$
- $\delta dt + \sigma dW_t^i$: stochastic depreciation (see e.g. Wälde'11, Feicht and Stummer'10 for stochastic depreciation modeling)
- $\{W_t^i, 1 \le i \le N\}$ are i.i.d. standard Brownian motions. The i.i.d. initial states $\{X_0^i, 1 \le i \le N\}$ are also independent of the N Brownian motions
- Take the standard choice $\gamma = 1 - \alpha$, i.e., equalizing the coefficient of the relative risk aversion to capital share

See Huang and Nguyen (2016)

Example 3 (ctn) The utility functional of agent i:

$$J_i(C^1, \dots, C^N) = E\left[\int_0^T e^{-\rho t} U(C_t^i, C_t^{(N,\gamma)})dt + e^{-\rho T} S(X_T)\right],$$

where $C_t^{(N,\gamma)} = \frac{1}{N}\sum_{i=1}^N (C_t^i)^\gamma$, $\gamma \in (0,1)$. Take $S(x) = \frac{\eta x^\gamma}{\gamma}$, $\eta > 0$, and

$$U(C_t^i, C_t^{(N,\gamma)}) = \frac{1}{\gamma}(C_t^i)^{\gamma(1-\lambda)}\left[\frac{(C_t^i)^\gamma}{C_t^{(N,\gamma)}}\right]^\lambda$$
$$\left(= \left[\frac{1}{\gamma}(C_t^i)^\gamma\right]^{1-\lambda}\left[\frac{1}{\gamma}\frac{(C_t^i)^\gamma}{C_t^{(N,\gamma)}}\right]^\lambda =: U_0^{1-\lambda}U_1^\lambda\right).$$

Interpretation of U: as a **weighted geometric mean** of U_0 (own utility) and U_1 (relative utility).

Try conventional methods to solve, for instance, Example 1':

$$dX_t^i = (A_{\theta_i} X_t^i + Bu_t^i + GX_t^{(N)})dt + DdW_t^i, \quad 1 \le i \le N$$

$$J_i(u^i, u^{-i}) = E\int_0^\infty e^{-\rho t}\{|X_t^i - (\Gamma X_t^{(N)} + \eta)|_Q^2 + (u_t^i)^T R u_t^i\}dt$$

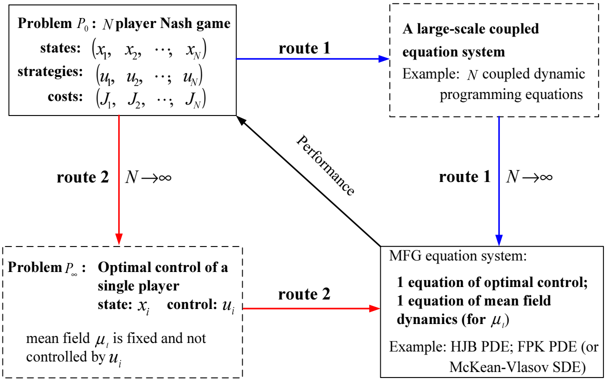
- Form large vector $X_t := [X_t^1, \dots, X_t^N]^T$. Similarly, $u_t = [u_t^1, \dots, u_t^N]^T$
- This gives a standard dynamic game

Immediate difficulties (for large N):

- Solve a **very large scale** dynamic programming problem
- In the resulting solution, each u_t^i will be a function of X_t, which is **too much information** and may be unnecessary

Such difficulties motivate the development of MFG theory!

The fundamental diagram of MFG theory



- blue: direct approach (Lasry and Lions, 06, 07); red: fixed point approach (Huang, Caines, Malhamé, 03, 07), (Huang, Malhamé, Caines, 06); see overview of the two approaches in (Caines, Huang, Malhamé, 17)

Before analyzing MFG, we mention the math toolkit for optimal control

- Dynamic programming (DP) (in Markov control models)
- Stochastic maximum principle (SMP)/calculus of variations

SMP is useful when the optimal control problem does not have Markov properties (example: random model coefficients)

Here we give brief preliminaries on continuous time DP

Consider the optimal control problem –

Dynamics and cost:

$$dX_t = (AX_t + Bu_t + G\bar{X}_t)ds + DdW_t,$$
$$J(u_{(\cdot)}) = E\int_0^T e^{-\rho t} L(X_t, u_t)dt,$$

where $L = |X_t - (\mathit{\Gamma}\bar{X}_t + \eta)|_Q^2 + u_t^T R u_t$, $\rho > 0$, $Q \geq 0$, $R > 0$. $\bar{X} \in C[0, T]$. Moreover $|\bar{X}_t| = O(e^{t(\rho-\epsilon)/2})$ for some small $\epsilon > 0$ (depending on $\bar{X}$) if $T = \infty$; denote $\bar{X} \in C_{\rho/2}[0, \infty)$.

u_t is $\mathcal{F}_t^W$-adapted (i.e., adapted to the filtration of the BM). Alternatively, one may take u_t to be X_t dependent (i.e. state feedback).

We consider two cases: i) T is finite, ii) $T = \infty$

By the method of DP, we try to solve a family of optimization problems with initial pair (t, x). Consider

$$dX_r = (AX_r + Bu_r + G\bar{X}_r)dr + DdW_r, \qquad r \geq t, \ X_t = x,$$
$$J(t, x, u_{(\cdot)}) = E_{t,x} \int_t^T e^{-\rho(r-t)} L(X_r, u_r) dr,$$

where $E_{t,x}$ indicates $X_t = x$, and u_r is $\mathcal{F}_r^W$-adapted.

Define the value function

$$V(t, x) = \inf_u E_{t,x} \int_t^T e^{-\rho(r-t)} L(X_r, u_r) dr.$$

By DP, for $\delta > 0$,

$$V(t,x) = \inf E[\int_t^{t+\delta} e^{-\rho(r-t)} L(X_r, u_r)dr + e^{-\rho\delta} V(t+\delta, X_{t+\delta})],$$

and we further obtain the HJB equation

$$\begin{aligned}
\rho V &= V_t + V_x^T(Ax + G\bar{X}_t) + |x - (\Gamma\bar{X}_t + \eta)|_Q^2 \\
&\quad + \frac{1}{2}Tr(D^T V_{xx} D) + \min_u [u^T R u + V_x^T B u] \\
&= V_t - \frac{1}{4} V_x^T B R^{-1} B^T V_x + V_x^T(Ax + G\bar{X}_t) \\
&\quad + |x - (\Gamma\bar{X}_t + \eta)|_Q^2 + \frac{1}{2}Tr(D^T V_{xx} D).
\end{aligned}$$

And $V(T,x) = 0$ when T is finite. V_t, V_x, V_{xx}: partial derivatives.

See e.g. Fleming and Rishel (1975), Yong and Zhou (1999)

i) For finite T, denote

$$V(t,x) = x^T P(t)x + 2s^T(t)x + q(t).$$

We derive differential Riccati equation (DRE)

$$\begin{cases} \rho P(t) = \dot{P}(t) - P(t)BR^{-1}B^T P(t) + P(t)A + A^T P(t) + Q, \\ P(T) = 0. \end{cases}$$

$$\begin{cases} \rho s(t) = \dot{s}(t) + (A^T - PBR^{-1}B^T)s(t) + PG\bar{X}_t - Q(\Gamma\bar{X}_t + \eta), \\ s(T) = 0. \end{cases}$$

Fact: The DRE has a unique solution (due to $Q \geq 0$ and $R > 0$). Use the minimizer in the HJB to give the optimal control

$$u_t^{\mathrm{opt}} = R^{-1}B^T(P(t)X_t + s(t)).$$

ii) For $T = \infty$,

- the DRE is replaced by an algebraic Riccati equation (ARE) which has a unique solution $\Pi \geq 0$.
- In this case, the ODE of s has no terminal condition;
- a growth condition can be imposed to determine a unique solution for s.

Solve Example 1' by the fixed point approach

- We use the LQ model to illustrate this methodology
- The procedure here is very robust and can be applied to different models

Fixed approach to the LQ MFG (as in Ex. 1'): Problem P_0 –

$$dX_t^i = (A_{\theta_i} X_t^i + Bu_t^i + GX_t^{(N)})dt + DdW_t^i, \quad 1 \le i \le N$$
$$J_i(u^i, u^{-i}) = E\int_0^\infty e^{-\rho t}\left\{|X_t^i - (\Gamma X_t^{(N)} + \eta)|_Q^2 + (u_t^i)^T R u_t^i\right\} dt$$

Assume $x_0^{(N)} = \frac{1}{N}\sum_{i=1}^N EX_0^i \to \bar{X}_0$.

Taking **approximation**: Optimal Control Problem P_∞ –

$$dX_t^i = (A_{\theta_i} X_t^i + Bu_t^i + G\bar{X}_t)dt + DdW_t^i, \quad 1 \le i \le N$$
$$J_i(u^i, u^{-i}) = E\int_0^\infty e^{-\rho t}\left\{|X_t^i - (\Gamma \bar{X}_t + \eta)|_Q^2 + (u_t^i)^T R u_t^i\right\} dt$$

- **Step 1.** Solve P_∞ assuming the deterministic $\bar{X}_t$ **were known**
- **Step 2.** Use **consistency condition** of MFG to find an equation for $\bar{X}_t$

Step 1. Denote (we consider $A_{\theta_i} \equiv A$ just for simplicity)

Riccati equation: $\rho \Pi = \Pi A + A^T \Pi - \Pi B R^{-1} B^T \Pi + Q$

and (for $s \in C_{\rho/2}$; no given initial condition)

$$\text{ODE:} \quad \rho s_t = \frac{ds_t}{dt} + (A^T - \Pi B R^{-1} B^T) s_t + \Pi G \bar{X}_t - Q(\Gamma \bar{X}_t + \eta)$$

Best response (BR): $\hat{u}_t^i = -R^{-1} B^T (\Pi X_t^i + s_t)$.

Step 2. **Closed-loop** for P_∞:

$$dX_t^i = (A - BR^{-1}B^T \Pi) X_t^i dt - BR^{-1}B^T s_t dt + G\bar{X}_t dt + D dW_t^i$$

$$\Longrightarrow \frac{d\bar{X}_t}{dt} = (A - BR^{-1}B^T \Pi)\bar{X}_t + G\bar{X}_t - BR^{-1}B^T s_t,$$

which is obtained from **averaging** X_t^i with $\bar{X}_0$ determined.

Summarizing, the MFG equation system is

$$\begin{cases} \frac{d\bar{X}_t}{dt} = (A - BR^{-1}B^T\Pi + G)\bar{X}_t - BR^{-1}B^T s_t, \\ \rho s_t = \frac{ds_t}{dt} + (A^T - \Pi BR^{-1}B^T)s_t + \Pi G\bar{X}_t - Q(\Gamma\bar{X}_t + \eta), \end{cases}$$

where $\bar{X}_0$ is given. Look for $s \in C_{\rho/2}[0,\infty)$.

Analysis and results:

This is carried out in the form of a list of Q&A's.

Question 1: Does there **exist** a unique solution $(\bar{X}, s)$?

Note: s only has a growth condition

Answer: under some conditions, there exists a unique solution.

Method: use fixed point theorem (write $\bar{X}$ as the fixed point of a linear operator), or subspace decomposition of the ODE system

Question 2. How is the solution used in the N player model?

Answer: provide **decentralized strategies**

The set of **decentralized strategies**, denoted as $\hat{u}$, for the N players:

$$\hat{u}_t^i = -R^{-1}B^T(\Pi X_t^i + s_t), \quad 1 \le i \le N.$$

Key Feature: Player i does not need the state or sample path information of other players.

Question 3: What is the **performance** of the set of decentralized strategies?

To analyze performance, we need to introduce strategy spaces.

$\mathcal{U}^i_{\rm centr}$ consists of strategies of the form $u^i(t, x^1, \ldots, x^N)$; $|u^i(t, x^1, \ldots, x^N)| \leq C(1 + |x|)$; and u^i is Lipschitz in $x = (x^1, \ldots, x^N)$. This ensures well defined SDEs.

- u^i is a **centralized strategy**, here depending on states of all players
- u^i is a Markov feedback strategy

Although decentralized strategies are desired, the consideration of $\mathcal{U}^i_{\rm centr}$ is useful for characterizing performance.

To answer Q3 on performance –

Main assumptions:

i) $X_0^i, i \geq 1$, are independent and also independent of the BMs $W^i, i \geq 1$.
ii) $\sup_i E|X_0^i|^2 \leq C$.
iii) The average of initial means $x_0^{(N)} \to \bar{X}_0$ as $N \to \infty$.

Question 3 (restated): What is the **performance** of the set of decentralized strategies?

Theorem: If the MFG equation system has a unique solution, the set of decentralized strategies is an ϵ-**Nash equilibrium**, i.e., for each i,

$$\inf_{u_i \in \mathcal{U}^i_{\mathrm{centr}}} J_i(u^i, \hat{u}^{-i}) \geq J_i(\hat{u}^i, \hat{u}^{-i}) - \epsilon,$$

where $\epsilon = O(|x_0^{(N)} - \bar{X}_0| + 1/\sqrt{N})$.

Interpretation: using **full information** by one deviating player offers little benefit to itself.

Question 4: How much **efficiency loss** occurs in the MFG?

In order to answer this question, we need to compare with a social optimization problem.

This is doable; we postpone this until introducing the social optimization problem to minimize $J_{\mathrm{soc}}^{(N)} = \sum_i J_i$.

Question 5: **Heterogeneous** agents?

Answer: the MFG equation system will be modified.

Assume empirical distribution for the parameter sequence $\{\theta_i, i \geq 1\}$

The intuition behind is to average non-uniform “particles”

Nonlinear case – Interacting particle modeling for MFG:

N agents (players), $1 \le i \le N$. Dynamics and costs:

$$dx_i = \frac{1}{N}\sum_{j=1}^{N} f(x_i, u_i, x_j)dt + \frac{1}{N}\sum_{j=1}^{N} \sigma(x_i, u_i, x_j)dw_i, \quad \text{each } x_j \in \mathbb{R}^n$$

$$J_i(u_i, u_{-i}) = E\int_0^T \frac{1}{N}\sum_{j=1}^{N} L(x_i, u_i, x_j)dt, \qquad 1 \le i \le N.$$

Assume i.i.d. initial states. (u_{-i}: controls of all players other than player i)

As in the LQ case, a key step is to **solve a special single agent optimal control problem**.

How to construct such a problem?

Recall

$$dx_i = \frac{1}{N}\sum_{j=1}^{N} f(x_i, u_i, x_j)dt + \frac{1}{N}\sum_{j=1}^{N} \sigma(x_i, u_i, x_j)dw_i, \quad \text{each } x_j \in \mathbb{R}^n$$

$$J_i(u_i, u_{-i}) = E\int_0^T \frac{1}{N}\sum_{j=1}^{N} L(x_i, u_i, x_j)dt, \qquad 1 \le i \le N.$$

Write $\delta_x^{(N)} = \frac{1}{N}\sum_{j=1}^{N} \delta_{x_j}$, $\delta_\bullet$: dirac measure. For $\varphi = f, \sigma, L$, denote

$$\frac{1}{N}\sum_{j=1}^{N} \varphi(x_i, u_i, x_j) = \int_{\mathbb{R}^n} \varphi(x_i, u_i, y)\delta_x^{(N)}(dy) := \varphi[x_i, u_i, \delta_x^{(N)}]$$

Idea: approximate $\delta_x^{(N)}$ by a deterministic measure μ_t

For simplicity, consider constant σ and scalar state. The optimal control problem with dynamics and cost:

$$dx_i = f[x_i, u_i, \mu_t]dt + \sigma dw_i, \quad \text{each } x_j \in \mathbb{R}$$
$$\bar{J}_i(u_i) = E\int_0^T L[x_i, u_i, \mu_t]dt,$$

where μ_t is called **the mean field**.

Again, following the method in the LQ MFG case –

- Assume $\{\mu_t, 0 \le t \le T\}$ were known; solve the optimal control $\hat{u}_i$.
- **Consistency condition** – the closed-loop state process under $\hat{u}_i$ has its marginal equal to μ_t.
- The analysis can be done for a finite classes/types of agents.

HJB–McKV/FPK for the nonlinear diffusion model:

- HJB equation (scaler state for simplicity):

$$\frac{\partial V}{\partial t} = \inf_{u\in U}\left\{ f[x,u,\mu_t]\frac{\partial V}{\partial x} + L[x,u,\mu_t]\right\} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2}$$
$$V(T,x) = 0, \qquad (t,x)\in[0,T)\times\mathbb{R}.$$
$$\Downarrow$$
$$\text{Best Response}: \quad u_t = \varphi(t,x|\mu_\cdot), \quad (t,x)\in[0,T]\times\mathbb{R}.$$

- Closed-loop McK-V equation (which can be written as a Fokker-Planck equation):

$$dx_t = f[x_t, \varphi(t,x|\mu_\cdot), \mu_t]dt + \sigma dw_t, \quad 0\le t\le T.$$

Find a solution (x_t, μ_t) in McK-V sense, i.e., $\text{Law}(x_t) = \mu_t$.

Refs. Huang, Caines and Malhame'06, Lasry and Lions (06, 07), Cardaliaguet'12

Lasry and Lions (2006, 2007) considered dynamics and costs

$$dx_t^i = -u^i dt + \sigma dw_t^i, \qquad x^i \in \mathbb{R}^d, \sigma > 0,$$

$$J^i(u^1, \ldots, u^N) = \liminf_{T\to\infty} \frac{1}{T} E \int_0^T (L(x^i, u^i) + V[x^i, \mu_t^{N,-i}])dt,$$

where $\mu_t^{N,-i}$ is the empirical distribution of states of other agents. The second argument of V means it is a functional of the distribution for given x^i. Assume periodicity of F and V in the cost. Denote $Q = [0,1]^d$.

Solve the N-player game, and show the solution converges to a limiting HJB-FPK equation system

$$-\frac{\sigma^2}{2}\Delta\nu(x) = H(x, \bigtriangledown\nu) + \lambda = V[x, m],$$

$$-\frac{\sigma^2}{2}\Delta m - div(\frac{\partial H}{\partial p}(x, \bigtriangledown\nu)m) = 0,$$

where $\int_Q \nu(x)dx = 0$, $\int_Q m(x) = 1$, $m \geq 0$.

H: Hamiltonian, λ: long run average cost, $\nu(x)$: differential cost

Finite horizon can also be considered for this model.

Dynamics of agent i:

$$dx_t^i = u_t^i dt + \sqrt{2} dw_t^i, \qquad x_t^i \in \mathbb{R}^d;$$

cost functional:

$$J_i^N(u_i, u_{-i}) = E \int_0^T [(u_t^i)^2 + F(x_t^i, \mu_t^{N,-i})] dt + EG(x_t^i, \mu_t^{N,-i}),$$

where $\mu_t^{N,-i}$ is the empirical distribution of the states of all other agents.

See Lasry and Lions'07, also notes Cardaliaguet'12

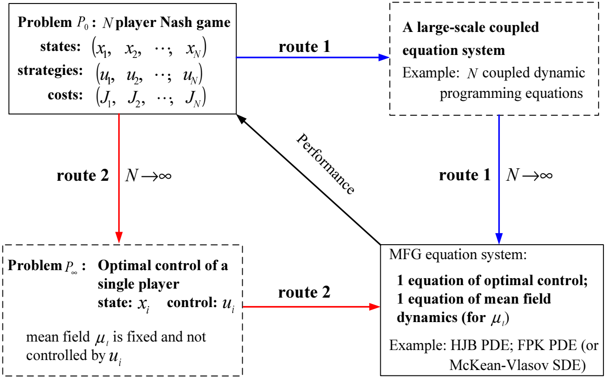
The MFG equations:

$$
\begin{aligned}
&\partial_t V - \Delta V + \frac{1}{2}|DV|^2 = F(x, m), \quad (x, t) \in \mathbb{R}^d \times (0, T) \\
&\partial_t m - \Delta m - \operatorname{div}(mDV) = 0, \\
&m(0) = m_0, \quad V(x, T) = G(x, m(T)), \quad x \in \mathbb{R}^d,
\end{aligned}
$$

where $V(t, x)$ and $m(t, x)$ are the value function and the density of the state distribution, respectively.

See an ϵ-Nash equilibrium result in Cardaliaguet'12

This relates the MFG solution to finite population games



So for nonlinear diffusion models, the diagram reduces to the structure on next page

The basic framework of MFGs: Math. Aspects

$$\begin{cases} P_0\text{—Game with } N \text{ players; Example} \\ dx_i = f(x_i, u_i, \delta_x^{(N)})dt + \sigma(\cdots)dw_i \\ J_i(u_i, u_{-i}) = E\int_0^T l(x_i, u_i, \delta_x^{(N)})dt \\ \delta_x^{(N)}: \text{empirical distribution of } (x_j)_{j=1}^N \end{cases} \xrightarrow{\text{solution}} \begin{array}{l} \text{HJBs coupled via densities } p_{i,t}^N,\ 1 \le i \le N \\ +N \text{ Fokker-Planck-Kolmogorov equations} \\ u_i \text{ adapted to } \sigma(w_i(s), s \le t) \\ (i.e., \text{ restrict to decentralized info} \\ \text{for } N \text{ players}); \text{ so giving } u_i^N(t, x_i) \end{array}$$

$\downarrow$ construct $\qquad \nwarrow$ performance? $\qquad$ (subseq. convergence) $\downarrow N \to \infty$

$$\begin{cases} P_\infty\text{—Limiting problem, 1 player} \\ dx_i = f(x_i, u_i, \mu_t)dt + \sigma(\cdots)dw_i \\ \bar{J}_i(u_i) = E\int_0^T l(x_i, u_i, \mu_t)dt \\ \text{Freeze } \mu_t, \text{ as approx. of } \delta_x^{(N)} \end{cases} \xrightarrow{\text{solution}} \begin{cases} \hat{u}_i(t, x_i): \text{ optimal response} \\ \text{HJB (with } v(T, \cdot) \text{ given):} \\ -v_t = \inf_{u_i}(f^T v_{x_i} + l + \frac{1}{2}Tr[\sigma\sigma^T v_{x_i x_i}]) \\ \text{Fokker-Planck-Kolmogorov:} \\ p_t = -div(fp) + \sum((\frac{\sigma\sigma^T}{2})_{jk} p)_{x_i^j x_i^k} \\ \text{Coupled via } \mu_t \text{ (w. density } p_t; p_0 \text{ given)} \end{cases}$$

- The consistency based approach (red) is more popular; related to ideas in statistical physics (McKean-Vlasov eqn); FPK can be replaced by an MV-SDE

Applications:

- Economic theory
- Finance
- Communication networks
- Social opinion formation
- Power systems
- Electric vehicle recharging control
- Public health (vaccination games)
- ......

Some applications of MFGs to economic growth and finance.

- Guéant, Lasry and Lions (2011): human capital optimization
- Lucas and Moll (2011): Knowledge growth and allocation of time
- Carmona and Lacker (2013): Investment of n brokers
- Huang (2013): capital accumulation with congestion effect
- Lachapelle et al. (2013): price formation
- Espinosa and Touzi (2013): Optimal investment with relative performance concern (depending on $\frac{1}{N-1}\sum_{j\neq} X_j$)
- Jaimungal and Nourian (2014): Optimal execution
- Chan and Sircar (2015): Bertrand and Cournot models
- more ......

References

Only some papers are sampled and inserted into the lecture.

For further literature, see these papers and books, and references therein.

Bensoussan, Frehse, and Yam. Springer Brief 2013.
Caines. Mean field games, in Encyclopedia of Systems and Control, Springer-Verlag, 2014.
Caines, Huang, and Malhame, in Springer dynamic game theory handbook, 2017.
Cardaliaguet. Notes on mean field games, 2012.
Carmona and Delarue. Probabilistic Theory of MFGs, vol I and II, Cham: Springer, 2018.
Gomes and Saud. Mean field games models – a brief survey. 2014.
Huang, Caines, and Malhame. 2003 CDC, IEEE TAC'07.
Huang, Malhame and Caines, CIS'06.
Lasry and Lions. Mean field games, 2006, 2007.
Weintraub, Benkard, and Van Roy (2008). Econometrica; for oblivious equilibria

Thank you!

Note added July 20, 2018

The primary goal of this tutorial (Workshop on Game Theory, Institute for Math. Sci., NUS, June 2018, organizing co-chairs: Profs. Y.-C. Chen and Y. Sun) is to present the basic ideas of MFGs to researchers who have interest but are non-specialists in this area. I try to make a modest usage of technical tools, and mostly use dynamic programming. The important approach via maximum principle is not covered here and can be checked from the references supplied at the end.

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