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Mean Field LQG Games with Mass Behavior Responsive to A Major Player

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Abstract— **We consider a linear-quadratic-Gaussian (LQG) game with a major player and a large number of minor players with mean field coupling. The state of the major player appears in the dynamics of the minor players, causing the mean field to be responsive to its control. We construct decentralized ε-Nash strategies. This is accomplished by combining (i) stochastic control with random coefficients and (ii) a procedure called anticipative variational calculations which addresses the major player's ability to simultaneously perturb its own state process and the mean field.**

I. INTRODUCTION

In the recent years there has been a very rapid accumulation of research on large population stochastic dynamic games with mean field coupling [1], [6], [11], [12], [14], [15], [16], [21], [23], [24], [26], [28]. Extensive efforts have been devoted to LQG models [3], [11], [12], [16], [21]. To obtain low complexity strategies, consistent mean field approximations provide a powerful approach, and in the resulting solution to a large but finite population model, each agent only needs to know its own state information and the off-line computable aggregate effect of the overall population. One may further establish an ε-Nash equilibrium property for the set of control strategies [12]. Consistent mean field approximations are also applicable to optimization with a social objective [7], [13], [23], estimation and filtering [22], [27], and recharging control of large populations of electric vehicles [17]. A maximum principle is developed for mean field control models in [2]. The survey [5] presents a timely report of recent progress in mean field game theory.

A natural generalization of the mean field game modeling has been introduced in [10] where a major player and a large number of minor players coexist. The major player model is extended to Markovian switching dynamics in [25]. Traditionally, games differentiating vastly different strengths of players have been well studied in cooperative game theory, and static models are usually considered [8], [9].

The LQG model in [10] contains minor players of a finite number of types. A state space augmentation approach was developed there by adding a new state which approximates the mean field and is described by a stochastic ordinary differential equation (ODE) driven by the major player's state. The use of this additional state Markovianizes the limiting decision problems. When the minor players are parametrized by a continuum set, the method in [10] faces challenges since it would lead to an infinite dimensional augmented state space. The work [18], [19], [20] treated a continuum parameter set by viewing the local control problems of the major player or a representative minor player as stochastic optimal control with random coefficient processes [4], and they were solved by use of adjoint equations in the form of linear backward stochastic differential equations (BSDEs) [4], [29], and subsequently consistent mean field approximations were developed. A limitation of the analysis in [18], [19], [20] is that it only deals with non-responsive mean field where the state of the major player does not appear in the dynamics of the minor players.

This paper considers responsive mean field by allowing the state of the major player to appear in the dynamics of the minor players as in [10]. Although we only consider homogeneous minor players, it is of interest to extend the BSDE based approach in [19], [20] since sufficient conditions on the existence of a solution can be developed by using linear operator techniques. By contrast, if the state space augmentation approach in [10] were applied to the finite horizon control problem in this paper, it would introduce a mean field ODE with time-varying coefficients due to the transient behavior of the mean field evolution. Then the consistent mean field approximation in [10] would lead to equality constraints on these coefficient functions. It is generally difficult to verify these constraints.

The paper is organized as follows. Section II introduces the game model. Section III describes anticipative variational calculations and the two limiting control problems. Sections IV and V solve the optimal control problems of the major player and the minor player, respectively. Consistent mean field approximations are analyzed in Section VI, and the equilibrium property is established in Section VII. Section VIII concludes the paper. Due to limited space, we omit all proofs of our results in this paper.

II. THE MEAN FIELD LQG GAME

The LQG game consists of a major player $\mathscr{A}_0$ and a population of minor players $\{\mathscr{A}_i, 1 \le i \le N\}$. At time $t \ge 0$, the state of player $\mathscr{A}_j$ is denoted by $x_j(t)$, $0 \le j \le N$. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge 0}, P)$ be a filtered probability space with the filtration $\{\mathscr{F}_t\}_{t\ge 0}$. The dynamics of the players are given by a system of linear stochastic differential equations (SDEs)

$$dx_0(t) = (A_0 x_0(t) + B_0 u_0(t) + F_0 x^{(N)}(t))dt + D_0 dW_0(t), \tag{1}$$

$$dx_i(t) = (Ax_i(t) + Bu_i(t) + Fx^{(N)}(t) + Gx_0(t))dt + DdW_i(t) \quad 1 \le i \le N, \tag{2}$$

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where $x^{(N)} = (1/N)\sum_{i=1}^{N} x_i$. When $G \neq 0$, the minor player receives a significant impact from the major player and any other minor player has only a negligible impact if N is large.

The states x_0, x_i and controls u_0, u_i are, respectively, n and n_1 dimensional vectors. The noise processes W_0, W_i are n_2 dimensional independent standard Brownian motions adapted to $\{\mathscr{F}_t\}_{t\geq 0}$, which are also independent of the initial states $\{x_j(0), 0 \leq j \leq N\}$. For simplicity, we may take the σ-algebra $\mathscr{F}_t = \sigma\{x_j(0), W_j(s), 0 \leq j \leq N, s \leq t\}$. The deterministic matrices A_0, B_0, F_0, D_0, A, B, F, G and D all have compatible dimensions. We will often drop time t in $x_0(t)$, $u_0(t)$, etc.

The cost function for $\mathscr{A}_0$ is given by

$$J_0 = E\int_0^T \left\{\left|x_0 - \Psi_0(x^{(N)})\right|_{Q_0}^2 + u_0^T R_0 u_0\right\}dt, \tag{3}$$

where $\Psi_0(x^{(N)}) = H_0 x^{(N)} + \eta_0$. Here and hereafter, we may write $z^T M z = |z|_M^2$ for a positive semi-definite matrix M. The cost function for $\mathscr{A}_i$, $1 \leq i \leq N$, is given by

$$J_i = E\int_0^T \left\{\left|x_i - \Psi(x_0, x^{(N)})\right|_{Q}^2 + u_i^T R u_i\right\}dt, \tag{4}$$

where $\Psi(x_0, x^{(N)}) = Hx_0 + \hat{H}x^{(N)} + \eta$. The component Hx_0 in the coupling term Ψ indicates the strong influence of the major player on each minor player. In (3) and (4), all the deterministic constant matrices or vectors H_0, H, $\hat{H}$, $Q_0 \geq 0$, $Q \geq 0$, $R_0 > 0$, $R > 0$, η_0 and η have compatible dimensions. We use $L^2_{\mathscr{F}}(0,T;\mathbb{R}^k)$ to denote all $\mathbb{R}^k$-valued random processes y defined on $[0,T]$ which are adapted to $\{\mathscr{F}_t\}_{t\geq 0}$ and $E\int_0^T |y(t)|^2 dt < \infty$.

A. Assumption

(**A**) The initial states $\{x_j(0), 0 \leq j \leq N\}$ are independent, $Ex_i(0) = 0$ for $1 \leq i \leq N$ and there is a constant C independent of N such that $\sup_{0\leq j\leq N} E|x_j(0)|^2 \leq C$. $\diamondsuit$

For simplicity we assume zero initial mean for the minor players and this condition may be relaxed.

III. Optimization with Anticipative Variational Calculations

We consider the approximation of the control problem of the major player. Let $x^{(N)}$ in (1) and (3) be approximated by a process z. This gives the dynamics

$$dx_0(t) = \big(A_0 x_0(t) + B_0 u_0(t) + F_0 z(t)\big)dt + D_0 dW_0(t) \tag{5}$$

and the cost

$$\bar{J}_0 = E\int_0^T \left\{\left|x_0 - \Psi_0(z)\right|_{Q_0}^2 + u_0^T R_0 u_0\right\}dt. \tag{6}$$

To avoid introducing too many variables, we still use x_0 to denote the state of the limiting control problem. The Brownian motion W_0 is the same as in (1). We may write $\bar{J}_0$ as $\bar{J}_0(x_0, u_0; z)$ and use (6) to define the cost associated with general processes (x_0', u_0', z') not necessarily satisfying (5).

Our objective is to find a solution pair $(\bar{x}_0, \bar{u}_0)$ such that the cost attains its minimum in some sense. Before solving the control problem, an immediate issue is how z should be specified and in what sense $\bar{J}_0$ is optimized. To proceed, let $P(t) \geq 0$ be the solution of the Riccati equation

$$\begin{cases} \dot{P} + PA + A^T P - PBR^{-1}B^T P + Q = 0, \\ P(T) = 0. \end{cases} \tag{7}$$

Definition 1: Let $z^* \in L^2_{\mathscr{F}}(0,T;\mathbb{R}^n)$ be given. We say $(\bar{x}_0, \bar{u}_0)$ is an *equilibrium solution* with respect to z^* if

(i) $(\bar{x}_0, \bar{u}_0, z^*)$ satisfies the SDE

$$d\bar{x}_0(t) = \big(A_0\bar{x}_0(t) + B_0\bar{u}_0(t) + F_0 z^*(t)\big)dt + D_0 dW_0(t),$$

(ii) $\bar{J}_0(\bar{x}_0, \bar{u}_0; z^*) \leq \bar{J}_0(\bar{x}_0 + \delta x_0, \bar{u}_0 + \delta u_0; z^* + \delta z)$ for all $\delta u_0 \in L^2_{\mathscr{F}}(0,T;\mathbb{R}^{n_1})$, where

$$d\delta x_0 = (A_0\delta x_0 + B_0\delta u_0 + F_0\delta z)dt, \tag{8}$$

$$d\delta z = \big((A - BR^{-1}B^T P + F)\delta z + G\delta x_0\big)dt, \tag{9}$$

$$\delta x_0(0) = \delta z(0) = 0. \tag{10}$$

$\diamondsuit$

We may simply call $\bar{u}_0$ an equilibrium solution. We have the relation

$$d(\bar{x}_0 + \delta x_0) = \big(A_0(\bar{x}_0 + \delta x_0) + B_0(\bar{u}_0 + \delta u_0) + F_0(z^* + \delta z)\big)dt + D_0 dW_0. \tag{11}$$

It should be noted that Definition 1 does not claim that

$$\bar{J}_0(\bar{x}_0, \bar{u}_0, z^*) \leq \bar{J}_0(x_0, u_0, z^*),\ \forall u_0 \in L^2_{\mathscr{F}}(0,T;\mathbb{R}^{n_1}),$$

where (x_0, u_0) satisfies (5) with $z = z^*$.

A. Some Heuristics on the Control Formulation

In order to enable an approximation to the strategy selection of the major player in the original game, the optimization of (x_0, u_0) in (5) should be anticipative with respect to z. Specifically, there is an implicit dependence of z on u_0. When u_0 changes to $u_0 + \delta u_0$, it causes a state variation δx_0 which in turn generates a state variation δx_i for a minor player. Subsequently, a large number of minor players contribute to a variation δz for the mean field. This responsive nature of the mean field does not appear in [19], [20] where the dynamics of each minor player do not contain x_0.

In order to specify δz as appearing in Definition 1, we describe the following replicating argument using N minor players. Suppose the minor player has a fixed control $u_i = -R^{-1}B^T(Px_i - \nu_i)$, where P is defined in (7) and ν_i is not affected by u_0. The state of the major player is x_0 corresponding to u_0. We have

$$dx_i = \big((A - BR^{-1}B^T P)x_i + BR^{-1}B^T\nu_i + Fx^{(N)} + Gx_0\big)dt + DdW_i.$$

Now we include a variation δx_0 due to δu_0 to obtain

$$d(x_i + \delta x_i) = \Big((A - BR^{-1}B^T P)(x_i + \delta x_i) + BR^{-1}B^T\nu_i + (1/N)F\sum_{i=1}^{N}(x_i + \delta x_i) + G(x_0 + \delta x_0)\Big)dt + DdW_i. \tag{12}$$

Hence $d\delta x^{(N)} = (A - BR^{-1}B^T P + F)\delta x^{(N)}dt + G\delta x_0 dt$. When $N \to \infty$, we replace $\delta x^{(N)}$ by δz and obtain (9).

B. The Limiting LQG Control Problems

1) Optimal Control Problem 1 (P1): Following the same reasoning as in [19], [20], we introduce a process of the form

$$\bar{z}(t) = f_1(t) + f_2(t)x_0(0) + \int_0^t g(t,s)dW_0(s), \quad (13)$$

where $f_1 \in C([0,T],\mathbb{R}^n)$, $f_2 \in C([0,T],\mathbb{R}^{n\times n})$ and $g \in C(\Delta,\mathbb{R}^{n\times n_2})$, where $\Delta = \{(t,s)|0 \le s \le t \le T\}$.

Now Problem (P1) is stated as: find an equilibrium solution $(\bar{x}_0,\bar{u}_0)$ with respect to $\bar{z}$ for (5)-(6).

2) Optimal Control Problem 2 (P2): Minimize the cost

$$\bar{J}_i = E\int_0^T \left\{\left|x_i - \Psi(x_i,\bar{z})\right|_Q^2 + u_i^T R u_i\right\}dt \quad (14)$$

subject to the system dynamics

$$dx_i = \big(Ax_i + Bu_i + F\bar{z} + G\bar{x}_0\big)dt + DdW_i,$$
$$d\bar{x}_0 = (A_0\bar{x}_0 + B_0\bar{u}_0 + F_0\bar{z})dt + D_0 dW_0,$$
$$\bar{z}(t) = f_1(t) + f_2(t)x_0(0) + \int_0^t g(t,s)dW_0(s),$$

where $(\bar{x}_0,\bar{u}_0)$ is determined from (P1) as the equilibrium solution with respect to $\bar{z}$.

IV. The Solution of Problem (P1)

Suppose $(\bar{x}_0,\bar{u}_0)$ is an equilibrium solution with respect to $\bar{z}$ for Problem (P1). Let $\delta u_0 \in L^2_{\mathscr{F}}(0,T;\mathbb{R}^{n_1})$ be a perturbation of $\bar{u}_0$, $\bar{x}_0 + \delta x_0$ the trajectory corresponding to the control $u_0 = \bar{u}_0 + \delta u_0$. The variations of $\bar{x}_0, \bar{z}$ are determined by (8)-(10). The first and second order variations of the cost are

$$\frac{\delta \bar{J}_0}{2} = E\int_0^T \Big[[\delta x_0 - H_0\delta z]^T Q_0[\bar{x}_0 - H_0\bar{z} - \eta_0] + \delta u_0^T R_0 \bar{u}_0\Big]dt,$$
$$\delta^2 \bar{J}_0 = E\int_0^T \Big[[\delta x_0 - H_0\delta z]^T Q_0[\delta x_0 - H_0\delta z] + \delta u_0^T R_0 \delta u_0\Big]dt.$$

Since $\delta^2\bar{J}_0 > 0$ for all δu_0 satisfying $E\int_0^T |\delta u_0|^2 dt > 0$, for $(\bar{x}_0,\bar{u}_0)$ to be an equilibrium solution to Problem (P1), a sufficient and necessary condition is $\delta\bar{J}_0 = 0$ for all δu_0.

Lemma 2: Assume $(\bar{x}_0, p_0, p_z, q_0, q_z) \in L^2_{\mathscr{F}}(0,T;\mathbb{R}^{3n} \times \mathbb{R}^{n\times n_2} \times \mathbb{R}^{n\times n_2})$ is a solution of the forward-backward stochastic differential equation (FBSDE)

$$\begin{cases} d\bar{x}_0 = \big[A_0\bar{x}_0 + B_0R^{-1}B_0^T p_0 + F_0\bar{z}\big]dt + D_0dW_0, \\ dp_0 = \big[Q_0(\bar{x}_0 - H_0\bar{z} - \eta_0) - A_0^T p_0 - G^T p_z\big]dt + q_0 dW_0, \\ dp_z = \big[-H_0^T Q_0(\bar{x}_0 - H_0\bar{z} - \eta_0) - F_0^T p_0 \\ \qquad -(A^T - PBR^{-1}B^T + F^T)p_z\big]dt + q_z dW_0, \\ \bar{x}_0(0) = x_0(0),\ p_0(T) = 0,\ p_z(T) = 0. \end{cases} \quad (15)$$

Then the pair $(\bar{x}_0,\bar{u}_0) \in L^2_{\mathscr{F}}(0,T;\mathbb{R}^{n+n_1})$, where $\bar{u}_0(t) = R_0^{-1}B_0^T p_0(t)$, is a solution to Problem (P1). □

A. Riccati Equations and State Feedback Control

Assume that $P_0(t)$, $P_z(t)$ are $\mathbb{R}^{n\times n}$-valued deterministic functions satisfying the system of ODEs of Riccati type

$$\begin{cases} \dot{P}_0 + P_0A_0 - P_0B_0R_0^{-1}B_0^T P_0 + A_0^T P_0 + Q_0 + G^T P_z = 0, \\ \dot{P}_z + P_zA_0 - P_zB_0R_0^{-1}B_0^T P_0 + (A^T - PBR^{-1}B^T + F^T)P_z \\ \quad + F_0^T P_0 - H_0^T Q_0 = 0, \\ P_0(T) = P_z(T) = 0. \end{cases} \quad (16)$$

To analyze (15), write $p_0 = -P_0\bar{x}_0 + \nu_0$, $p_z = -P_z\bar{x}_0 + \nu_z$, where the two processes ν_0, ν_z are to be determined with the terminal conditions $\nu_0(T) = \nu_z(T) = 0$. Note that we can write $\bar{u}_0 = -R_0^{-1}B_0^T(P_0\bar{x}_0 - \nu_0)$. By Ito's formula, it can be shown that the coupled equation system (15) is equivalent to the FBSDE

$$\begin{aligned} d\bar{x}_0 &= \big[(A_0 - B_0R^{-1}B_0^T P)\bar{x}_0 + B_0R^{-1}B_0^T\nu_0 + F_0\bar{z}\big]dt + D_0dW_0, \\ d\nu_0 &= \big[(P_0B_0R_0^{-1}B_0^T - A_0^T)\nu_0 - G^T\nu_z + (P_0F_0 - Q_0H_0)\bar{z} \\ &\quad - Q_0\eta_0\big]dt + (q_0 + P_0D_0)dW_0, \qquad (17) \\ d\nu_z &= \big[-(A^T - PBR^{-1}B^T + F^T)\nu_z + (P_zB_0R_0^{-1}B_0^T - F_0^T)\nu_0 \\ &\quad + (H_0^T Q_0H_0 - P_zF_0)\bar{z} + H_0^T Q_0\eta_0\big]dt \\ &\quad + (q_z + P_zD_0)dW_0, \qquad (18) \\ \bar{x}_0(0) &= x_0(0), \quad \nu_0(T) = \nu_z(T) = 0. \end{aligned}$$

Theorem 3: If there is a unique solution to (16), the FBSDE (15) has a unique solution. □

To proceed, we will find a representation of $\bar{x}_0$ determined by (17) in the form

$$\bar{x}_0(t) = f_{\bar{x}_0,1}(t) + f_{\bar{x}_0,2}(t)x_0(0) + \int_0^t g_{\bar{x}_0}(t,s)dW_0(s), \quad (19)$$

where $f_{\bar{x}_0,1} \in C([0,T],\mathbb{R}^n)$, $f_{\bar{x}_0,2} \in C([0,T],\mathbb{R}^{n\times n})$, and $g_{\bar{x}_0} \in C(\Delta,\mathbb{R}^{n\times n_2})$ are to be determined.

To solve the equations (17) and (18), denote

$$\nu_{0z} = \begin{bmatrix}\nu_0 \\ \nu_z\end{bmatrix}, \mathbb{M}_{0z}^T = \begin{bmatrix} A_0^T - P_0B_0R_0^{-1}B_0^T & G^T \\ F_0^T - P_zB_0R_0^{-1}B_0^T & A^T - PBR^{-1}B^T + F^T\end{bmatrix},$$

$$\mathbb{F}_0 = \begin{bmatrix} P_0F_0 - Q_0H_0 \\ H_0^T Q_0H_0 - P_zF_0\end{bmatrix}, \mathbb{Q}_0 = \begin{bmatrix} -Q_0 \\ H_0^T Q_0\end{bmatrix}, \mu_{0z} = \begin{bmatrix} P_0D_0 + q_0 \\ P_zD_0 + q_z\end{bmatrix}.$$

Denote $\zeta_{0z} = \mathbb{F}_0\bar{z} + \mathbb{Q}_0\eta_0$. Then by (13),

$$\zeta_{0z}(t) = f_{\zeta_{0z},1}(t) + f_{\zeta_{0z},2}(t)x_0(0) + \int_0^t g_{\zeta_{0z}}(t,s)dW_0(s),$$

where $f_{\zeta_{0z},1}(t) = \mathbb{F}_0 f_1(t) + \mathbb{Q}_0\eta_0$, $f_{\zeta_{0z},2}(t) = \mathbb{F}_0 f_2(t)$, $g_{\zeta_{0z}}(t,s) = \mathbb{F}_0 g(t,s)$ and

$$d\nu_{0z} = (\zeta_{0z} - \mathbb{M}_{0z}^T(t)\nu_{0z})dt + \mu_{0z}dW_0.$$

Let $\Phi_{0z}(t,s)$ be the solution of the following system

$$\begin{cases} d\Phi_{0z}(t,s) = \mathbb{M}_{0z}(t)\Phi_{0z}(t,s)dt, \\ \Phi_{0z}(s,s) = I, \qquad t \ge 0,\ s \ge 0. \end{cases} \quad (20)$$

Then by [19, Lemma A.1 (ii)],

$$\nu_{0z}(t) = f_{\nu_{0z},1}(t) + f_{\nu_{0z},2}(t)x_0(0) + \int_0^t g_{\nu_{0z}}(t,s)dW_0(s), \quad (21)$$

where

$$f_{\nu_{0z},1}(t) = \int_t^T \Phi_{0z}^T(s_1,t)\Big[\mathbb{F}_0 f_1(s_1) + \mathbb{Q}_0\eta_0\Big]ds_1, \quad (22)$$

$$f_{\nu_{0z},2}(t) = \int_t^T \Phi_{0z}^T(s_1,t)\mathbb{F}_0 f_2(s_1)ds_1, \quad (23)$$

$$g_{\nu_{0z}}(t,s) = \int_t^T \Phi_{0z}^T(s_1,t)\mathbb{F}_0 g(s_1,s)ds_1. \quad (24)$$

We continue to solve the first equation in (17). Let $\xi_0(t) = B_0R_0^{-1}B_0^T\nu_0(t) + F_0\bar{z}(t)$. Denote $\mathbb{I} = \begin{bmatrix} I & 0 \end{bmatrix} \in \mathbb{R}^{n\times 2n}$. Then $\nu_0 = \mathbb{I}\nu_{0z}$. Therefore, by (13) and (21)-(24),

$$\xi_0(t) = f_{\xi_0,1}(t) + f_{\xi_0,2}(t)x_0(0) + \int_0^t g_{\xi_0}(t,s)dW_0(s),$$

where $f_{\xi_0,j}(t) = B_0R_0^{-1}B_0^T\mathbb{I}f_{\nu_{0z},j}(t) + F_0f_j(t),\ j = 1,2,$ and $g_{\xi_0}(t,s) = B_0R_0^{-1}B_0^T\mathbb{I}g_{\nu_{0z}}(t,s) + F_0g(t,s)$.

Denote $\mathbb{A}_0 = A_0 - B_0R^{-1}B_0^TP_0$. We have

$$d\bar{x}_0(t) = (\xi_0(t) + \mathbb{A}_0(t)\bar{x}_0(t))dt + D_0dW_0(t).$$

Let $\Phi_0(t,s)$ be the solution of the following system

$$\begin{cases} d\Phi_0(t,s) = \mathbb{A}_0(t)\Phi_0(t,s)dt, \\ \Phi_0(s,s) = I, \qquad t\geq 0,\ s\geq 0. \end{cases} \tag{25}$$

Therefore, by [19, Lemma A.1 (i)] we obtain (19), where

$$\begin{aligned} f_{\bar{x}_0,1}(t) &= \int_0^t \Phi_0(t,s_1)f_{\xi_0,1}(s_1)ds_1 \\ &= \int_0^t\int_{s_1}^T \Phi_0(t,s_1)B_0R_0^{-1}B_0^T\mathbb{I}\Phi_{0z}^T(s_2,s_1) \\ &\quad \times \big(\mathbb{F}_0(s_2)f_1(s_2) + \mathbb{Q}_0\eta_0\big)ds_2ds_1 + \int_0^t \Phi_0(t,s_1)F_0f_1(s_1)ds_1 \\ &=: [\Gamma_{0,1}f_1](t), \end{aligned} \tag{26}$$

$$\begin{aligned} f_{\bar{x}_0,2}(t) &= \Phi_0(t,0) + \int_0^t \Phi_0(t,s_1)f_{\xi_0,2}(s_1)ds_1 \\ &= \int_0^t\int_{s_1}^T \Phi_0(t,s_1)B_0R_0^{-1}B_0^T\mathbb{I}\Phi_{0z}^T(s_2,s_1)\mathbb{F}_0(s_2)f_2(s_2)ds_2ds_1 \\ &\quad + \int_0^t \Phi_0(t,s_1)F_0f_2(s_1)ds_1 + \Phi_0(t,0) \\ &=: [\Gamma_{0,2}f_2](t), \end{aligned} \tag{27}$$

$$\begin{aligned} g_{\bar{x}_0}(t,s) &= \int_s^t \Phi_0(t,s_1)g_{\xi_0}(s_1,s)ds_1 + \Phi_0(t,s)D_0 \\ &= \int_s^t\int_{s_1}^T \Phi_0(t,s_1)B_0R_0^{-1}B_0^T\mathbb{I}\Phi_{0z}^T(s_2,s_1)\mathbb{F}_0(s_2)g(s_2,s)ds_2ds_1 \\ &\quad + \int_s^t \Phi_0(t,s_1)F_0g(s_1,s)ds_1 + \Phi_0(t,s)D_0 \\ &=: [\Lambda_0 g](t,s). \end{aligned} \tag{28}$$

V. The Solution of Problem (P2)

Lemma 4: Suppose Problem (P1) has an equilibrium solution with respect to $\bar{z}$. Then the following holds:
(i) There exists a unique optimal control to Problem (P2).
(ii) The pair $(\bar{x}_i,\bar{u}_i)$ is the optimal solution to Problem (P2) if and only if $\bar{u}_i(t) = R^{-1}B^Tp_i(t)$, where $(\bar{x}_i(t),p_i(t),q_i(t),r_i(t))$ is the solution of the FBSDE

$$\begin{cases} d\bar{x}_i = \big(A\bar{x}_i + BR^{-1}B^Tp_i + F\bar{z} + G\bar{x}_0\big)dt + DdW_i, \\ dp_i = \big[Q\big(\bar{x}_i - H\bar{x}_0 - \hat{H}z - \eta\big) - A^Tp_i\big]dt + q_idW_i + r_idW_0, \\ \bar{x}_i(0) = x_i(0), \quad p_i(T) = 0. \end{cases} \tag{29}$$

(iii) (29) has a unique solution $(\bar{x}_i,p_i,q_i,r_i)$. □

Let $P(t)\geq 0$ be the solution of the Riccati equation (7). Write $p_i(t) = -P(t)x_i(t) + \nu_i(t)$, where $\nu_i(t)$ will be determined later satisfying the terminal condition $\nu_i(T) = 0$.

Denote $\mathbb{A}(t) = A - BR^{-1}B^TP(t)$. Similar to (17)-(18), the coupled equation system (29) is equivalent to the FBSDE

$$\begin{cases} d\bar{x}_i = \big(\mathbb{A}\bar{x}_i + BR^{-1}B^T\nu_i + F\bar{z} + G\bar{x}_0\big)dt + DdW_i, \\ d\nu_i = \big\{-\mathbb{A}^T\nu_i + (PF - Q\hat{H})\bar{z} + (PG - QH)\bar{x}_0 - Q\eta\big\}dt \\ \qquad\quad + (q_i + PD)dW_i + r_idW_0, \\ \bar{x}_i(0) = x_i(0), \quad \nu_i(T) = 0. \end{cases} \tag{30}$$

We will represent $\bar{x}_i(t)$ in the form

$$\begin{aligned} \bar{x}_i(t) =& f_{\bar{x}_i,1}(t) + f_{\bar{x}_i,2}(t)x_0(0) + f_{\bar{x}_i,3}(t)x_i(0) \\ &+ \int_0^t g_{\bar{x}_i}(t,s)dW_0(s) + \int_0^t h_{\bar{x}_i}(t,s)dW_i(s), \end{aligned} \tag{31}$$

where $f_{\bar{x}_i,1} \in C([0,T],\mathbb{R}^n)$, $f_{\bar{x}_i,2}, f_{\bar{x}_i,3} \in C([0,T],\mathbb{R}^{n\times n})$, and $g_{\bar{x}_i}, h_{\bar{x}_i} \in C([0,T],\mathbb{R}^{n\times n_2})$ are to be determined.

Let $\zeta_i(t) = (P(t)F - Q\hat{H})\bar{z}(t) + (P(t)G - QH)\bar{x}_0(t) - Q\eta$, $\lambda_i(t) = q_i(t) + P(t)D$. Then by (19),

$$\zeta_i(t) = f_{\zeta_i,1}(t) + f_{\zeta_i,2}(t)x_0(0) + \int_0^t g_{\zeta_i}(t,s)dW_0(s),$$

where

$$\begin{aligned} f_{\zeta_i,1}(t) &= \big[P(t)F - Q\hat{H}\big]f_1(t) + \big[P(t)G - QH\big]f_{\bar{x}_0,1}(t) - Q\eta, \\ f_{\zeta_i,2}(t) &= \big[P(t)F - Q\hat{H}\big]f_2(t) + \big[P(t)G - QH\big]f_{\bar{x}_0,2}(t), \\ g_{\zeta_i}(t,s) &= \big[P(t)F - Q\hat{H}\big]g(t,s) + \big[P(t)G - QH\big]g_{\bar{x}_0}(t,s). \end{aligned}$$

By (30), we have

$$d\nu_i(t) = [\zeta_i(t) - \mathbb{A}^T\nu_i(t)]dt + r_i(t)dW_0(t) + \lambda_i(t)dW_i(t).$$

Let $\Phi(t,s)$ be the solution of

$$\begin{cases} d\Phi(t,s) = \mathbb{A}(t)\Phi(t,s)dt, \\ \Phi(s,s) = I, \qquad t,s\geq 0. \end{cases} \tag{32}$$

Then by [19, Lemma A.2 (ii)],

$$\nu_i(t) = f_{\nu_i,1}(t) + f_{\nu_i,2}(t)x_0(0) + \int_0^t g_{\nu_i}(t,s)dW_0(s), \tag{33}$$

where

$$\begin{aligned} f_{\nu_i,1}(t) = \int_t^T \Phi^T(s_1,t)\Big[&\big(Q\hat{H} - P(s_1)F\big)f_1(s_1) \\ &+ \big(QH - P(s_1)G\big)f_{\bar{x}_0,1}(s_1) + Q\eta\Big]ds_1, \end{aligned} \tag{34}$$

$$\begin{aligned} f_{\nu_i,2}(t) = \int_t^T \Phi^T(s_1,t)\Big[&\big(Q\hat{H} - P(s_1)F\big)f_2(s_1) \\ &+ \big(QH - P(s_1)G\big)f_{\bar{x}_0,2}(s_1)\Big]ds_1, \end{aligned} \tag{35}$$

$$\begin{aligned} g_{\nu_i}(t,s) = \int_t^T \Phi^T(s_1,t)\Big[&\big(Q\hat{H} - P(s_1)F\big)g(s_1,s) \\ &+ \big(QH - P(s_1)G\big)g_{\bar{x}_0}(s_1,s)\Big]ds_1. \end{aligned} \tag{36}$$

Next, let $\xi_i(t) = BR^{-1}B^T\nu_i(t) + F\bar{z}(t)$. Then by (13) and (33), $\xi_i(t) = f_{\xi_i,1}(t) + f_{\xi_i,2}(t)x_0(0) + \int_0^t g_{\xi_i}(t,s)dW_0(s)$ with

$$f_{\xi_i,j}(t) = BR^{-1}B^Tf_{\nu_i,j}(t) + Ff_j(t),\ j = 1,2, \tag{37}$$

$$g_{\xi_i}(t,s) = BR^{-1}B^Tg_{\nu_i}(t,s) + Fg(t,s). \tag{38}$$

We have $d\bar{x}_i(t) = \big(\xi_i(t) + \mathbb{A}(t)\bar{x}_i(t)\big)dt + DdW_i(t)$. Therefore from [19, Lemma A.2 (i)], we obtain (31), where

$$
\begin{aligned}
f_{\bar{x}_i,1}(t) &= \int_0^t \Phi(t,s_1) f_{\xi_i,1}(s_1)ds_1 \\
&= \int_0^t \Phi(t,s_1)BR^{-1}B^T \int_{s_1}^T \Phi^T(s_2,s_1)\times \\
&\Big[(Q\hat{H} - P(s_2)F)f_1(s_2) + (QH - P(s_2)G)f_{\bar{x}_0,1}(s_2) + Q\eta\Big]ds_2ds_1 \\
&\quad + \int_0^t \Phi(t,s_1)Ff_1(s_1)ds_1 =: [\Gamma_1 f_1](t), \qquad (39)
\end{aligned}
$$

$$
\begin{aligned}
f_{\bar{x}_i,2}(t) &= \int_0^t \Phi(t,s_1) f_{\xi_i,2}(s_1)ds_1 \\
&= \int_0^t \Phi(t,s_1)BR^{-1}B^T \int_{s_1}^T \Phi^T(s_2,s_1)\times \\
&\Big[\big(Q\hat{H} - P(s_2)F\big)f_2(s_2) + \big(QH - P(s_2)G\big)f_{\bar{x}_0,2}(s_2)\Big]ds_2ds_1 \\
&\quad + \int_0^t \Phi(t,s_1)Ff_2(s_1)ds_1 =: [\Gamma_2 f_2](t), \qquad (40)
\end{aligned}
$$

$$
\begin{aligned}
g_{\bar{x}_i}(t,s) &= \int_s^t \Phi(t,s_1) g_{\xi_i}(s_1,s)ds_1 \\
&= \int_s^t \Phi(t,s_1)BR^{-1}B^T \int_{s_1}^T \Phi^T(s_2,s_1)\times \\
&\Big[\big(Q\hat{H} - P(s_2)F\big)g(s_2,s) + \big(QH - P(s_2)G\big)g_{\bar{x}_0}(s_2,s)\Big]ds_2ds_1 \\
&\quad + \int_s^t \Phi(t,s_1)Fg(s_1,s)ds_1 =: [\Lambda g](t,s), \qquad (41)
\end{aligned}
$$

and furthermore, $f_{\bar{x}_i,3}(t) = \Phi(t,0)$, $h_{\bar{x}_i}(t,s) = \Phi(t,s)D$.

VI. The Consistency Condition

Lemma 5: We have
(i) Γ_1 is a mapping from $C([0,T],\mathbb{R}^n)$ to $C([0,T],\mathbb{R}^n)$.
(ii) Γ_2 is a mapping from $C([0,T],\mathbb{R}^{n\times n})$ to $C([0,T],\mathbb{R}^{n\times n})$.
(iii) Λ is a mapping from $C(\Delta,\mathbb{R}^{n\times n_2})$ to $C(\Delta,\mathbb{R}^{n\times n_2})$. □

Denote the product space

$$
C_{\text{NCE}} = C([0,T],\mathbb{R}^n)\times C([0,T],\mathbb{R}^{n\times n})\times C(\Delta,\mathbb{R}^{n\times n_2}).
$$

The definition below characterizes the consistency condition for the mean field approximation. When the controls obtained in Section III-B are applied, the mean field replicated by the closed loop in the population limit should coincide with the one assumed at the beginning.

Definition 6: A triple $(f_1, f_2, g) \in C_{\text{NCE}}$ is called a consistent solution to the Nash certainty equivalence (NCE) equation system if

$$
\begin{cases} f_j(t) = [\Gamma_j f_j](t), & 0 \le t \le T,\ j = 1,2, \\ g(t,s) = [\Lambda g](t,s), & 0 \le s \le t \le T. \end{cases} \qquad (42)
$$

◇

Denote the linear operators $\bar{\Gamma}_{0,1}$, $\bar{\Gamma}_{0,2}$ and $\bar{\Lambda}_0$ on $C([0,1],\mathbb{R}^n)$, $C([0,T],\mathbb{R}^{n\times n})$ and $C(\Delta,\mathbb{R}^{n\times n_2})$, respectively, as follows:

$$
\begin{aligned}
[\bar{\Gamma}_{0,j} f_j](t) &= \int_0^t \int_{s_1}^T \Phi_0(t,s_1)B_0R_0^{-1}B_0^T \mathbb{I}\Phi_{0z}^T(s_2,s_1)\times \\
&\quad \times \mathbb{F}_0(s_2)f_j(s_2)ds_2ds_1 + \int_0^t \Phi_0(t,s_1)F_0 f_j(s_1)ds_1,\ j=1,2,
\end{aligned}
$$

$$
\begin{aligned}
[\bar{\Lambda}_0 g](t,s) &= \int_s^t \int_{s_1}^T \Phi_0(t,s_1)B_0R_0^{-1}B_0^T \mathbb{I}\Phi_{0z}^T(s_2,s_1)\times \\
&\quad \times \mathbb{F}_0(s_2)g(s_2,s)ds_2ds_1 + \int_s^t \Phi_0(t,s_1)F_0 g(s_1,s)ds_1,
\end{aligned}
$$

which are obtained by retaining the linear term of the affine operators $\Gamma_{0,j}$ and Λ_0, respectively.

Corresponding to Γ_1, Γ_2 and Λ, define the linear operators $\bar{\Gamma}_1$, $\bar{\Gamma}_2$, and $\bar{\Lambda}$ on $C([0,1],\mathbb{R}^n)$, $C([0,T],\mathbb{R}^{n\times n})$ and $C(\Delta,\mathbb{R}^{n\times n_2})$, respectively, as follows:

$$
\begin{aligned}
[\bar{\Gamma}_j f_j](t) &= \int_0^t \int_{s_1}^T \Phi(t,s_1)BR^{-1}B^T\Phi^T(s_2,s_1)\times \\
&\times \Big[\big(Q\hat{H} - P(s_2)F\big)f_j(s_2) + \big(QH - P(s_2)G\big)[\bar{\Gamma}_{0,j}f_j](s_2)\Big]ds_2ds_1 \\
&\quad + \int_0^t \Phi(t,s_1)Ff_j(s_1)ds_1, \quad j = 1,2,
\end{aligned}
$$

$$
\begin{aligned}
[\bar{\Lambda} g](t,s) &= \int_s^t \int_{s_1}^T \Phi(t,s_1)BR^{-1}B^T\Phi^T(s_2,s_1)\times \\
&\times \Big[\big(Q\hat{H} - P(s_2)F\big)g(s_2,s) + \big(QH - P(s_2)G\big)[\bar{\Lambda}_0 g](s_2,s)\Big]ds_2ds_1 \\
&\quad + \int_s^t \Phi(t,s_1)Fg(s_1,s)ds_1.
\end{aligned}
$$

Let $C([0,1],\mathbb{R}^n)$, $C([0,T],\mathbb{R}^{n\times n})$ and $C(\Delta,\mathbb{R}^{n\times n_2})$ be endowed with the usual sup-norms $\|\cdot\|_\infty$ so that they are all Banach spaces. For $f_2 \in C([0,T],\mathbb{R}^{n\times n})$, we write $f_2 = [f_{2,1},\dots,f_{2,n}]$, where $f_{2,i} \in C([0,T],\mathbb{R}^n)$ for each $i = 1,\dots,n$. Then $[\bar{\Gamma}_2 f_2](t) = \big[[\bar{\Gamma}_1 f_{2,1}](t),\dots,[\bar{\Gamma}_1 f_{2,n}](t)\big]$. We have the following result for the NCE equation system.

Theorem 7: If the norms of $\bar{\Gamma}_1$ and $\bar{\Lambda}$ satisfy $\|\bar{\Gamma}_1\| < 1$ and $\|\bar{\Lambda}\| < 1$, (42) has a unique solution $(f_1,f_2,g) \in C_{\text{NCE}}$. □

VII. Asymptotic Equilibrium Analysis

Throughout this section we assume that there exists a solution $(f_1,f_2,g) \in C_{\text{NCE}}$ to the NCE equation system (42).

Consider the system (1)-(2). Let the control laws of $\mathscr{A}_0$ and $\mathscr{A}_i$, $1 \le i \le N$, be given by

$$
\hat{u}_0(t) = R_0^{-1}B_0^T\big(-P_0(t)\hat{x}_0(t) + \nu_0(t)\big), \qquad (43)
$$

$$
\hat{u}_i(t) = R^{-1}B^T\big(-P(t)\hat{x}_i(t) + \nu_i(t)\big), \qquad (44)
$$

where $\nu_0(t)$, $\nu_z(t)$ and $\nu_i(t)$ are determined by (17), (18) and (29) corresponding to the solution (f_1,f_2,g) to (42). Their explicit solutions are given by (21) and (33). After the control laws (43)-(44) are applied, the dynamics of $\mathscr{A}_0$ and $\mathscr{A}_i$ may be written in the form

$$
d\hat{x}_0 = \Big(\mathbb{A}_0\hat{x}_0 + B_0R_0^{-1}B_0^TP_0\nu_0 + F_0\hat{x}^{(N)}\Big)dt + D_0dW_0,
$$

$$
d\hat{x}_i = \Big(\mathbb{A}\hat{x}_i + BR^{-1}B^TP\nu_i + F\hat{x}^{(N)} + G\hat{x}_0\Big)dt + DdW_i,
$$

where $1 \le i \le N$ and $\hat{x}^{(N)} = (1/N)\sum_{i=1}^N \hat{x}_i$.

We now construct the limiting equation system for the $N+1$ players

$$
d\bar{x}_0 = \Big(\mathbb{A}_0\bar{x}_0 + B_0R_0^{-1}B_0^TP_0\nu_0 + F_0\bar{z}\Big)dt + D_0dW_0,
$$

$$
d\bar{x}_i = \Big(\mathbb{A}\bar{x}_i + BR^{-1}B^TP\nu_i + F\bar{z} + G\bar{x}_0\Big)dt + DdW_i,
$$

where $1 \le i \le N$ and the initial conditions are $\bar{x}_i(0) = x_i(0)$. We have the error estimate of the mean field approximation.

Proposition 8: Assume (**A**). We have

$$E\int_0^T \left|\bar{z}(t)-\bar{x}^{(N)}(t)\right|^2 dt = O\Big(\frac{1}{N}\Big),$$
$$E\int_0^T \Big[\left|\bar{z}(t)-\hat{x}^{(N)}(t)\right|^2+\sup_{0\le j\le N}\left|\hat{x}_j(t)-\bar{x}_j(t)\right|^2\Big]dt = O\Big(\frac{1}{N}\Big),$$

where $\bar{x}^{(N)}=(1/N)\sum_{i=1}^N \bar{x}_i$. □

Consider the system of $N+1$ agents described by (1) and (2). Let the class $\mathscr{U}_W$ consist of all processes y_W of the form

$$y_W(t)=\int_0^t [h_0(t,s)dW_0(s),\dots,h_N(t,s)dW_N(s)]^T$$

where each h_j is an $\mathbb{R}^{n\times n_2}$-valued bounded measurable function on Δ.

For any $j=0,...,N$, the admissible control set $\mathscr{U}_j$ of agent $\mathscr{A}_j$ consists of control u_j as a time dependent function linear in $x_0,x_1,\dots,x_N,y_W$ for some $y_W\in\mathscr{U}_W$. The resulting control of a player may not be purely in a feedback form since the noise process may be used via y_W; this more general form of controls is necessary in order to include the decentralized controls (43)-(44) that we have derived. Since the control still uses the players' states, $(u_0,u_1,\dots,u_N)$ is in a partial feedback form, and will be called a set of partial feedback strategies. Note that $\mathscr{U}_j$ is not restricted to be decentralized. Given $(u_0,u_1,\dots,u_N)\in\mathscr{U}_0\times\dots\times\mathscr{U}_N$, the closed-loop system has a unique strong solution. For $j=0,...,N$, denote $u_{-j}=(u_0,u_1,...,u_{j-1},u_{j+1},...,u_N)$.

Definition 9: A set of controls $u_j\in\mathscr{U}_j$, $0\le j\le N$, for the $N+1$ players is called an ε-Nash equilibrium with respect to the costs J_j, $0\le j\le N$, where $\varepsilon\ge 0$, if for any j, $0\le j\le N$, we have $J_j(u_j,u_{-j})\le J_j(u'_j,u_{-j})+\varepsilon$, when any alternative u'_j is applied by player $\mathscr{A}_j$. ◊

Theorem 10: Assume (**A**). Then the set of controls $\hat{u}_j$, $0\le j\le N$, for the $N+1$ players is an ε-Nash equilibrium, i.e., for $0\le j\le N$,

$$J_j(\hat{u}_j,\hat{u}_{-j})-\varepsilon\le \inf_{u_j\in\mathscr{U}_j} J_j(u_j,\hat{u}_{-j})\le J_j(\hat{u}_j,\hat{u}_{-j}),$$

where $0<\varepsilon=O(1/\sqrt{N})$. □

VIII. Conclusion

This paper has extended the BSDE based approach in [18], [19], [20] to treat responsive mean field in large population stochastic dynamic games involving a major player. The key step is the development of a procedure called anticipative variational calculations for the control analysis of the major player. The consistent mean field approximation reduces to fixed point analysis of linear operators on function spaces.

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