Introduction to Mean Field Game Theory Part II

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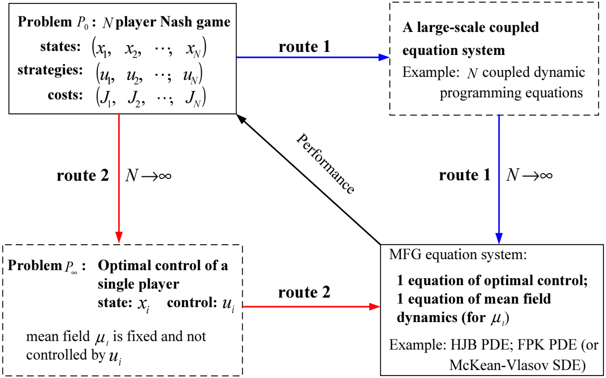
Outline of talk

- Nonlinear case (ctn), application to stochastic growth
- MFG with a major player
- Mean field social optimization
- Relation of two approaches in the fundamental diagram
- References

In Part I, main materials have been organized around this diagram:



For nonlinear diffusion models –

- We may formalize a single agent optimal control problem; we further elaborate below
- Lasry and Lions (2007) solve the N player games and show the solutions converge along a subsequence to a limiting HJB-FPK equation system

$$-\frac{\sigma^2}{2}\Delta\nu(x) = H(x, \nabla\nu) + \lambda = V[x, m],$$
$$-\frac{\sigma^2}{2}\Delta m - div(\frac{\partial H}{\partial p}(x, \nabla\nu)m) = 0,$$

where $\int_Q \nu(x)dx = 0$, $\int_Q m(x) = 1$, density $m \geq 0$. $Q = [0,1]^d$.
H: Hamiltonian, λ: long run average cost, $\nu(x)$: differential cost

How to construct the optimal control problem?

Dynamics and costs for N agents:

$$dx_i = \frac{1}{N}\sum_{j=1}^{N} f(x_i, u_i, x_j)dt + \frac{1}{N}\sum_{j=1}^{N} \sigma(x_i, u_i, x_j)dw_i, \quad \text{each } x_j \in \mathbb{R}^n$$

$$J_i(u_i, u_{-i}) = E\int_0^T \frac{1}{N}\sum_{j=1}^{N} L(x_i, u_i, x_j)dt, \qquad 1 \le i \le N.$$

Assume i.i.d. initial states. (u_{-i}: controls of all other players)

Write $\delta_x^{(N)} = \frac{1}{N}\sum_{j=1}^N \delta_{x_j}$. Approximate $\delta_x^{(N)}$ by a deterministic meas. μ_t.

The idea is to approximate summation by integration

The optimal control problem:

$$dx_i = f[x_i, u_i, \mu_t]dt + \sigma[x_i, u_i, \mu_t]dw_i,$$
$$\bar{J}_i(u_i) = E\int_0^T L[x_i, u_i, \mu_t]dt,$$

where μ_t is called **the mean field**. For instance,

$$f[x_i, u_i, \mu_t] = \int_y f(x_i, u_i, y)\mu_t(dy)$$

- The solution gives an HJB equation
- By the fixed point approach, apply the **consistency condition**
- This can be generalized to multiple classes/types of agents

HJB–McKV/FPK for the nonlinear diffusion model:

- HJB equation (scaler state for simplicity):

$$\frac{\partial V}{\partial t} = \inf_{u \in U} \left\{ f[x, u, \mu_t] \frac{\partial V}{\partial x} + L[x, u, \mu_t] \right\} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2}$$
$$V(T, x) = 0, \qquad (t, x) \in [0, T) \times \mathbb{R}.$$
$$\Downarrow$$
$$\text{Best Response}: \quad u_t = \varphi(t, x | \mu_\cdot), \quad (t, x) \in [0, T] \times \mathbb{R}.$$

- Closed-loop McK-V equation (which can be written as a Fokker-Planck equation):

$$dx_t = f[x_t, \varphi(t, x|\mu_\cdot), \mu_t]dt + \sigma dw_t, \quad 0 \le t \le T.$$

Find a solution (x_t, μ_t) in McK-V sense, i.e., $\mathrm{Law}(x_t) = \mu_t$.
Show ϵ-Nash equilibrium

Refs. Huang, Caines and Malhame'06, Lasry and Lions (06, 07), Cardaliaguet'12

Now we apply the fixed point approach to solve the

stochastic growth problem with relative performance

as described in Example 3 of Part I.

The idea of relative performance in economic literature:

- Abel (1990), Amer. Econ. Rev.
- Hori and Shibata (2010), J. Optim. Theory Appl.
- Turnovsky and Monterio (2007), Euro. Econ. Rev.
- ...

MFG with relative performance (Espinosa and Touzi, 2013)

- Geometric BM dynamics for risky assets
- The performance of agent (manager) i ($i = 1, 2, \dots, N$):

$$EU\left[(1-\lambda)X_T^i + \lambda(X_T^i - X_T^{(-i)})\right], \qquad X_T^{(-i)} = \frac{1}{N-1}\sum_{j\neq i} X_T^j,\ 0<\lambda<1$$

Feature: This is difference-like comparison

Growth with relative performance

Dynamics of N agents:

$$dX_t^i = \left[A(X_t^i)^\alpha - \delta X_t^i - C_t^i\right] dt - \sigma X_t^i dW_t^i, \quad 1 \le i \le N$$

- X_t^i: capital stock, $X_0^i > 0$, $EX_0^i < \infty$, C_t^i: consumption rate
- Ax^α, $\alpha \in (0,1)$: Cobb-Douglas production function, $0 < \alpha < 1$, $A > 0$
- $\delta dt + \sigma dW_t^i$: stochastic depreciation (see e.g. Wälde'11, Feicht and Stummer'10 for stochastic depreciation modeling)
- $\{W_t^i, 1 \le i \le N\}$ are i.i.d. standard Brownian motions. The i.i.d. initial states $\{X_0^i, 1 \le i \le N\}$ are also independent of the N Brownian motions
- Take the standard choice $\gamma = 1 - \alpha$, i.e., equalizing the coefficient of the relative risk aversion to capital share

See Huang and Nguyen (2016)

The utility functional of agent i:

$$J_i(C^1,\dots,C^N) = E\left[\int_0^T e^{-\rho t} U(C_t^i, C_t^{(N,\gamma)})dt + e^{-\rho T} S(X_T)\right],$$

where $C_t^{(N,\gamma)} = \frac{1}{N}\sum_{i=1}^N (C_t^i)^\gamma$, $\gamma \in (0,1)$. Take $S(x) = \frac{\eta x^\gamma}{\gamma}$, $\eta > 0$, and

$$U(C_t^i, C_t^{(N,\gamma)}) = \frac{1}{\gamma}(C_t^i)^{\gamma(1-\lambda)} \left[\frac{(C_t^i)^\gamma}{C_t^{(N,\gamma)}}\right]^\lambda$$

$$\left(= \left[\tfrac{1}{\gamma}(C_t^i)^\gamma\right]^{1-\lambda} \left[\tfrac{1}{\gamma}\tfrac{(C_t^i)^\gamma}{C_t^{(N,\gamma)}}\right]^\lambda =: U_0^{1-\lambda} U_1^\lambda\right).$$

Interpretation of U: as a **weighted geometric mean** of U_0 (own utility) and U_1 (relative utility).

The infinite population limit:
i) A representative agent

$$dX_t = (AX_t^{1-\gamma} - \delta X_t - C_t)dt - \sigma X_t dW_t, \quad t \ge 0.$$

No need of superscript i to label the agent. $X_0 > 0$.
ii) The utility functional

$$\bar{J}(C(\cdot)) = E\left[\int_0^T e^{-\rho t} U(C_t, \bar{C}_t^{(\gamma)})dt + e^{-\rho T} S(X_T)\right],$$

$$U(C_t, \bar{C}_t^{(\gamma)}) = \frac{1}{\gamma}\left[C_t^\gamma\right]^{1-\lambda}\left[\frac{C_t^\gamma}{\bar{C}_t^{(\gamma)}}\right]^\lambda, \quad S(X_T) = \frac{\eta X_T^\gamma}{\gamma}.$$

Consider $\bar{C}^{(\gamma)}(\cdot) \in C([0,T];\mathbb{R}^+)$.

- $\bar{C}_t^{(\gamma)}$ is used to approximate $C^{(N,\gamma)} = \frac{1}{N}\sum_{i=1}^N (C_t^i)^\gamma$
- The best responses of all agents regenerate $\bar{C}_t^{(\gamma)}$ (giving a fixed point)
- These two steps derive an appropriate **fixed point equation (FPE)** $b_t = \Gamma(b)_t, \quad t \in [0,T]$, where $b \in C([0,T],\mathbb{R}^+)$

Detail: The fixed point problem to **determine** $\bar{C}_t^{(\gamma)}$. Define $B_t = \left(\bar{C}_t^{(\gamma)}\right)^\lambda$ and $b_t = B_t^{\frac{1}{\gamma-1}}$. It is more convenient to work with b.

Then we derive a **fixed point equation** (FPE)

$$b_t = \Gamma(b)_t, \quad t \in [0, T].$$

Here Γ is obtained as follows:

$$\begin{aligned} \Gamma_0(b)_t &= p^{\frac{1}{\gamma-1}}(t) = \left[e^{a(t-T)}\eta^{\frac{1}{1-\gamma}} + e^{at}\int_t^T e^{-as} b_s ds\right]^{-1} \\ \Gamma_1(b)_t &= b_t \Gamma_0(b)_t \\ \Lambda(b)_t &= (EX_t^\gamma)^{\frac{1}{\gamma}}, \quad 0 \le t \le T. \end{aligned}$$

$a > 0$ is a constant determined from $(\gamma, \sigma^2, \delta, \rho)$. EX_t^γ satisfies a linear ODE! Finally

$$\Gamma(b)_t = [\Gamma_1(b)_t \Lambda(b)_t]^{\frac{\lambda\gamma}{\gamma-1}}$$

The individual strategy is a linear feedback

$$\hat{C}_t^i = b_t \Big[e^{a(t-T)} \eta^{\frac{1}{1-\gamma}} + e^{at} \int_t^T e^{-as} b_s ds \Big]^{-1} X_t^i, \qquad 1 \le i \le N.$$

Theorem 1 Suppose that $b \in C([0,T]; \mathbb{R}^+)$ is a solution of **FPE** with $\lambda > 0$ and the i.i.d. initial conditions X_0^i satisfy $E|X_0^i|^{2\gamma} < \infty$. Then

$$E|\hat{C}_t^{(N,\gamma)} - \bar{C}_t^{(\gamma)}|^2 = O\Big(\frac{1}{N}\Big).$$

Theorem 2 (ϵ-**Nash Equilibrium**) Under the conditions of Theorem 1, we have

$$J_i(\hat{C}^i, \hat{C}^{-i}) \le \sup_{C^i(\cdot)\in\mathcal{U}_i} J_i(C^i, \hat{C}^{-i}) \le J_i(\hat{C}^i, \hat{C}^{-i}) + \varepsilon_N,$$

where $\varepsilon_N = O(1/\sqrt{N})$. $\mathcal{U}_i$ is the set of centralized strategies.

Note: Existence of a solution to FPE is established under a contraction condition

Numerics:

We solve the fixed equation $b = \Gamma(b)$ with the following parameters

$$T = 2,\ A = 1,\ \delta = 0.05,\ \gamma = 0.6,\ \eta = 0.2,\ \rho = 0.04,\ \sigma = 0.08$$

- λ will take three different values $0.1, 0.3, 0.5$ for comparisons.
- See Feicht and Stummer (2010) for typical parameter values in capital growth models with stochastic depreciation.
- Time is discretized with step size 0.01.

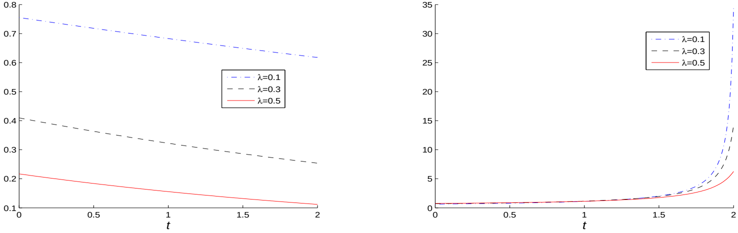


Figure : *Left*: b_t solved from $b = \Gamma(b)$; *right*: $b_t \Gamma_0(b)_t$ (as control gain)

When the agent is more concerned with the relative utility (i.e., taking larger λ), it tends to consume with more caution during the late stage

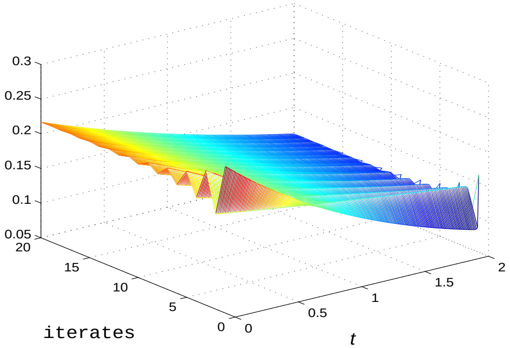


Figure : The computation of b_t in the first 20 iterates by operator Γ, $\lambda = 0.5$.

MFG with a major player – Dynamics:

$$dx_0(t) = \left[A_0 x_0(t) + B_0 u_0(t) + F_0 x^{(N)}(t)\right] dt + D_0 dW_0(t), \quad t \geq 0,$$
$$dx_i(t) = \left[A(\theta_i) x_i(t) + B u_i(t) + F x^{(N)}(t) + G x_0(t)\right] dt + D dW_i(t),$$

$x^{(N)} = \frac{1}{N}\sum_{i=1}^{N} x_i$ mean field term (average state of minor players).

- Major player $\mathcal{A}_0$ with state $x_0(t)$, minor player $\mathcal{A}_i$ with state $x_i(t)$.
- W_0, W_i are independent standard Brownian motions, $1 \leq i \leq N$.

We introduce the following assumption:
(**A1**) θ_i takes its value from a finite set $\Theta = \{1, \dots, K\}$ with an empirical distribution $F^{(N)}$, which converges weakly when $N \to \infty$.

Individual costs:

The cost for $\mathcal{A}_0$:

$$J_0(u_0, \dots, u_N) = E \int_0^\infty e^{-\rho t} \Big\{ \big| x_0 - \Phi(x^{(N)}) \big|_{Q_0}^2 + u_0^T R_0 u_0 \Big\} dt,$$

$\Phi(x^{(N)}) = H_0 x^{(N)} + \eta_0$: cost coupling term

The cost for $\mathcal{A}_i$, $1 \le i \le N$:

$$J_i(u_0, \dots, u_N) = E \int_0^\infty e^{-\rho t} \Big\{ \big| x_i - \Psi(x_0, x^{(N)}) \big|_{Q}^2 + u_i^T R u_i \Big\} dt,$$

$\Psi(x_0, x^{(N)}) = H x_0 + \hat{H} x^{(N)} + \eta$: cost coupling term.

- The presence of x_0 in the dynamics and cost of $\mathcal{A}_i$ shows the strong influence of the major player $\mathcal{A}_0$.

A matter of "sufficient statistics"

One might conjecture asymptotic Nash equilibrium strategies of the form:

- $x_0(t)$ *would be sufficient statistic for* $\mathcal{A}_0$*'s decision* $\Longrightarrow u_0(t, x_0(t))$;
- $(x_0(t), x_i(t))$ *would be sufficient statistics for* $\mathcal{A}_i$*'s decision* $\Longrightarrow u_i(t, x_0(t), x_i(t))$.

Fact: The above conjecture fails!

Theorem (ε-Nash equilibrium) Under some technical conditions, for the $N+1$ players, a set of decentralized strategies of the form

$$(u_0[t, x_0(t), z(t)],\ u_i[t, x_0(t), z(t), x_i(t)],\ \ 1 \le i \le N)$$

is an ε-Nash equilibrium as $N \to \infty$. (see Huang, SICON'10 for detail.)

See (Nguyen and Huang, 12) for continuum parameter θ; (Nourian and Caines, 2013), (Carmona and Zhu, 2016) for nonlinear models.

The model structure is very rich.

- It allows different information/interaction structures
- In particular, leadership can be addressed. Moon and Basar (2014); Bensoussan et al (2016)
- Cooperation or partial cooperation: Buckdahn et al (2014), Huang and Nguyen (2016)

Other connections

The major player model has connection with the so-called common noise model introduced in Lions' lecture (see e.g. Gomes and Saude'14).

The master equation is a useful tool for analyzing MFG equations, particularly for models with common noise. The idea is to introduce a single equation to describe the solution of the MFG. It can be viewed as an abstract dynamic programming equation.

The LQ mean field social optimization problem

- Individual dynamics (N agents):

$$dx_i = A(\theta_i)x_i dt + Bu_i dt + DdW_i, \qquad 1 \leq i \leq N$$

- Individual costs:

$$J_i = E \int_0^\infty e^{-\rho t} \left\{ |x_i - \Phi(x^{(N)})|_Q^2 + u_i^T R u_i \right\} dt,$$

where $\Phi(x^{(N)}) = \Gamma x^{(N)} + \eta$

- Specification
  - $\theta_i, 1 \leq i \leq N$: dynamic parameter with empirical distr. $F_N \to F$ weakly, $Ex_i(0) = m_0$, u_i: control, W_i: noise
  - $x^{(N)} = (1/N) \sum_{i=1}^N x_i$: mean field coupling term
- The social cost: $J_{\rm soc}^{(N)} = \sum_{i=1}^N J_i$
- The objective: minimize $J_{\rm soc}^{(N)} \Longrightarrow$ Pareto optima

The SCE equation system

- The Social Certainty Equivalence (SCE) equation system:

$$
\begin{aligned}
\rho s_\theta &= \frac{ds_\theta}{dt} + (A_\theta^T - \Pi_\theta B R^{-1} B^T) s_\theta \\
&\quad - [(\Gamma^T Q + Q\Gamma - \Gamma^T Q \Gamma)\bar{x} + (I - \Gamma^T) Q\eta], \\
\frac{d\bar{x}_\theta}{dt} &= A_\theta \bar{x}_\theta - B R^{-1} B^T (\Pi_\theta \bar{x}_\theta + s_\theta), \\
\bar{x} &= \int \bar{x}_\theta dF(\theta),
\end{aligned}
$$

where $\bar{x}_\theta(0) = m_0$ and s_θ is sought within $C_{\rho/2}([0,\infty), \mathbb{R}^n)$ (i.e. continuous with a growth rate slower than $e^{\rho t/2}$)

$\Pi_\theta \geq 0$: uniquely solved from an Algebraic Riccati equation

How is the SCE equation system constructed?

- Different from the (Nash) MFG case since now agents are not selfish
- Key idea: the "right" balance of self interest and it social impact
- Indeed, agent i's effect on any other agent is negligible; however, its social impact may be significant. Think of road resource usage by selfish drivers; Wardrop equilibria

The social optimality theorem

Theorem Under technical conditions, the set of SCE based decentralized control laws

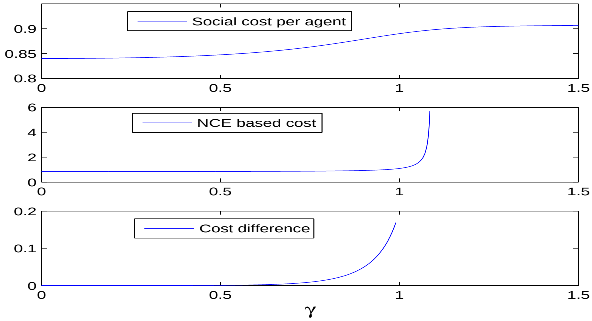
$$\hat{u}_i = -R^{-1}B^T(\Pi_{\theta_i}\hat{x}_i + s_{\theta_i}), \qquad 1 \le i \le N$$

has asymptotic social optimality, i.e., for $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_N)$,

$$|(1/N)J_{\rm soc}^{(N)}(\hat{u}) - \inf_{u \in \mathcal{U}_o}(1/N)J_{\rm soc}^{(N)}(u)| = O(1/\sqrt{N} + \bar{\epsilon}_N),$$

where $\lim_{N\to\infty}\bar{\epsilon}_N = 0$ (related to the weak convergence of $F_N(\theta)$) and $\mathcal{U}_o$ is defined as a set of centralized information based controls

Once we have solved the social optimization problem, we can return to the early question in Part I: **efficiency loss** in MFGs



Middle: per agent cost in MFG

Another basic question:

What is the relation of the two approaches

- direct approach
- fixed point approach

in the fundamental diagram?

Compare the two fundamental approaches

It is seen that the two approaches can be applied under **various sufficient conditions**.
It is unlikely to have a definite comparison if no concrete model classes are specified.

Here we consider **symmetric LQ game models** on $[0, T]$.

- The direct approach can be applied if and only if a nonsymmetric Riccati equation is solvable on $[0, T]$. This result is obtained by a **re-scaling technique** (Huang and Zhou, 2018).
- The fixed point approach determines a unique solution if and only if a certain algebraic equation is uniquely solvable.
- We compare **their domains of applicability**.

The LQ model – Dynamics:

$$dX_i(t) = \big(AX_i(t) + Bu_i(t) + GX^{(N)}(t)\big)dt + DdW_i(t), \quad 1 \le i \le N,$$

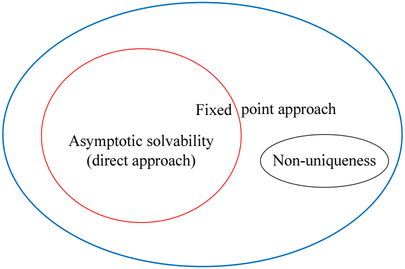
where the state $X_i \in \mathbb{R}^n$, control $u_i \in \mathbb{R}^{n_1}$, $X^{(N)} = \frac{1}{N}\sum_{k=1}^N X_k$, $W_i \in \mathbb{R}^{n_2}$: N independent Brownian motions (so, white noise).

Cost:

$$\begin{aligned} J_i = E\int_0^T \Big(|X_i(t) - \Gamma X^{(N)}(t) - \eta|_Q^2 + u_i^T(t)Ru_i(t)\Big)dt \\ + E|X_i(T) - \Gamma_f X^{(N)}(T) - \eta_f|_{Q_f}^2. \end{aligned}$$

The matrices (or vectors) A, B, G, D, Γ, Q, R, Γ_f, Q_f, η, η_f have compatible dimensions, and $Q \ge 0$, $R > 0$, $Q_f \ge 0$.

Relation of the two approaches –



Theorem. Asymptotic solvability (i.e., the direct approach is feasible) implies that the fixed point approach gives a unique solution.

References

Only some papers are sampled and inserted into the lecture.

For further literature, see these papers and books, and references therein.

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Thank you!

Note added July 20, 2018

The primary goal of this tutorial (Workshop on Game Theory, Institute for Math. Sci., NUS, June 2018, organizing co-chairs: Profs. Y.-C. Chen and Y. Sun) is to present the basic ideas of MFGs to researchers who have interest but are non-specialists in this area. I try to make a modest usage of technical tools, and mostly use dynamic programming. The important approach via maximum principle is not covered here and can be checked from the references supplied at the end.

The financial support of Institute for Math. Sci., NUS, and the organizers is gratefully acknowledged.