Twisted endoscopy in miniature*

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Abstract

We give an exposition of twisted endoscopy through the examples of $\mathrm{GL}(2, \mathbf{R})$ and the unitary group $\mathrm{SU}(2,1)$. The exposition is centred on the representation-theoretic aspects of the theory. We include detailed descriptions of L-packets, z-extensions, and correspondences between endoscopic groups.

1 Introduction

Anyone espying the Langlands program will quickly encounter the word "endoscopy". One could ponder how this word spread into its mathematical usage (*p.* 19 [Lan83]), but it may be more instructive to go back to an earlier time. Influential early papers use the terms "stable conjugacy" or "L-indistinguishability" instead ([Lan79a], [Lan79b], [Lan79c]). Stable conjugacy has to do with conjugacy classes of algebraic groups over a field, sitting inside conjugacy classes over an algebraic closure (2.3 [Lab08], [Art97]). L-indistinguishability has to do with representations of algebraic groups and how they sit inside L-packets. To say what an L-packet is, or should be, is a subject of its own. There are many expository works on the subject (III [Bor79], 9 [Kna97], [Kud94], [Mez09] to name a few), but the beginner is likely to find him/herself at sea without first anchoring the theory to some well worked out examples. It is best to work things out over the field of the real numbers, where everything is known ([Lan89], [Kna94]). An excellent extension of this approach to the theory of endoscopy may be found in [Lab08]. Continuing with this approach, we work through the representation theory of *twisted* endoscopy for the general linear group $\mathrm{GL}(2, \mathbf{R})$ and the special unitary group $\mathrm{SU}(2,1)$.

The foundations of twisted endoscopy are given by Kottwitz and Shelstad in [KS99]. The significance of the theory is manifest in [Art] and nicely described in 30 [Art05]. Our goal is not to motivate the theory of twisted endoscopy. Rather, we wish to give someone who is familiar with some basic expository work (such as [Bor79]) sufficient grounding to begin reading [KS99] and related representation theoretic works.

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We have made efforts to restrict our examples to the perspective of L-indistinguishability, ignoring stable conjugacy entirely. This was done in part to make the exposition concise. It was also done to allow for greater detail in the discussion of L-packets. For this reason those interested in reading Labesse's introductory article ([Lab08]) may wish to consult our examples, especially in sections dealing with the "dual picture".

The dual picture in the case of ordinary endoscopy over the real numbers may be sketched as follows. One begins with a connected reductive algebraic group G which is defined over $\mathbf{R}$. From this one defines the *dual group* $\hat{G}$ which is a complex connected reductive algebraic group. The real structure of G is partially encoded in a Galois action on G. This action is usually extendend trivially from the Galois group to the Weil group $W_{\mathbf{R}} = \mathbf{C}^{\times} \cup \mathbf{C}^{\times}\tau$, where

$$\tau z \tau^{-1} = \bar{z},\ \tau^2 = -1,\ z \in \mathbf{C}^{\times}$$

(9.4 [Bor79]). The resulting semidirect product ${}^LG = \hat{G} \rtimes W_{\mathbf{R}}$ is called the the *L-group* of the group of $\mathbf{R}$-points $G(\mathbf{R})$ (or simply of G). At heart, the dual picture is a correspondence between irreducible representations of $G(\mathbf{R})$ and data pertaining to ${}^LG$. This is known as the real case of the Local Langlands Correspondence ([Lan89]). It is a correspondence between homomorphisms $\varphi : W_{\mathbf{R}} \to {}^LG$ on the one hand, and finite sets of irreducible representations Π_{φ}–the L-packets–on the other. The homomorphisms φ must satisfy additional properties, making them *admissible* homomorphisms. Furthermore, the correspondence only depends on φ up to $\hat{G}$-conjugacy, a rule we will use repeatedly in the sequel.

The image of an admissible homomorphism $\varphi : W_{\mathbf{R}} \to {}^LG$ might sit inside a smaller group ${}^LH \subset^L G$. If this is so, then the resulting admissible homomorphism $\varphi_H : W_{\mathbf{R}} \to {}^LH$ corresponds to an L-packet Π_{φ_H} of representations of $H(\mathbf{R})$. One would then expect there to be some relationship between the characters of the representations in Π_{φ_H} and the characters of the representations in Π_{φ}. This expectation was formulated as a character identity and proven by Shelstad in [She82] (see also [She10]). We shall call identities of this kind *spectral transfer*, as they may be interpreted to transfer information about characters in Π_{φ_H} to information about characters in Π_{φ}. Regarded in this way, it is apt to call the group H an *endoscopic group* of G. Indeed, it is a tool used to analyze Π_{φ} which arises from looking inside ${}^LG$.

The theory of twisted endoscopy enriches the dual picture by including an $\mathbf{R}$-automorphism θ of G, or a quasicharacter ω of $G(\mathbf{R})$. For the sake of brevity, we shall ignore twisting by ω. In this twisted dual picture, the only L-packets Π_{φ} which are relevant are those which are preserved under composition by θ. In addition, the only representations $\pi \in \Pi_{\varphi}$ of such an L-packet which contribute to spectral transfer are those which are equivalent to $\pi \circ \theta$. These θ-stable representations have *twisted* characters which occur in spectral transfer identities with characters of Π_{φ_H} ([Mez13]).

We elucidate the twisted dual picture and spectral transfer for $\mathrm{GL}(2, \mathbf{R})$ and $\mathrm{SU}(2, 1)$, each being twisted by a variant of the automorphism given by inverse-transpose. For each of these examples we follow the same template. We begin with the definition of the L-group. The non-split structure of $\mathrm{SU}(2,1)$ makes the L-group noteworthy. With the L-group in place we compute admissible homomorphisms φ. Again in the interest of brevity, we restrict our attention to *discrete* admissible homomorphisms. These are the admissible homomorphisms which correspond to L-packets consisting of discrete series representations.

Given a discrete admissible homomorphism, we turn to finding endoscopic groups and their accompanying data. We follow the outline given on *p.* 24 [KS99]. Bearing in mind that φ matters only up to $\hat{G}$ conjugacy and that we desire $\Pi_\varphi \circ \theta = \Pi_\varphi$, it is sensible to seek $s \in \hat{G}$ such that

$$\mathrm{Int}(s) \circ \hat{\theta} \circ \varphi = \varphi. \tag{1}$$

Here, $\mathrm{Int}(s)$ denotes conjugation by s and $\hat{\theta}$ is an automorphism of ${}^L G$ induced by θ. (This is the precursor to the "S-groups" of 6.6 [Lab08] or 6 [Art08].) By virtue of φ being discrete, the number of such s is finite modulo the centre of $\hat{G}$. Each element s yields the dual of an endoscopic group by decreeing $\hat{H}$ to equal the identity component of the fixed-point group

$$\hat{G}^{s\hat{\theta}} = \{x \in \hat{G} : s\hat{\theta}(x)s^{-1} = x\}.$$

The Galois action on $\hat{H}$ is determined by conjugation of $\varphi(W_{\mathbf{R}})$ on $\hat{H}$. More specifically, we define the subgroup $\mathcal{H} = \hat{H}\varphi(W_{\mathbf{R}}) \subseteq {}^L G$ and obtain a split exact sequence

$$1 \to \hat{H} \to \mathcal{H} \to W_{\mathbf{R}} \to 1.$$

By choosing a splitting of this sequence, we obtain a Galois action on $\hat{H}$ which extends trivially to $\rho_{\mathcal{H}} : W_{\mathbf{R}} \to \mathrm{Aut}(\hat{H})$. This fixes the L-group ${}^L H = \hat{H} \rtimes_{\rho_{\mathcal{H}}} W_{\mathbf{R}}$ and thereby a unique quasisplit endoscopic group H up to isomorphism.

In the case of $\mathrm{GL}(2,\mathbf{R})$ we shall see that s is unique modulo the centre of $\hat{G}$, but is of two kinds depending on the nature of φ. This produces two possible endoscopic groups $H(\mathbf{R})$, namely $\mathrm{PGL}(2,\mathbf{R})$ and the unitary group $U(1)$. In both cases ${}^L H$ is isomorphic to $\mathcal{H}$ so that there is an embedding of ${}^L H$ in ${}^L G$. We define $\varphi_H : W_{\mathbf{R}} \to {}^L H$ so that its composition with this embedding is φ.

In the case of $\mathrm{SU}(2,1)$ it turns out that every s as in (1) yields the endoscopic group $\mathrm{PGL}(2,\mathbf{R})$. However, the L-group of this endoscopic group is *not* isomorphic to $\mathcal{H}$. As a result, there is no embedding of the endoscopic L-group into the L-group of $\mathrm{SU}(2,1)$ and there is no obvious way to define φ_H from φ. This predicament gives us the opportunity to present a substitute for H which is a *z-extension.* We continue by defining an associated admissible homomorphism φ_1 for the z-extension in place of φ_H.

After specifying the endoscopic data, we describe the L-packets of all groups concerned. In doing this we follow *p.* 132 [Lan89]. We express the characters of the representations in these L-packets as functions on the regular elements of maximally compact tori. For the representations of $\mathrm{GL}(2,\mathbf{R})$ and $\mathrm{SU}(2,1)$ we also compute the relevant twisted characters as functions on the regular elements of subtori.

Spectral transfer is an identity between twisted characters and characters of endoscopic groups (or their z-extensions). Both types of characters are expressed on maximally compact tori. In order to achieve the desired identities then, we require a means of comparison between the maximally compact tori of the endoscopic groups and the maximally compact tori of $\mathrm{GL}(2, \mathbf{R})$ or $\mathrm{SU}(2,1)$. The means of comparison takes the shape of a map defined from conjugacy classes of the endoscopic groups to *twisted* conjugacy classes. This map is computed explicitly in each of the two examples. Since the characters involved are invariant under (twisted) conjugacy, we compare the twisted character values of the representations in Π_φ with the character values of the representations in Π_{φ_H} or Π_{φ_1}. These comparisons are tantamount to the character identities of spectral transfer.

One must not forget that we are investigating spectral transfer "in miniature". The examples we work through fall short of the cardinal labour of finding transfer factors ([She]), and it is facile to invert the character identities (6 [Art08], [She08]). Finally, we wish to remark that there is an alternative formalism to twisted endoscopy based on the concept of a *twisted space* (I.3 [Lab04]). We encourage the reader to convert our examples into this polished formalism.

2 Twisted endoscopy for $\mathrm{GL}(2, \mathbf{R})$

2.1 The L-group and discrete L-parameters

Let $G = \mathrm{GL}_2$. The reader may identify the absolute *algebraic* group GL_2 with the complex *matrix* group $\mathrm{GL}(2, \mathbf{C})$. The dual group of G is $\hat{G} = \mathrm{GL}_2$ (6.6.3 [Mez09]). In this example we are taking the real structure of G to be given so that $G(\mathbf{R})$ is equal to the split group $\mathrm{GL}(2, \mathbf{R})$. This implies that the L-group ${}^L G$ is the direct product $\hat{G} \times W_{\mathbf{R}}$ (1.3 and 2.3 [Bor79]). Now, one of the defining properties of an admissible homomorphism $\varphi : W_{\mathbf{R}} \to \hat{G} \times W_{\mathbf{R}}$ is that its projection onto $W_{\mathbf{R}}$ is the identity map (8.2 [Bor79]). Since the product in the codomain is direct, an admissible homomorphism φ is completely determined by its values in $\hat{G}$. We will therefore abusively abbreviate an admissible homomorphism to a map $\varphi : W_{\mathbf{R}} \to \hat{G}$.

Suppose $\varphi : W_{\mathbf{R}} \to \hat{G}$ is such an admissible homomorphism. To get a better hold on the image of φ, let $\hat{T}$ be the diagonal subgroup of $\hat{G} = \mathrm{GL}_2$. It is a maximal torus of $\hat{G}$ and all such tori are conjugate to $\hat{T}$. Since $\varphi(\mathbf{C}^\times)$ is an abelian group consisting of semisimple elements (8.2 [Bor79]) and we are only interested in φ up to conjugacy in $\hat{G}$, we may assume that $\varphi(\mathbf{C}^\times)$ is contained in $\hat{T}$. Using the supersolvability of $W_{\mathbf{R}}$, we may extend this inclusion to $\varphi(W_{\mathbf{R}})$ being contained in the normalizer $N_{\hat{G}}(\hat{T})$ of $\hat{T}$ (*p.* 126 [Lab08]). We write this normalizer as $N_{\hat{G}}(\hat{T}) = \hat{T} \cup w_0 \hat{T}$, where

$$w_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Let us now assume that φ is *discrete*. By this we mean that the image of

φ is not contained in a proper Levi subgroup of $\hat{G}$ (10.3 (3) [Bor79]). As $\hat{T}$ is a proper Levi subgroup, this forces $\varphi(\tau) = w_0 t$ for some $t \in \hat{T}$. In order to determine φ, we must find $t \in \hat{T}$ and the values of $\varphi(z)$ for all $z \in \mathbf{C}^\times$.

Let us begin with the latter. The restriction of φ to $\mathbf{C}^\times$ amounts to a homomorphism of $\mathbf{C}^\times$ into $\hat{T} \cong \mathbf{C}^\times \times \mathbf{C}^\times$. Observe that $\mathbf{C}^\times$ is isomorphic to the direct product of $\mathbf{R}$ and the circle group $U(1)$. Using this observation it is a simple exercise to show that any homomorphism from $\mathbf{C}^\times$ to itself is of the form

$$re^{ia} \mapsto r^s e^{ima}, \ r, a \in \mathbf{R}, \, r > 0$$

for some $s \in \mathbf{C}$ and $m \in \mathbf{Z}$. Equivalently, any such homomorphism is of the form

$$z \mapsto z^\mu \bar{z}^\nu, \ z \in \mathbf{C}^\times$$

where $\mu, \nu \in \mathbf{C}$ satisfy $\mu + \nu = s$ and $\mu - \nu = m \in \mathbf{Z}$ from above. We may therefore express the values of φ on $\mathbf{C}^\times$ by

$$\varphi(z) = \begin{bmatrix} z^{\mu_1} \bar{z}^{\nu_1} & 0 \\ 0 & z^{\mu_2} \bar{z}^{\nu_2} \end{bmatrix}, \ z \in \mathbf{C}^\times$$

for appropriate $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathbf{C}$. From the equation

$$\varphi(\bar{z}) = \varphi(\tau)\varphi(z)\varphi(\tau)^{-1} = w_0 t \, \varphi(z) \, (w_0 t)^{-1}$$

one may then deduce that $\nu_1 = \mu_2$, $\nu_2 = \mu_1$ and that $\mu_1 - \mu_2 = \mu_1 - \nu_1 \in \mathbf{Z}$. From the equation

$$\begin{bmatrix} (-1)^{\mu_1 - \mu_2} & 0 \\ 0 & (-1)^{\mu_2 - \mu_1} \end{bmatrix} = \varphi(-1) = \varphi(\tau^2) = (w_0 t)^2$$

one may deduce that

$$t = \begin{bmatrix} t_1 & 0 \\ 0 & (-1)^{\mu_1 - \mu_2 + 1} \, t_1^{-1} \end{bmatrix}$$

for some $t_1 \in \mathbf{C}^\times$. After conjugating φ by

$$\begin{bmatrix} t_1^{1/2} & 0 \\ 0 & t_1^{-1/2} \end{bmatrix} \tag{2}$$

we see that φ may be expressed as

$$\varphi(z) = \begin{bmatrix} z^{\mu_1} \bar{z}^{\mu_2} & 0 \\ 0 & z^{\mu_2} \bar{z}^{\mu_1} \end{bmatrix}, \ z \in \mathbf{C}^\times,$$

where $\mu_1 - \mu_2 \in \mathbf{Z}$ and

$$\varphi(\tau) = \begin{bmatrix} 0 & (-1)^{\mu_1 - \mu_2} \\ 1 & 0 \end{bmatrix}.$$

The integer $\mu_1 - \mu_2$ must be non-zero. Otherwise the image of φ is contained in an abelian subgroup of semisimple elements and so must belong to a maximal torus, which is a proper Levi subgroup of $\hat{G}$.

Lastly, after possibly conjugating φ with

$$\begin{bmatrix} 0 & (-1)^{\mu_1-\mu_2} \\ 1 & 0 \end{bmatrix} \tag{3}$$

we may assume that $\mu_1 - \mu_2$ is a positive integer. The reader might wish to compare our results with 3.2 [Lab08], where our $\mu_1 - \mu_2$ is denoted by n and our $\mu_1 + \mu_2$ is denoted by s.

2.2 Endoscopic data

Let θ be the outer automorphism of G given by composing inverse-transpose and conjugation by w_0, that is

$$\theta(x) = w_0 (x^{-1})^{\intercal} w_0^{-1}, \; x \in G.$$

Conjugation by w_0 is included in the definition of θ so that it preserves the familiar upper-triangular Borel subgroup of G. The automorphism θ induces an automorphism $\hat{\theta}$ on the dual group $\hat{G} = \mathrm{GL}_2$ which acts in the same fashion (1.2 [KS99]).

We wish to compute $s \in \mathrm{GL}_2$ as in (1) for the discrete parameter $\varphi : W_{\mathbf{R}} \to \mathrm{GL}_2$ computed in section 2.1. To this end we compute

$$\hat{\theta}\left(\begin{bmatrix} z^{\mu_1} \bar{z}^{\mu_2} & 0 \\ 0 & z^{\mu_2} \bar{z}^{\mu_1} \end{bmatrix} \right) = \begin{bmatrix} z^{-\mu_2} \bar{z}^{-\mu_1} & 0 \\ 0 & z^{-\mu_1} \bar{z}^{-\mu_2} \end{bmatrix}, \; z \in \mathbf{C}^{\times} \tag{4}$$

and

$$\hat{\theta}\left(\begin{bmatrix} 0 & (-1)^{\mu_1-\mu_2} \\ 1 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & -1 \\ (-1)^{\mu_1-\mu_2+1} & 0 \end{bmatrix}. \tag{5}$$

If s satisfies (1) then equation (4) and $\mu_1 \neq \mu_2$ imply that $s \in \hat{T}$ and $\mu_1 = -\mu_2$. Equation (5) then implies that

$$s \in \begin{bmatrix} 1 & 0 \\ 0 & (-1)^{2\mu_1+1} \end{bmatrix} Z_{\hat{G}}.$$

It is evident that s exists if and only if $\mu_1 = -\mu_2$. The centre $Z_{\hat{G}}$ will be irrelevant in computing endoscopic groups as these come about by taking the centralizer of s. We may therefore identify s with one of two matrices. In the case that $2\mu_1$ is odd we say that μ_1 is a *half-integer* and s may be identified with the identity matrix in GL_2. Otherwise, we say that μ_1 is an integer, and the set s may be identified with

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathrm{GL}_2. \tag{6}$$

We separate the computation of endoscopic data into these two cases.

2.2.1 The case where μ_1 is a half-integer

In this case φ simplifies to

$$\begin{aligned} \varphi(z) &= \begin{bmatrix} (z/\bar{z})^{\mu_1} & 0 \\ 0 & (z/\bar{z})^{-\mu_1} \end{bmatrix}, \ z \in \mathbf{C}^\times, \\ \varphi(\tau) &= \begin{bmatrix} 0 & (-1)^{2\mu_1} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \end{aligned} \tag{7}$$

and $s \in \mathrm{GL}_2$ is the identity matrix.

The dual group $\hat{H}$ is equal to the identity component of the fixed point subgroup $\mathrm{GL}_2^{\hat{\theta}} = \mathrm{SL}_2$, which is connected. Clearly, the image of φ is contained in SL_2 so that the group $\mathcal{H} = \mathrm{SL}_2\varphi(W_{\mathbf{R}}) = \mathrm{SL}_2$. We should exercise some caution here, because our abusive identification of the codomain of φ with GL_2. The extended Galois action $\rho_{\mathcal{H}} : W_{\mathbf{R}} \to \mathrm{Aut}(\hat{H})$ only makes sense if we revert to the codomain being the direct product $\mathrm{GL}_2 \times W_{\mathbf{R}}$ in which case $\mathcal{H} = \mathrm{SL}_2 \times W_{\mathbf{R}}$. By definition, the automorphism $\rho_{\mathcal{H}}(\tau)$ acts on SL_2 by conjugation with an element $(x, \tau) \in \mathcal{H}$ which preserves a Borel subgroup and a set of positive root vectors (2.3 [Bor79]). Conjugation by $(1, \tau)$ is trivial and the only $x \in \mathrm{SL}_2$ which preserve a Borel subgroup and the root vectors are $x = \pm I$. Thus, $\rho_{\mathcal{H}}(\tau)$ and $\rho_{\mathcal{H}}$ itself are trivial. From this it follows that the L-group ${}^L H$ is equal to the direct product $\mathrm{SL}_2 \times W_{\mathbf{R}}$, $H = \mathrm{PGL}_2$ (6.6.4 [Mez09]) and $H(\mathbf{R})$ is the split form $\mathrm{PGL}(2, \mathbf{R})$. We define $\xi_H : \mathcal{H} \to {}^L H$ to be the identity map.

2.2.2 The case where μ_1 is an integer

In the case at hand

$$\begin{aligned} \varphi(z) &= \begin{bmatrix} (z/\bar{z})^{\mu_1} & 0 \\ 0 & (z/\bar{z})^{-\mu_1} \end{bmatrix}, \ z \in \mathbf{C}^\times, \\ \varphi(\tau) &= \begin{bmatrix} 0 & (-1)^{2\mu_1} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned} \tag{8}$$

The dual group $\hat{H}$ of the desired endoscopic group is given by the identity component of the fixed-point subgroup $\{x \in \hat{G} : x = s\hat{\theta}(x)s^{-1}\}$ where s is equal to (6). Elementary computations reveal that the fixed-point subgroup is the semidirect product

$$\left\{ \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix} : x \in \mathbf{C}^\times \right\} \rtimes \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

whose identity component is the rank one torus

$$\hat{H} = \left\{ \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix} : x \in \mathbf{C}^\times \right\} \cong \mathrm{GL}_1.$$

If we revert again to the codomain of φ being $\mathrm{GL}_2 \times W_{\mathbf{R}}$ then

$$\varphi(\tau) = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tau \right)$$

and

$$\mathcal{H} = \hat{H}\varphi(W_{\mathbf{R}}) \subset \mathrm{GL}_2 \times W_{\mathbf{R}}.$$

The automorphism $\rho_{\mathcal{H}}(\tau)$ acts on $\hat{H}$ by conjugation with $\varphi(\tau)$. This action is inversion on $\hat{H}$ and implies that $H \cong \mathrm{GL}_1$ is a rank one torus such that $H(\mathbf{R})$ is isomorphic to $U(1)$ (9.4 [Bor79]). One may verify that there is an isomorphism $\xi_H : \mathcal{H} \to {}^L H$ satisfying $\xi_H(\varphi(\tau)) = (I, \tau)$ and $\xi_H(h) = (h, 1)$ for all $h \in \hat{H}$.

2.3 Spectral data

We continue with the assumption that $\varphi : W_{\mathbf{R}} \to \mathrm{GL}_2$ is a discrete admissible homomorphism as in section 2.2. From the isomorphism $\xi_H : \mathcal{H} \to {}^L H$ we obtain an admissible homomorphism $\varphi_H = \xi_H \circ \varphi$ for H. The Langlands Correspondence prescribes L-packets Π_φ and Π_{φ_H} to φ and φ_H respectively.

The homomorphism φ is parameterized by positive $\mu_1 \in \frac{1}{2}\mathbf{Z}$. When μ_1 is a half-integer the group $H(\mathbf{R})$ is equal to $\mathrm{PGL}(2, \mathbf{R})$, and when μ_1 is an integer $H(\mathbf{R})$ equals $U(1)$. We compute the L-packets Π_φ and Π_{φ_H}, beginning with Π_φ and separating the remaining computations of Π_{φ_H} into the half-integral and integral cases.

2.3.1 L-packets of $\mathrm{GL}(2, \mathbf{R})$

The admissible homomorphism φ is completely determined by the positive element $\mu_1 \in \frac{1}{2}\mathbf{Z}$. Our first task is to trace the path from this element to a distribution character of a representation in Π_φ as argued on page 132 [Lan89]. The crux of this argument is the pioneering work of Harish-Chandra, which attaches to every (dominant and analytically integral) weight of a maximally compact torus of $\mathrm{GL}(2, \mathbf{R})$ a distribution character of an essentially square-integrable representation of $\mathrm{GL}(2, \mathbf{R})$ (Theorem 9.20 [Kna86]).

In order to apply this work we must identify μ_1 with a weight of some maximally compact torus. Let S be the maximal torus of GL_2 defined over $\mathbf{R}$ such that $S(\mathbf{R})$ is equal to the set of elements of the form

$$\begin{bmatrix} \cos(a) & \sin(a) \\ -\sin(a) & \cos(a) \end{bmatrix} \begin{bmatrix} e^b & 0 \\ 0 & e^b \end{bmatrix}, \; a, b \in \mathbf{R}. \tag{9}$$

The group $S(\mathbf{R})$ is a maximally compact $\mathbf{R}$-torus in $\mathrm{GL}(2, \mathbf{R})$. We need some way of getting a weight of this torus from μ_1, or equivalently, from φ. Since the image of φ is attached to $\hat{T}$, one might expect to get a weight of S by associating S with T. The dual group $\hat{S}$ is isomorphic to the diagonal subgroup $\hat{T}$, as both are complex rank two tori. This isomorphism does not detect the real structure of S or T though, and these real structures are different. The real structures of S and T are encoded in their L-groups. As a result, the L-group ${}^L S = S \rtimes W_{\mathbf{R}}$ is not isomorphic to ${}^L T = \hat{T} \times W_{\mathbf{R}}$. In fact ${}^L S$ is isomorphic to $\hat{T} \rtimes \varphi(W_{\mathbf{R}})$ (*cf.* 9.4 [Bor79]). The trick then, is to identify ${}^L S$ with $\hat{T} \rtimes \varphi(W_{\mathbf{R}})$. In making this identification, the cocharacter group $X_*(\hat{S}) = \mathrm{Hom}(\mathrm{GL}_1, \hat{S})$ is identified with $X_*(\hat{T})$. By definition, the character group $X^*(S) = \mathrm{Hom}(S, \mathrm{GL}_1)$ is equal to $X_*(\hat{S})$ and so may also be identified with $X_*(\hat{T}) = X^*(T)$.

Taking these identifications into consideration, let us forget about S for a moment. The cocharacters in $X_*(\hat{T})$ are of the form

$$z \mapsto \begin{bmatrix} z^{m_1} & 0 \\ 0 & z^{m_2} \end{bmatrix},\ z \in \mathbf{C}^\times, \tag{10}$$

where m_1 and m_2 are integers. Extending $X_*(\hat{T})$ to $X_*(\hat{T}) \otimes \mathbf{C}$ has the effect of allowing m_1 and m_2 to belong to $\mathbf{C}$. The element $\mu_1 \in \frac{1}{2}\mathbf{Z}$ characterizing φ is assigned to the element in $X_*(\hat{T}) \otimes \mathbf{C}$ given by $m_1 = \mu_1 - (1/2)$ and $m_2 = -m_1$ ($-1/2$ should be thought of as the half-sum of the positive root(s)). The weights of T lie in $X^*(T) \otimes \mathbf{C} = X_*(\hat{T}) \otimes \mathbf{C}$. Langlands shows that

$$z \mapsto \begin{bmatrix} z^{\mu_1 - 1/2} & 0 \\ 0 & z^{-\mu_1 + 1/2} \end{bmatrix},\ z \in \mathbf{C}^\times$$

corresponds to a (analytically integral) weight of $S(\mathbf{R})$.

To say what this weight of $S(\mathbf{R})$ is, let us first give its values on T. Since every element of T may be written as a product

$$\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix} \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix},\ x, y \in \mathbf{C}^\times, \tag{11}$$

it suffices to say what the value of this weight is on each of these two factors. On the first factor the value is $x^{m_1}(x^{-1})^{-m_1} = x^{2\mu_1 - 1}$, and on the second factor the value is $y^{m_1} y^{-m_1} = 1$ (*cf.* Lemma 3.2.11 [Spr98]). To pass from T to $S(\mathbf{R})$ we identify the first (resp. second) factor in (11) with the complexification of the first (resp. second) factor in (9) (*cf.* the Cayley transform in section 2.4.1). Under this identification we obtain the weight of $S(\mathbf{R})$ which maps an element in (9) to $e^{i(2\mu_1 - 1)a}$ (page 131 and Lemma 2.8 [Lan89]).

It is clear from (9) that $S(\mathbf{R})\mathrm{SL}(2, \mathbf{R}) = Z\mathrm{SL}(2, \mathbf{R})$. This is a subgroup of $\mathrm{GL}(2, \mathbf{R})$ of index two. One applies Harish-Chandra's theory by first associating the above weight of $S(\mathbf{R})$ to an essentially square integrable representation $\pi^+_{\mu_1}$ of $Z\mathrm{SL}(2, \mathbf{R})$. The value of the distribution character of $\pi^+_{\mu_1}$ on regular elements of the form (9) is

$$-\frac{e^{i(2\mu_1 - 1)a}}{1 - e^{-2ia}} = -\frac{e^{2i\mu_1 a}}{e^{ia} - e^{-ia}} \tag{12}$$

(page 134 [Lan89]), and these values uniquely determine the representation (Theorem 12.6 [Kna86]). Finally, the representation π_{μ_1} of $\mathrm{GL}(2, \mathbf{R})$ obtained from $\pi^+_{\mu_1}$ by induction is irreducible. Its distribution character on regular elements of the form (9) has values

$$-\frac{e^{2i\mu_1 a} - e^{-2i\mu_1 a}}{e^{ia} - e^{-ia}}. \tag{13}$$

This is a representation in Π_φ prescribed by φ.

It is also the only representation in Π_φ. To explain why, we must return to our isomorphism of $\hat{T} \rtimes \varphi(W_{\mathbf{R}})$ with ${}^L S$. This isomorphism is not unique. Indeed, composition of this isomorphism with an inner automorphism normalizing

$\hat{T}$ yields another isomorphism. In particular, composition by the action of the Weyl group of $\hat{T}$ yields isomorphisms. This action of the Weyl group may be accounted for by fixing the isomorphism as we have done and acting on φ instead. For example one could act on φ by the inner automorphism of the non-trivial Weyl group representative given in (3). The reader might find it worthwhile to repeat the steps we have taken on the resulting admissible homomorphism. Is so doing one verifies that the roles of $\mu_1$and μ_2 are reversed. In turn we have that $\mu_2 - \mu_1$ is a *negative* integer, and Harish-Chandra's theory attaches a representation $\pi^-_{\mu_1}$ of $Z\mathrm{SL}(2, \mathbf{R})$ to this the *negative* number $\mu_2 = -\mu_1 \in \frac{1}{2}\mathbf{Z}$. The character values on regular elements of (9) are

$$-\frac{e^{i(2\mu_2+1)a}}{1-e^{2ia}} = \frac{e^{-2i\mu_1 a}}{e^{ia} - e^{-ia}}. \tag{14}$$

However, upon induction to $\mathrm{GL}(2, \mathbf{R})$ one again recovers (13) and therefore the same representation π_{μ_1}. This independence of Weyl group action is a special property of $\mathrm{GL}(2, \mathbf{R})$. In general, these Weyl group actions do produce inequivalent representations which comprise the L-packet (7.1 [Lab08]).

The reader familiar with the theory of discrete series representations of semisimple groups may wish to compare our notation with familiar sources. The representation $\pi^+_{\mu_1}$ is trivial on the identity component of the central group Z and is therefore readily identified with a discrete series representation of $\mathrm{SL}(2, \mathbf{R})$ which is usually denoted by either $\mathcal{D}^+_{2\mu_1+1}$ (II.5 [Kna86]) or $\mathcal{D}^+_{2\mu_1}$ (4.2 [Lab08]). The representation $\pi^-_{\mu_1}$ may similarly be identified with the discrete series representation denoted by $\mathcal{D}^-_{2\mu_1+1}$ or $\mathcal{D}^-_{2\mu_1}$.

2.3.2 L-packets of $H(\mathbf{R})$ when μ_1 is a half-integer

We are assuming that μ_1 is a half-integer so that $H(\mathbf{R}) = \mathrm{PGL}(2, \mathbf{R})$. The admissible homomorphism φ_H is identical to φ in (7) save for its codomain, which may be identified with the subgroup $\hat{H} = \mathrm{SL}_2$ of GL_2.

Only minor changes are required in section 2.3.1 to obtain Π_{φ_H}, although there are some conceptual differences worth highlighting. Setting $Z = Z_G(\mathbf{R})$ equal to the centre of $\mathrm{GL}(2, \mathbf{R})$, the maximal torus of PGL_2 may be taken to be $S_H = S/Z_G$. Its group of real points is

$$S_H(\mathbf{R}) = \left\{ \begin{bmatrix} \cos(a) & \sin(a) \\ -\sin(a) & \cos(a) \end{bmatrix} Z, \ a \in \mathbf{R} \right\}.$$

We wish to pass from the parameter $\mu_1 \in \frac{1}{2}\mathbf{Z}$ to a character of $S_H(\mathbf{R})$. As before, we may think of μ_1 as an element in $X_*(\hat{T}) \otimes \mathbf{C}$. Actually, this element is fixed under $\mathrm{Int}(s)\hat{\theta}$, which is equal to $\hat{\theta}$ on $\hat{T}$. Consequently, we may think of μ_1 as belonging to the fixed-point set $X_*(\hat{T})^{\hat{\theta}} \otimes \mathbf{C}$. We would like to proceed from here to an element of $X^*(T/Z_G) \otimes \mathbf{C}$ and thence to $X^*(S_H) \otimes \mathbf{C}$ and a character of $S_H(\mathbf{R})$. Some abstract algebra is required. Let us write the homomorphism

$$t \mapsto t\theta(t)^{-1}, \ t \in T$$

additively as $1-\theta$. Then the exact sequence of tori

$$1 \to (1-\theta)T \to T \to T/(1-\theta)T \to 1$$

gives rise to an exact sequence of free $\mathbf{Z}$-modules

$$1 \to X^*(T/(1-\theta)T) \to X^*(T) \to X^*((1-\theta)T) \to 1$$

(Proposition 8.2 (c) [Bor91]). The third map in the latter sequence is restriction to $(1-\theta)T$, and so the image of the second map is the θ-fixed subgroup $X^*(T)^\theta$. It is easily verified that $(1-\theta)T = Z_G$ and thus it follows that

$$X_*(\hat{T})^{\hat{\theta}} \otimes \mathbf{C} \cong X^*(T)^\theta \otimes \mathbf{C} \cong X^*(T/Z_G) \otimes \mathbf{C}. \tag{15}$$

Now returning to μ_1 being thought of as an element in $X_*(\hat{T})^{\hat{\theta}} \otimes \mathbf{C}$, we make the same adjustments to μ_1 as in section 2.3.1, namely setting $m_1 = \mu_1 - 1/2 = -m_2$. Next, we use (15) and identify T/Z_G with S_H to obtain the weight of $S_H(\mathbf{R})$ which has values of the form $e^{i(2\mu_1-1)a}$.

This weight corresponds to a discrete series representation of the identity component of $\mathrm{PGL}(2,\mathbf{R})$ (in the real manifold topology). This identity component is isomorphic to $\mathrm{PSL}(2,\mathbf{R}) = \mathrm{SL}(2,\mathbf{R})/\{\pm I\}$ and is of index two[1]. The character of this discrete series representation has values of the form

$$-\frac{e^{i(2\mu_1-1)a}}{1-e^{-2ia}} = -\frac{e^{2i\mu_1 a}}{e^{ia}-e^{-ia}}$$

on the regular elements of $S_H(\mathbf{R})$. Inducing this representation to $\mathrm{PGL}(2,\mathbf{R})$ produces a discrete series representation whose character is again of the form (13) on regular elements of $S_H(\mathbf{R})$. It is the only representation in the L-packet Π_{φ_H}.

2.3.3 L-packets of $H(\mathbf{R})$ when μ_1 is an integer

We are assuming that μ_1 is an integer so that $H(\mathbf{R}) = U(1)$. The admissible homomorphism $\varphi_H : W_{\mathbf{R}} \to {}^L H$ satisfies $\varphi_H(\tau) = (I,\tau)$ which acts by inversion on $\hat{H} \cong \mathrm{GL}_1$. Furthermore,

$$\varphi_H(z) = \begin{bmatrix} (z/\bar{z})^{\mu_1} & 0 \\ 0 & (z/\bar{z})^{-\mu_1} \end{bmatrix}, \ z \in \mathbf{C}^\times.$$

The path from this admissible homomorphism to Π_{φ_H} does not require Harish-Chandra's theory of discrete series for there is no semisimple Lie group associated to $U(1)$ from which one could induce. In this case we follow the more direct correspondence for tori (2 [Lan89], 9.2 [Bor79]). In the correspondence for tori we continue think of μ_1 as an element of $X_*(\hat{T})^{\hat{\theta}} \otimes \mathbf{C}$, but no adjustment

[1] Warning! $\mathrm{SL}(2,\mathbf{R})/\{\pm I\}$ is not an algebraic group so that it is not equal to the split real form of PSL_2.

is made to μ_1. That is, we take $m_1 = \mu_1 = -m_2$ as in (10). The isomorphisms of (15) remain valid in the present case and they yield the element in $X^*(T/Z_G) \subset X^*(T/Z_G) \otimes \mathbf{C}$ defined by

$$\left[\begin{array}{cc} x & 0 \\ 0 & x^{-1} \end{array}\right] Z_G \mapsto x^{2\mu_1}. \tag{16}$$

In order to connect this element of $X^*(T/Z_G)$ to $X^*(H)$ we remind ourselves that $\hat{H} = (\hat{T}^{\hat{\theta}})^0$. As consequences we have that $X_*(\hat{H})$ is isomorphic to the free abelian subgroup of $X_*(\hat{T}^{\hat{\theta}})$, and $X^*(H) \otimes \mathbf{C} \cong X^*(T/Z_G) \otimes \mathbf{C}$. The image of the element (16) under the latter isomorphism determines the character

$$e^{ia} \mapsto e^{2i\mu_1 a}, \ a \in \mathbf{R} \tag{17}$$

on $H(\mathbf{R}) = U(1)$ (9.4 (a) [Bor79]), and this character is equal to the unique representation in Π_{φ_H}.

2.3.4 Twisted characters for $\mathrm{GL}(2, \mathbf{R})$

Having determined the L-packets for the endoscopic groups we revisit the L-packets for $\mathrm{GL}(2, \mathbf{R})$. In section 2.3.1 we saw that the L-packet Π_φ consists of a single representation $\pi = \pi_{\mu_1}$ whose character is determined by (13). It is not this character which is comparable with the character of the representation in Π_{φ_H}. Rather it is the *twisted* character of π which is comparable.

To define the twisted character we need to introduce the twisting automorphism θ into the picture. Equation (1) implies that π is equivalent to the representation $\pi^\theta = \pi \circ \theta$ (Lemma 1 [Mez13]). Let us denote an intertwining operator which exhibits this equivalence by U, *i.e.*

$$\mathsf{U}\, \pi(x) = \pi^\theta(x)\, \mathsf{U}, \ x \in \mathrm{GL}(2, \mathbf{R}).$$

By Schur's Lemma, the intertwining operator U is unique up to scalar multiplication. For $\mathrm{GL}(2, \mathbf{R})$ the circumstances are simple enough to describe a canonical unitary operator U in terms of the inducing data in the definition of π. We shall do this shortly. What this allows us to do is define the twisted character of π as the distribution

$$f \mapsto \mathrm{tr} \int_{\mathrm{GL}(2,\mathbf{R})} f(x)\pi(x)\mathsf{U}\, dx \tag{18}$$

defined on smooth compactly supported functions $f : \mathrm{GL}(2, \mathbf{R}) \to \mathbf{C}$ (5.2 [Mez13]). As in the case of ordinary distribution characters (*cf.* (13)), this twisted character may be identified with a function defined on the regular elements of the torus $S(\mathbf{R})$ (Theorem 2.1.1 [Bou87], Theorem 15.1 [Ren97]).

Let us describe the operator U and compute the resulting twisted character. Set $\pi^\pm = \pi^\pm_{\mu_1}$ and let $V^\pm$ be the space of the representations $\pi^\pm$. Recall that $\pi = \mathrm{ind}_{Z\mathrm{SL}(2,\mathbf{R})}^{\mathrm{GL}(2,\mathbf{R})} \pi^+$. The subgroup $Z\,\mathrm{SL}(2, \mathbf{R})$ is of index two in $\mathrm{GL}(2, \mathbf{R})$, and

$$\alpha = \left[\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array}\right]$$

is a representative of the non-trivial coset. By virtue of the finite index, Mackey's decomposition theorem applies to π with the result that

$$\pi_{|Z\,\mathrm{SL}(2,\mathbf{R})} \cong \pi^+ \oplus (\pi^+)^\alpha.$$

Here $(\pi^+)^\alpha$ is defined as $\pi^+ \circ \mathrm{Int}(\alpha)$. If one evaluates $\mathrm{Int}(\alpha)$ on regular elements of (9) and composes with π^+'s character (12), one arrives at π^-'s character (14). It follows that $(\pi^+)^\alpha \cong \pi^-$, and we may identify the space of π with

$$V^+ \oplus \pi(\alpha)V^+ \cong V^+ \oplus V^-,$$

where $\pi(x)$ acts as π^+ on V^+ for all $x \in Z\,\mathrm{SL}(2,\mathbf{R})$.

Let U be the unitary operator on $V^+ \oplus \pi(\alpha)V^+$ defined by

$$\mathsf{U}(v_1 + \pi(\alpha)v_2) = v_1 + (-1)^{2\mu_1+1}\pi(\alpha)v_2,\ v_1, v_2 \in V^+.$$

The factor $(-1)^{2\mu_1+1}$ is significant as it is equal to $\pi(-I)$. Observe that if $y \in \mathrm{SL}(2,\mathbf{R})$ then $\theta(y) = y$ and

$$\theta(y\alpha) = y(-\alpha) = -y\alpha.$$

Therefore

$$\mathsf{U}\pi^\theta(y)(v_1 + \pi(\alpha)v_2) = \pi(y)v_1 + (-1)^{2\mu_1+1}\pi(y\alpha)v_2 = \pi(y)\mathsf{U}(v_1 + \pi(\alpha)v_2),$$

and

$$\begin{aligned}\mathsf{U}\pi^\theta(y\alpha)(v_1 + \pi(\alpha)v_2) &= (-1)^{2\mu_1+1}\mathsf{U}(\pi(y)\pi(\alpha)v_1 + \pi(y)v_2)\\ &= (-1)^{2\mu_1+1}\left((-1)^{2\mu_1+1}\pi(y)\pi(\alpha)v_1 + \pi(y)v_2\right)\\ &= \pi(y\alpha)(v_1 + (-1)^{2\mu_1+1}\pi(\alpha)v_2)\\ &= \pi(y\alpha)\mathsf{U}(v_1 + \pi(\alpha)v_2).\end{aligned}$$

This proves that U intertwines π^θ with π.

We may now compute the twisted character of π relative to U. For f as in (18) we compute

$$\mathrm{tr} \int_G f(x)\pi(x)\mathsf{U}\,dx = \mathrm{tr} \int_{Z\,\mathrm{SL}(2,\mathbf{R})} f(x)\pi(x)\mathsf{U} + f(x\alpha)\pi(x\alpha)\mathsf{U}\,dx.$$

Fixing an orthonormal basis $\mathcal{B}$ relative to an inner product $\langle\cdot,\cdot\rangle$ on V^+, this becomes

$$
\begin{aligned}
&\sum_{v\in\mathcal{B}}\sum_{j,k=0}^{1}\int f(x\alpha^j)\ \langle \pi(x\alpha^j)\mathsf{U}\pi(\alpha^k)v,\pi(\alpha^k)v\rangle\,dx\\
&\quad = \sum_{v\in\mathcal{B}}\sum_{j,k=0}^{1}\int f(x\alpha^j)\ \langle \pi(x)\pi(\alpha^j)(-1)^{k(2\mu_1+1)}\pi(\alpha^k)v,\pi(\alpha^k)v\rangle\,dx\\
&\quad = \sum_{v\in\mathcal{B}}\sum_{k=0}^{1}(-1)^{k(2\mu_1+1)}\int f(x)\ \langle \pi(x)\pi(\alpha^k)v,\pi(\alpha^k)v\rangle\,dx\\
&\quad = \mathrm{tr}\int f(x)\pi(x)_{|V^+}\,dx+(-1)^{2\mu_1+1}\ \mathrm{tr}\int f(x)\pi(x)_{|\pi(\alpha)V^+}\,dx\\
&\quad = \mathrm{tr}\int f(x)\pi^+(x)\,dx+(-1)^{2\mu_1+1}\mathrm{tr}\int f(x)\pi^-(x)\,dx,
\end{aligned}
$$

where all integrals are over $Z\,\mathrm{SL}(2,\mathbf{R})$.

This computation shows that we may identify the twisted character of π with the *sum* of the characters of π^+ and π^- when μ_1 is half-integral, and the *difference* of the characters of π^+ and π^- when μ_1 is integral. In both cases, the support of the twisted character lies in the subgroup $Z\,\mathrm{SL}(2,\mathbf{R})$.

When μ_1 is half-integral the values of the twisted character on the regular elements of (9) are again equal to (13), which is the sum of (12) and (14). By contrast, when μ_1 is integral the values of the twisted character on the regular elements of (9) are equal to

$$
-\frac{e^{2i\mu_1 a}+e^{-2i\mu_1 a}}{e^{ia}-e^{-ia}}, \tag{19}
$$

which is the difference of (12) and (14). It is interesting to note that the latter twisted character vanishes on the split tori of $Z\,\mathrm{SL}(2,\mathbf{R})$ (Proposition 10.14 [Kna86]). As with ordinary characters, the twisted characters are determined by (13) and (19) (Theorem 15.1 [Ren97]).

2.4 Transfer

In this section we shall spell out the correspondence given in Theorem 3.3.A [KS99] in the special case of $G=\mathrm{GL}_2$. This correspondence is given by a map from semisimple conjugacy classes of an endoscopic group H and the *twisted* semisimple conjugacy classes of GL_2. The twisted conjugacy class of an element $x\in\mathrm{GL}_2$ relative to the automorphism θ is by definition the set $\{g^{-1}x\theta(g):g\in\mathrm{GL}_2\}$. It is more precise to call this a θ-conjugacy class. One calls this a semisimple θ-conjugacy class if x is a semisimple element of GL_2.

Once this correspondence is made clear we may use it to compare the representation in Π_φ with representations in Π_{φ_H}. The comparison is effected by identifying the representation in Π_φ with its *twisted* character as in section 2.3.4. Since this twisted character is determined by its values on semisimple

θ-conjugacy classes, one may pull back this twisted character to a class function on H using the above correspondence. The resulting class function may be then be compared with the character of the representation in Π_{φ_H}. Through such a comparison, a precise character identity will become apparent and this identity constitutes a notion of transferring the representation in Π_{φ_H} to the representation in Π_{φ}.

2.4.1 The correspondence of twisted conjugacy classes

Recall from section 2.3.4 that the twisted character of $\pi \in \Pi_{\varphi}$ is completely determined by its values on the regular elements of $S(\mathbf{R})$. We therefore identify the twisted character of π with (13) when μ_1 is a half-integer, and with (19) when μ_1 is an integer. We wish to pull back these twisted characters to the endoscopic group $H(\mathbf{R})$. In general, the torus S is not directly related to any torus in H. There is an indirect relationship through the intermediate maximal torus T, which relates θ-conjugacy classes of elements in S to conjugacy classes of elements in a torus of H (3.3 [KS99]). The approach is to start with the conjugacy class of a toral element in H, move to a twisted conjugacy class of an element in T, and then end up with a twisted conjugacy class of an element in S. Working backwards, we will present the passage from T to S in this approach first.

Although it is mathematically pointless, we will make some notational distinctions which have the advantage of being in line with the exposition of 3.3 [KS99]. Set $G^* = G = \mathrm{GL}_2$ and $\theta^* = \theta$. We assemble two triples, (G, θ, S) and (G^*, θ^*, T). Our task is to specify a bijection between the set $Cl_{ss}(G, \theta)$ of semisimple θ-conjugacy classes of G and the set $Cl_{ss}(G^*, \theta^*)$ of of semisimple θ^*-conjugacy classes of G^*. Upon completion, we shall do away with the notational distinctions and recover the passage between T and S sought above.

Let $\psi = \mathrm{Int}(c)$ where

$$c = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \in \mathrm{GL}_2.$$

We may regard ψ as an isomorphism $\psi : G \to G^*$. Some readers will recognize this map as a *Cayley transform* (*pp.* 417-418 [Kna86]) and that $\psi(S) = T$. The reader may wish to verify that $\theta^* = \psi\theta\psi^{-1}$ (*cf.* 1.2 [KS99]), and that ψ induces a bijection

$$Cl_{ss}(G, \theta) \to Cl_{ss}(G^*, \theta^*) \tag{20}$$

(*cf.* 3.1 [KS99]) by way of the map

$$gx\theta(g^{-1}) \mapsto g(cxc^{-1})\,\theta(g^{-1}) = (gc)\,x\,\theta^*((gc)^{-1}),\ \ x, g \in G.$$

Let us now begin to remove the artificial distinction between G^* and G by expressing bijection (20) in terms of the tori S and T. Let $\Omega(G, S)$ and $\Omega(G, T)$ be the Weyl groups of S and T respectively. It follows from the fact that all maximal tori are conjugate that the semisimple conjugacy classes of G are in bijection with the equivalence classes $T/\Omega(G,T)$ (or $S/\Omega(S,T)$). Lemma 3.2.A [KS99] extends this result to a bijection between $Cl_{ss}(G,\theta)$ and $T_\theta/\Omega(G,T)^\theta$. Here, T_θ is the quotient $T/(1-\theta)T$ and $\Omega(G,T)^\theta$ is the subgroup elements in $\Omega(G,T)$ whose action on T commutes with θ. Recall from section 2.3.2 that $T/(1-\theta)T = T/Z_G$. It is easily verified that $\Omega(G,T) = \Omega(G,T)^\theta$. Similarly, $S/(1-\theta)S = S/Z_G$ and $\Omega(G,S) = \Omega(G,S)^\theta$. In conclusion, the bijection (20) of semisimple θ-conjugacy classes may be expressed as a bijection

$$(S/Z_G)/\Omega(G,S) \to (T/Z_G)/\Omega(G,T).$$

We now make the final connection between $(T/Z_G)/\Omega(G,T)$ and the conjugacy classes of toral elements in H. In the case that μ_1 is a half-integer we have an isomorphism between T/Z_G and S_H which is again given by $\mathrm{Int}(c)$. Let us call this isomorphism ψ_H and let us denote the set of semisimple conjugacy classes of H by $Cl_{ss}(H)$. Then we may combine our results to form the map

(21)

$$Cl_{ss}(H) \to S_H/\Omega(H,S_H) \overset{\psi_H}{\to} (T/Z_G)/\Omega(G,T) \overset{\psi^{-1}}{\to} (S/Z_G)/\Omega(G,S) \to Cl_{ss}(G,\theta).$$

This is the map $\mathcal{A}_{H/G}$ of Theorem 3.3.A [KS99]. If we truncate this map to $S_H/\Omega(H,S_H) \to (S/Z_G)/\Omega(G,S)$ we see that it is defined over $\mathbf{R}$, and given by

$$\begin{bmatrix} \cos(a) & \sin(a) \\ -\sin(a) & \cos(a) \end{bmatrix} Z \mapsto \begin{bmatrix} \cos(a) & \sin(a) \\ -\sin(a) & \cos(a) \end{bmatrix} Z,\ a \in \mathbf{R}. \tag{22}$$

This exposes that the middle portion of our map $\mathcal{A}_{H/G}$ is the identity map and that one could have just as well done without ψ and ψ_H. This also hints at a more general phenomenon which ensures that $\mathcal{A}_{H/G}$ may be defined from a map of tori defined over $\mathbf{R}$ (Lemma 3.3.B [KS99]).

In the case that μ_1 is an integer, the endoscopic group H itself is a rank one torus so that $Cl_{ss}(H) = H$. In this case the map $\mathcal{A}_{H/G}$ is of the form

$$H \overset{\psi_H}{\to} (T/Z_G)/\Omega(G,T) \overset{\psi^{-1}}{\to} (S/Z_G)/\Omega(G,S) \to Cl_{ss}(G,\theta), \tag{23}$$

where ψ_H is defined by

$$e^{ia} \mapsto \begin{bmatrix} e^{ia} & 0 \\ 0 & e^{-ia} \end{bmatrix} Z,\ a \in \mathbf{C} \tag{24}$$

(*cf.* (16)). The truncated map $H \to (S/Z_G)/\Omega(G,S)$ is again defined over $\mathbf{R}$ and is defined by

$$e^{ia} \mapsto \begin{bmatrix} \cos(a) & \sin(a) \\ -\sin(a) & \cos(a) \end{bmatrix} Z,\ a \in \mathbf{R}. \tag{25}$$

Unlike the half-integral case, the map $H \to (S/Z_G)/\Omega(G,S)$ is not injective. Indeed, conjugation by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

acts as inversion on S and lies in the Weyl group $\Omega(G,S)$. Therefore both e^{ia} and e^{-ia} in $H(\mathbf{R})$ have the same image in $(S/Z_G)/\Omega(G,S)$.

2.4.2 Spectral transfer

We are now ready to compare the twisted character of $\pi \in \Pi_\varphi$ with the character of $\pi_H \in \Pi_{\varphi_H}$. The twisted character is determined by its values on the real representatives of $Cl_{ss}(G, \theta)$, or alternatively, by its values on the real representatives of $(S/Z_G)/\Omega(G, S)$ (section 2.4.1). Analogously, the ordinary character of π_H is determined by its values on the real representatives of $S_H/\Omega(H, S_H)$ when μ_1 is half-integral, and by its values on $H(\mathbf{R})$ when μ_1 is integral.

If we compose the twisted character of π with the map $\mathcal{A}_{H/G}$ we obtain an invariant distribution on $H(\mathbf{R})$. Let us compare this distribution with the distribution of π_H in the separate cases of μ_1 being half-integral and integral.

When μ_1 is a half-integer, the twisted character on $S(\mathbf{R})$ is (13), and its composition with (22) is equal to itself. This is also equal the character of π_H on $S_H(\mathbf{R})$ (section 2.3.2). The twisted character identity between π and π_H in this case is deceptively obvious.

On the other hand, when μ_1 is an integer the twisted character is equal to (19) on $S(\mathbf{R})$ and its composition with (25) is the distribution on $H(\mathbf{R})$ given by the function

$$e^{ia} \mapsto -\frac{e^{2i\mu_1 a} + e^{-2i\mu_1 a}}{e^{ia} - e^{-ia}}, \; a \in \mathbf{R}. \tag{26}$$

We are to compare this function with (17). There are two apparent distinguishing features. First, there is the distinguishing factor of $-1/(e^{ia} - e^{-ia})$. This is the negative reciprocal of the Weyl denominator for $\mathrm{GL}(2, \mathbf{R})$ (*p.* 141 [Kna86]). The corresponding Weyl denominator does not occur for $H(\mathbf{R})$ for it has no roots. Thus, one sees that more generally a comparison of characters requires adjustment by Weyl denominators.

The second distinguishing feature is that the numerator of (26) contains not only $e^{2i\mu_1 a}$, but also its inverse $e^{-2i\mu_1 a}$. This is mirrored by the two-to-one nature of map (25). The occurrence of both $e^{2i\mu_1 a}$ and $e^{-2i\mu_1 a}$ is necessitated by the principle that endoscopy should be invariant under conjugation. One can see this by hearkening back to the end of section 2.3.1 where conjugation of φ by (3) had the effect of replacing the positive number μ_1 by the negative number $-\mu_1$. If one were to follow through our arguments using this modification of φ then one would arrive to the *same* L-packet of G. By contrast, one would arrive to the L-packet of of H containing the representation $e^{ia} \mapsto e^{-2\mu_1 a}$ (*cf.* (17)).

Yet another reason of seeing why both $e^{2i\mu_1 a}$ and $e^{-2i\mu_1 a}$ must occur is in the choice of the map (24). One could equally well have chosen

$$e^{-ia} \mapsto \begin{bmatrix} e^{ia} & 0 \\ 0 & e^{-ia} \end{bmatrix} Z, \; a \in \mathbf{C}$$

and this modification too would have again forced the representation $e^{ia} \mapsto e^{-2\mu_1 a}$ into the picture. The choice of map (24) is actually again a manifestation principle of invariance under conjugation. Indeed, the choices of map stem from choices of Borel subgroups (see proof of Theorem 3.3.A [KS99]), and these are all equivalent under conjugation.

In conclusion, one obtains an identity between (26) and (17) after accounting for the Weyl denominator and the invariance under conjugation. In general there is much more to account for. For example, we have said nothing about the minus sign in front of the Weyl denominator.

3 Twisted endoscopy for $\mathrm{SU}(2,1)$

The special unitary group $\mathrm{SU}(2,1)$ may be defined as

$$\mathrm{SU}(2,1) = \{g \in \mathrm{SL}_3 : \bar{g}^{\intercal} I_{2,1} g = I_{2,1}\}$$

where $I_{2,1}$ is the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Its real Lie algebra $\mathfrak{su}(2,1)$ is given by the set of elements of the form

$$\begin{bmatrix} ia & X_{12} & X_{13} \\ -\overline{X_{12}} & -ia+ib & X_{23} \\ \overline{X_{13}} & \overline{X_{23}} & -ib \end{bmatrix}, \ a, b \in \mathbf{R}, \ X_{12}, X_{13}, X_{23} \in \mathbf{C}$$

(I.8, I.14 [Kna96]). The group $\mathrm{SU}(2,1)$ is a real form of SL_3. This is evident from the isomorphism $\mathfrak{su}(2,1) \otimes \mathbf{C} \cong \mathfrak{sl}(2, \mathbf{C})$. We therefore set $G' = \mathrm{SL}_3$ and $G'(\mathbf{R}) = \mathrm{SU}(2,1)$. The dual group $\hat{G}'$ may be computed to equal PGL_3 by modifying the computations 6.6.4 [Mez09].

3.1 The L-group and discrete parameters

Unlike the case of $\mathrm{GL}(2, \mathbf{R})$, the group $\mathrm{SU}(2,1)$ is not split and so the L-group is no longer a direct product of $\hat{G}'$ with $W_{\mathbf{R}}$. Indeed, the maximally $\mathbf{R}$-split tori of $\mathrm{SU}(2,1)$ are all conjugate and not split (*p.* 432 [Kna86]). It is an elementary computation to show that the centralizer of any maximally $\mathbf{R}$-split torus is equal to itself. From this it follows that $\mathrm{SU}(2,1)$ is *quasisplit*, *i.e.* that there is a Borel subgroup B' of G' which is defined over $\mathbf{R}$ (Proposition 16.2.2 [Spr98]). The Galois group $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ acts on B' through the resulting $\mathbf{R}$-structure and one can show that this action is the restriction of an outer automorphism of G' of order two (1.3 [Bor79]). This outer automorphism determines an algebraic automorphism of $\hat{G}'$ (2.3 [Bor79]). The L-group ${}^L G'$ of $\mathrm{SU}(2,1)$ is defined as the semidirect product

$$\hat{G}' \rtimes W_{\mathbf{R}} = \mathrm{PGL}_3 \rtimes W_{\mathbf{R}}$$

in which $\mathbf{C}^{\times} \subset W_{\mathbf{R}}$ acts trivially on $\hat{G}'$ and τ acts on $\hat{G}'$ by the outer automorphism above.

In order to pin down the outer automorphism above, one could explicitly compute B' and make the necessary choices in 1-2 [Bor79]. To avoid these gruesome computations, one may note that there is a unique outer automorphism of PGL_3 which fixes a Borel subgroup and a choice of simple root spaces thereof (27.4 [Hum75]). If one were to take the Borel subgroup to be the upper-triangular subgroup, with corresponding simple roots $\{\alpha_1, \alpha_2\}$, then the outer automorphism is given by composing inverse, transpose and the action of the longest Weyl group element. For reasons to be made clear later, we shall choose our Borel subgroup $\hat{B}' \subset \hat{G}'$ to correspond to the pair of roots $\{-\alpha_2, \alpha_1 + \alpha_2\}$. Hence, the elements of $\hat{B}'$ look like

$$\begin{bmatrix} * & * & * \\ 0 & * & 0 \\ 0 & * & * \end{bmatrix} Z'$$

where Z' is the centre of GL_3, and the outer automorphism of $W_{\mathbf{R}}$ on PGL_3 is defined by

$$\tau \cdot gZ' = w_1 (g^{\intercal})^{-1} w_1^{-1} Z', \; g \in \mathrm{PGL}_3 \tag{27}$$

where

$$w_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

This completes our description of the L-group and so we turn to finding discrete admissible homomorphisms. The basic requirements given in section 2.1 lead us to seek homomorphisms $\varphi : W_{\mathbf{R}} \to {}^L G'$ satisfying

1. For all $w \in W_{\mathbf{R}}$ there exists $gZ' \in \mathrm{PGL}_3$ such that $\varphi(w) = (gZ', w)$.
2. There exists $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathbf{C}$, with $\mu_k - \nu_k \in \mathbf{Z}$ for each $k = 1, 2$, such that $g = \mathrm{diag}(w^{\mu_1} \overline{w}^{\nu_1}, w^{\mu_2} \overline{w}^{\nu_2}, 1)$ when $w \in \mathbf{C}^\times$ in 1.
3. The first coordinate of $\phi(\tau)$ normalizes the diagonal subgroup in PGL_3.
4. The diagonal subgroup in PGL_3 has a finite number of fixed points under conjugation by $\phi(\tau)$.

The first three items in this list originate from φ being an admissible homomorphism (8.2 [Bor79]). The final item originates from φ being discrete. Indeed, if the fixed-point subgroup is infinite then it contains a torus, and the centralizer of this torus would be a proper Levi subgroup of ${}^L G'$ containing the image of φ (Lemma 3.5 [Bor79]).

A healthy calculation under the above four constraints reveals that, up to conjugation by elements of $\hat{G}'$, the admissible homomorphism φ is given by $\varphi(\tau) = (g' Z', \tau)$ where

$$g' = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and for all $z \in \mathbf{C}^\times$

$$\varphi(z) = \left(\begin{bmatrix} (z/\overline{z})^{\mu_1} & 0 & 0 \\ 0 & (z/\overline{z})^{\mu_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} Z', z \right). \tag{28}$$

From the equation $\varphi(-1) = \varphi(\tau^2) = (z, -1)$, one finds that $\mu_k \in \mathbf{Z}$ for each $k = 1, 2$. Finally, we discard the possibility that $\mu_1 = \mu_2 = 0$, for this implies that the image of φ is semisimple and abelian, and is thus contained in a minimal Levi subgroup of ${}^L G'$.

3.2 Endoscopic data

Let θ' be the outer automorphism of $G' = \mathrm{SL}_3$ given by composing inverse-transpose with conjugation by $w_1 \in \mathrm{SU}(2,1)$, that is

$$\theta'(x) = w_1 (x^{-1})^{\intercal} w_1^{-1}, \ x \in \mathrm{SL}_3.$$

Conjugation by w_1 is included in this definition so that θ' preserves the Borel subgroup

$$B = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & 0 \\ 0 & * & * \end{bmatrix} \right\} \subset G'$$

dual to $\hat{B}'$. It is straightforward to verify that θ' is an algebraic involution which preserves $\mathrm{SU}(2,1)$, *i.e.* θ' is defined over $\mathbf{R}$. The automorphism $\hat{\theta}'$ of $\hat{G}'$ induced by θ' coincides with the action of τ in (27). We shall also denote by $\hat{\theta}'$ the automorphism of ${}^L G'$ extended trivially on $W_{\mathbf{R}}$.

To find endoscopic data we shall solve (1) where φ is an admissible homomorphism defined in the previous section. We are looking for

$$s^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ k & h & j \end{bmatrix} \in \mathrm{GL}_3$$

such that $\hat{\theta}' \circ \varphi(w) = \mathrm{Int}(s^{-1}) \circ \varphi(w)$ for all $w \in W_{\mathbf{R}}$. When $w = z \in \mathbf{C}^\times$ this is equivalent to

$$\begin{bmatrix} az^{\mu_1} & bz^{\mu_2} & c \\ dz^{\mu_1} & ez^{\mu_2} & f \\ kz^{\mu_1} & hz^{\mu_2} & j \end{bmatrix} = \lambda \begin{bmatrix} az^{-\mu_2} & bz^{-\mu_2} & cz^{-\mu_2} \\ dz^{-\mu_1} & ez^{-\mu_1} & fz^{-\mu_1} \\ k & h & j \end{bmatrix}$$

for some $\lambda \in \mathbf{C}^\times$ with $\lambda^3 = z^{2(\mu_1+\mu_2)}$. By discarding the case that $\mu_1 = \mu_2 = 0$, one may deduce from this equation that $\mu_1 \neq 0$ and $\mu_2 \neq 0$. The latter in turn implies that $\lambda = 1$, $s = \mathrm{diag}(a, e, j)$ and $\mu_1 = -\mu_2$. Next, from the equation $\hat{\theta}' \circ \varphi(\tau) = \mathrm{Int}(s^{-1}) \circ \varphi(\tau)$ we deduce that $a^2 = e^2 = j^2$. As we are only interested in s modulo Z', we may take $j = 1$ and $s^{-1} = \mathrm{diag}(a, e, 1)$ with $a^2 = e^2 = 1$. In conclusion, equation (1) has solutions if and only if $\mu_1 = -\mu_2 \in \mathbf{Z}$ is non-zero, and in this case $s = \mathrm{diag}(\pm 1, \pm 1, 1) Z'$.

We are now ready to find the group $\mathcal{H}$ attached to each s. The dual group $\hat{H}'$ is equal to the identity component of the fixed-point subgroup $\{x \in \hat{G}' : x = s\hat{\theta}'(x)s^{-1}\}$. As it happens, the group $\hat{H}'$ is isomorphic to SL_2 for every choice of s. We shall justify this isomorphism only for the case $s = \mathrm{diag}(a, e, 1)$ with $a = -e$. The Borel subgroup $\hat{B}'$ contains the diagonal subgroup $\hat{T}'$ in $\hat{G}'$, and $\hat{\theta}'$ preserves each of these subgroups. Work of Steinberg (Theorem 1.1.A [KS99], Lemma I.1.1 [Lab04]) tells us that the $\hat{\theta}'$-fixed-point subgroups of $\hat{B}'$ and $\hat{T}'$ are respectively a Borel subgroup and maximal torus of $\hat{H}'$. It is simple to compute the Lie algebras of these fixed-point subgroups. If X is in the Lie algebra of $\hat{B}'$ and $sw_1(-X^\intercal)w_1^{-1}s^{-1} = X$ then it is of the form

$$X = \begin{bmatrix} x_1 & x_2 & 0 \\ 0 & -x_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ x_1, x_2 \in \mathbf{C}.$$

Clearly, the Borel subalgebra generated by such X is isomorphic to the upper-triangular Borel subalgebra in $\mathfrak{sl}(2,\mathbf{C})$. It is simple to verify that $X^\intercal$ also belongs to $\hat{H}'$ and generates the opposite Borel subalgebra under the isomorphism. The group $\hat{H}'$ is generated by the exponentials of these two subalgebras (Proposition 8.1.1 (ii) [Spr98]) so that

$$\hat{H}' = \left\{ \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} Z' : A \in \mathrm{SL}_2 \right\} \cong \mathrm{SL}_2.$$

We appear to be in the same situation as in section 2.2.1. Indeed, the dual endoscopic group $\hat{H}'$ is isomorphic to SL_2. Moreover, since SL_2 has no outer automorphisms (27.4 [Hum75]), the L-group must be a direct product ${}^L H' = \hat{H}' \times W_{\mathbf{R}}$ (2.3 [Bor79]). It follows that $H' = \mathrm{PGL}_2$ is split and $H'(\mathbf{R}) = \mathrm{PGL}(2,\mathbf{R})$. Despite these similarities with section 2.2.1, the L-group ${}^L H'$ is *not* isomorphic to $\mathcal{H}$. This can be proved by first observing that $\varphi(\tau)$ acts on $\hat{H}'$ by inverse-transpose. This action is the same as conjugation on $\hat{H}'$ by

$$w_2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} Z' \in \hat{H}'.$$

Consequently, the only elements of $\mathcal{H} = \hat{H}'\varphi(W_{\mathbf{R}})$ whose second coordinate is τ and whose action on $\hat{H}'$ under conjugation is trivial are the elements

$$\left(\pm \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} Z', \tau \right) = (\pm w_2, 1)\, \varphi(\tau) \in \hat{H}'. \tag{29}$$

If there were an isomorphism $\iota : {}^L H' \rightarrow \mathcal{H}$ then $\iota(1,\tau)$ would equal one of the two elements in (29) and

$$\iota(1,-1) = \iota(1,\tau)^2 = ((\pm w_2, 1)\, \varphi(\tau))^2 = \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} Z', -1 \right).$$

On the other hand, when $z \in \mathbf{C}^\times$ we would have $\iota(1,z) = (\epsilon(z), z)$ for some continuous homomorphism $\epsilon : \mathbf{C}^\times \rightarrow Z_{\hat{H}'}$ into the centre of $\hat{H}'$. Since the centre of $\hat{H}'$ is a group of order two, ϵ is trivial and $\iota(1,-1) = (Z',-1)$–a contradiction.

To remedy this situation, one turns to a *z-extension* of $H' = \mathrm{PGL}_2$ (2.2 [KS99]), which we may choose to be GL_2. There is an obvious exact sequence

$$1 \rightarrow Z_G \rightarrow \mathrm{GL}_2 \rightarrow \mathrm{PGL}_2 \rightarrow 1$$

in which Z_G is the centre of GL_2. Now we may define a monomorphism ξ_1 from $\mathcal{H}$ into ${}^L\mathrm{GL}_2 = \mathrm{GL}_2 \times W_{\mathbf{R}}$ by

$$\xi_1((w_2,1)\,\varphi(\tau)) = (iI,\tau),$$
$$\xi_1(Z',z) = (I,z),\ z \in \mathbf{C}^\times \text{ and}$$
$$\xi_1\left(\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} Z', 1\right) = (A,1),\ A \in \mathrm{SL}_2.$$

This monomorphism may be regarded as an extension of the inclusion of $\hat{H}' \cong \mathrm{SL}_2$ into $\hat{\mathrm{GL}}_2 = \mathrm{GL}_2$ (*cf.* Lemma 2.2.A [KS99]). The so-called z-pair (GL_2, ξ_1) will give us a way of defining an admissible homomorphisms through φ and $\mathcal{H}$ in much the same way that the pair (H, ξ_H) did in section 2.3.

3.3 Spectral data

We assume that $\varphi : W_{\mathbf{R}} \to \mathrm{SL}_3 \rtimes W_{\mathbf{R}}$ is an admissible homomorphism as in section 3.2 with $\mu_1 = -\mu_2 \in \mathbf{Z}$ being non-zero (*cf.* (28)). Set $\varphi_1 = \xi_1 \circ \varphi$ so that $\varphi_1 : W_{\mathbf{R}} \to \mathrm{GL}_2 \times W_{\mathbf{R}}$ is an admissible homomorphism of $\mathrm{GL}(2,\mathbf{R})$. One may simplify the discussion by noting that conjugation of φ by w_1 has the sole effect of replacing μ_1 by $-\mu_1$. We may therefore assume that $\mu_1 \in \mathbf{Z}$ is positive (*cf.* section 2.1).

As before, our aim is to describe the L-packets Π_φ and Π_{φ_1} in terms of the characters of the representations which they contain. Most of this has already been done in section 2.3.1 for Π_{φ_1}. If one truncates φ_1 to its image in GL_2 then it is given by

$$\varphi_1(z) = \begin{bmatrix} (z/\bar{z})^{\mu_1} & 0 \\ 0 & (z/\bar{z})^{-\mu_1} \end{bmatrix},\ z \in \mathbf{C}^\times$$
$$\varphi_1(\tau) = \xi_1(w_2,1)^{-1}\,\xi_1((w_2,1)\varphi(\tau)) = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

After conjugating φ_1 by (2) with $t_1 = -i$ we arrive at the admissible homomorphism of (8). As L-packets are insensitive to conjugation, the conclusions of sections 2.2.2 and 2.3.1 apply to φ_1 as well. Hence, the L-packet of Π_{φ_1} consists of the representation π_{μ_1} whose character is given by (13).

We continue with the description of Π_φ. As in section 2.3.1, we will describe the representations in Π_φ by giving character values on a maximally compact torus. A maximal torus in $\mathrm{SU}(2,1)$ is given by

$$T'(\mathbf{R}) = \left\{ \begin{bmatrix} e^{ia} & 0 & 0 \\ 0 & e^{i(-a+b)} & 0 \\ 0 & 0 & e^{-ib} \end{bmatrix} : a,b \in \mathbf{R} \right\} \tag{30}$$

Here, T' is the diagonal subgroup of SL_3 and $T'(\mathbf{R})$ is actually compact. We are to identify ${}^LT'$ with $\hat{T}' \rtimes \varphi(W_{\mathbf{R}})$, where $\hat{T}'$ is the diagonal subgroup of PGL_3. The lattice of cocharacters $X_*(\hat{T}')$ is isomorphic to $\langle \lambda_1, \lambda_2 \rangle / \langle \lambda_1\lambda_2\lambda_3 \rangle$ where

$$\lambda_1(z) = \begin{bmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \lambda_2(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \lambda_3(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{bmatrix},\ z \in \mathbf{C}^\times.$$

Recall that we have fixed a positive system of roots $\{-\alpha_2, \alpha_1+\alpha_2, \alpha_1\}$ corresponding to the Borel subgroup $\hat{B}'$ in section 3.1. The half-sum of these roots is α_1. One assigns the parameter μ_1 to the element $\lambda_1^{m_1}\lambda_2^{-m_1}\langle\lambda_1\lambda_2\lambda_3\rangle$ with $m_1 = \mu_1 - 1$. This element of $X_*(\hat{T}')\otimes \mathbf{C}$ yields a weight of T' by identifying $\langle\lambda_1, \lambda_2\rangle/\langle\lambda_1\lambda_2\lambda_3\rangle$ with $\langle\varepsilon_1, \varepsilon_2\rangle/\langle\varepsilon_1\varepsilon_2\varepsilon_3\rangle$, where

$$\varepsilon_j\left(\left[\begin{array}{ccc} z_1 & 0 & 0\\ 0 & z_2 & 0\\ 0 & 0 & z_3\end{array}\right]\right) = z_j \in \mathbf{C}^\times,\ j=1,2,3.$$

On an element of the form $\mathrm{diag}(z_1, z_1^{-1}z_2, z_2^{-1}) \in T'$ the element $\varepsilon_1^{m_1}\varepsilon_2^{-m_1}\langle\varepsilon_1\varepsilon_2\varepsilon_3\rangle$ takes the value $z_1^{2\mu_1-2}z_2^{-\mu_1+1}$. Upon passing to $T'(\mathbf{R})$, one obtains a weight which has values $e^{i(2\mu_1-2)a}e^{i(-\mu_1+1)b}$ on the elements of (30). This weight may be written more compactly as $e^{(\mu_1-1)\alpha_1}$ by abusively identifying the root α_1 of $\hat{T}'$ with its obvious counterpart on the Lie algebra of T'.

This weight fixes a unique discrete series representation $\pi'_{\mu_1\alpha_1}$ of $\mathrm{SU}(2,1)$ up to equivalence. The distribution character of $\pi'_{\mu_1\alpha_1}$ is determined by its values on the regular elements of $T'(\mathbf{R})$ or its real Lie algebra $\mathfrak{t}'$. On $\mathfrak{t}'$ it is equal to

$$\frac{-e^{\mu_1\alpha_1}+e^{-\mu_1\alpha_1}}{(e^{\alpha_2/2}-e^{-\alpha_2/2})(e^{(\alpha_1+\alpha_2)/2}-e^{-(\alpha_1+\alpha_2)/2})(e^{\alpha_1/2}-e^{-\alpha_1/2})} \tag{31}$$

(Theorem 12.7 [Kna86]). Here, α_1 and α_2 are (abusively) the simple roots of $\mathfrak{sl}(3,\mathbf{C})$ coming from the Borel subalgebra of upper-triangular matrices, the denominator is the Weyl denominator in which the factors are taken over the positive system $\{-\alpha_2, \alpha_1+\alpha_2, \alpha_1\}$ (Remark 2 *p.* 105 [Kna86]). The sum in the numerator is taken over elements in the the Weyl group Ω_K of the maximal compact subgroup

$$K = \left\{\left[\begin{array}{cc} A & 0\\ 0 & \det(A)^{-1}\end{array}\right] : A \in \mathrm{U}(2)\right\} \cong \mathrm{U}(2)$$

((1.123) [Kna96]).

There is an equivalence of $\pi'_{\mu_1\alpha_1}$ under Ω_K (Theorem 9.20 [Kna86]), but this equivalence does not extend to the Weyl group $\Omega(G', T')$ of SL_3. In fact, the representations occurring in the L-packet Π_φ are, by definition, representations in the orbit of $\pi'_{\mu_1\alpha_1}$ under $\Omega(G',T')/\Omega_K$. The group Ω_K is generated by the reflection w_{α_1} corresponding to α_1, and the nontrivial representatives of $\Omega(G',T')/\Omega_K$ may be taken to be w_{α_2} and $w_{\alpha_1+\alpha_2}$. Consequently, the L-packet is equal to

$$\Pi_\varphi = \{\pi'_{\mu_1\alpha_1}, \pi'_{\mu_1 w_{\alpha_2}\cdot\alpha_1}, \pi'_{\mu_1 w_{\alpha_1+\alpha_2}\cdot\alpha_1}\} = \{\pi'_{\mu_1\alpha_1}, \pi'_{\mu_1(\alpha_1+\alpha_2)}, \pi'_{-\mu_1\alpha_2}\}.$$

The discrete series representations $\pi'_{\mu_1(\alpha_1+\alpha_2)}$ and $\pi'_{-\mu_1\alpha_2}$ are determined by the values of their characters on the regular elements of $T'(\mathbf{R})$ and have analogues to (31) when α_1 is replaced by $\alpha_1+\alpha_2$ or $-\alpha_2$.

For the remainder of this section we will examine how the automorphism θ' acts on Π_φ and the corresponding characters. The differential of θ' sends α_1 to itself, and sends $\alpha_1 + \alpha_2$ to $-\alpha_2$. This implies that the character of $\pi'_{\mu_1\alpha_1}$ is invariant under composition by θ', and the characters of $\pi'_{\mu_1(\alpha_1+\alpha_2)}$ and $\pi'_{-\mu_1\alpha_2}$ are interchanged under composition by θ'. This means that $(\pi'_{\mu_1\alpha_1})^{\theta'} \cong \pi'_{\mu_1\alpha_1}$ and $(\pi'_{\mu_1(\alpha_1+\alpha_2)})^{\theta'} \cong \pi'_{-\mu_1\alpha_2}$ so that we may write $\Pi_\varphi \circ \theta' = \Pi_\varphi$. Since $\pi'_{\mu_1\alpha_1}$ is the only representation stable under θ', it is the only representation of Π_φ with a twisted character. The other two representations do not contribute to twisted spectral transfer.

We close by displaying the values of the twisted character of $\pi'_{\mu_1\alpha_1}$ on the θ'-fixed subgroup

$$(T')^{\theta'}(\mathbf{R}) = \left\{ \begin{bmatrix} e^{ia} & 0 & 0 \\ 0 & e^{-ia} & 0 \\ 0 & 0 & 1 \end{bmatrix} : a \in \mathbf{R} \right\}$$

of $T'(\mathbf{R})$. The methods of section 2.3.4 are not applicable in the present context as the representation $\pi'_{\mu_1\alpha_1}$ is not induced from some convenient subgroup of $\mathrm{SU}(2,1)$. Luckily, Bouaziz provides the desired formula using the characterization of discrete series representations of Duflo (Proposition 6.1.2 [Bou87], III [Duf82]). There are several details to work out (*cf.* (47) [Mez13]), but the final result is that on the regular set of the real Lie algebra $(\mathfrak{t}')^{\theta'} \cong (T')^{\theta'}(\mathbf{R})$ the character is

$$\pm \frac{e^{\mu_1\alpha_1} - e^{-\mu_1\alpha_1}}{e^{\alpha_1} - e^{-\alpha_1}}. \tag{32}$$

The denominator of this quotient may be thought of as a Weyl denominator for the disconnected group $\mathrm{SU}(2,1) \rtimes \langle\theta'\rangle$. The numerator of this expression is a sum over θ'-invariant Weyl group elements in Ω_K. The indeterminacy in sign is a remnant of the possible choices for an operator intertwining $(\pi'_{\mu_1\alpha_1})^{\theta'}$ with $\pi'_{\mu_1\alpha_1}$.

As an aside, let us justify the peculiar choice of Borel subgroup $\hat{B}'$ and the appearance of w_1. It may be verified that the only Ω_K-orbits of weights of $\mathfrak{t}'$ which are preserved by negative-transpose are of the form $\{m\alpha_1, -m\alpha_1\}$, $m \in \mathbf{Z}$. These are Ω_K-orbits of positive multiples of α_1 and the Weyl chamber containing these multiples fixes in turn the unique positive system $\{-\alpha_2, \alpha_1+\alpha_2, \alpha_1\}$ and the Borel subgroup $\hat{B}'$. As a consequence, it is tidier to tailor the theory about the representations $\pi'_{m\alpha}$ which are stable under an outer automorphism by choosing $\hat{B}'$ as we have and to inlcude conjugation by w_1 in the outer automorphism θ' so that $\hat{B}'$ is preserved.

3.4 Transfer

3.4.1 The correspondence of twisted conjugacy classes

The purpose of this section is to define a sequence of maps analogous to (21), when considering $G' = \mathrm{SL}_3$ in the place of $G = \mathrm{GL}_2$. In this consideration the task is both simplified and complicated. It is simplified by the fact that there

is no distinction between tori in G' analogous to the distinction between S and T in G. There is only T' and so one would expect a sequence
(33)

$$Cl_{ss}(H') \to S_{H'}/\Omega(H', S_{H'}) \overset{\psi_{H'}}{\to} (T'/(1-\theta')T')/\Omega(G', T')^{\theta'} \to Cl_{ss}(G', \theta').$$

Things are complicated by the fact that we have introduced the z-extension GL_2 of $H' = \mathrm{PGL}_2$. In our spectral transfer we shall have to compare character values on semisimple conjugacy classes of $\mathrm{GL}(2, \mathbf{R})$, not $\mathrm{PGL}(2, \mathbf{R})$. We therefore require a bridge between $Cl_{ss}(\mathrm{PGL}_2)$ and $Cl_{ss}(\mathrm{GL}_2)$.

Let us tend to (33) first. In the half-integral case of section 2.4.1 $H = \mathrm{PGL}_2 = H'$. Thus, we define $S_{H'} = S_H = S/Z_G$ and the leftmost map of (33) is the usual bijection. Towards defining $\psi_{H'}$, one may calculate that

$$(1-\theta')T' = \left\{ \begin{bmatrix} y & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & y^{-2} \end{bmatrix} : y \in \mathbf{C}^\times \right\}.$$

This implies that there is an isomorphism $T/Z_G \to T'/(1-\theta')T'$ defined by

$$\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix} Z_G \mapsto \begin{bmatrix} x & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} (1-\theta')T', \; x \in \mathbf{C}^\times.$$

Define the map $\psi_{H'}$ to be the composition of this isomorphism with the isomorphism $\psi_H : S_H \to T/Z_G$ of section 2.4.1 induced by $\mathrm{Int}(c)$. The isomorphism $\psi_{H'} : S_{H'} \to T'/(1-\theta')T'$ is defined over $\mathbf{R}$ as one may verify that

$$\psi_{H'}\left(\begin{bmatrix} \cos(a) & \sin(a) \\ -\sin(a) & \cos(a) \end{bmatrix} Z \right) = \begin{bmatrix} e^{-ia} & 0 & 0 \\ 0 & e^{ia} & 0 \\ 0 & 0 & 1 \end{bmatrix} (1-\theta')T'.$$

We must still show that $\psi_{H'}$ passes to a map of equivalence classes under the action of the groups $\Omega(H', S_{H'})$ and $\Omega(G', T')^{\theta'}$. The former group is generated by $w_0 Z_G$ and latter group is

$$\{w \in \Omega(G', T') : \theta' w (\theta')^{-1}_{|T'} = w\} = \{w \in \Omega(G', T') : w_1 w w_1^{-1} = w\} = \langle w_1 \rangle.$$

Each equivalence class with respect to either of these groups is comprised of an element and its inverse. Apparently, the isomorphism $\psi_{H'}$ passes to a bijection of equivalence classes, as desired.

The rightmost map of (33) is again the bijection of Lemma 3.2.A [KS99], so we have completed the correspondence between twisted conjugacy classes of G' and conjugacy classes of H'. One could denote (33) by $\mathcal{A}_{H'/G'}$ as in Lemma 3.3.A [KS99]. Note that this map is a bijection.

We now define the bridge between the conjugacy classes $Cl_{ss}(G)$ of the z-extension $G = \mathrm{GL}_2$ and $Cl_{ss}(H')$ through a map

$$Cl_{ss}(G) \to S/\Omega(G, S) \to S_{H'}/\Omega(H', S_{H'}) \to Cl_{ss}(H'). \tag{34}$$

The maps on the left and right are the usual bijections. The map in the middle is the surjection induced by $S \to S/Z_G = S_{H'}$. This second correspondence is defined over $\mathbf{R}$ in the sense that the conjugacy class of an element in (9) is mapped to the conjugacy class of

$$\begin{bmatrix} \cos(a) & \sin(a) \\ -\sin(a) & \cos(a) \end{bmatrix} Z \in H'(\mathbf{R}). \tag{35}$$

3.4.2 Spectral transfer

The two maps (33) and (34) allow us to compare the twisted character of the representation $\pi'_{\mu_1\alpha_1} \in \Pi_\varphi$ with the character of the representation $\pi_{\mu_1} \in \Pi_{\varphi_1}$ of $\mathrm{GL}(2, \mathbf{R})$. The twisted character of $\pi'_{\mu_1\alpha_1}$ is determined by (32) and it's values are invariant under θ'-conjugacy. Let us begin with the real θ'-conjugacy class of

$$\begin{bmatrix} e^{-ia} & 0 & 0 \\ 0 & e^{ia} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in (T')^{\theta'}(\mathbf{R}).$$

The pre-image of this element under (33) is the conjugacy class of (35) in $\mathrm{PGL}(2, \mathbf{R})$. The pre-image of (35) under (34) is the collection of conjugacy classes in $\mathrm{GL}(2, \mathbf{R})$ of the elements in (9) for fixed $a \in \mathbf{R}$. Fortunately and not coincidentally, the character (13) of π_{μ_1} is constant on each conjugacy class in this collection over $b \in \mathbf{R}$. In consequence, we may compare the twisted character value of (32)

$$\pm \frac{e^{2i\mu_1 a} - e^{-2i\mu_1 a}}{e^{2ia} - e^{-2ia}}. \tag{36}$$

with the character value (13). There are two obvious differences. The first difference is in the Weyl denominators: $e^{2ia} - e^{-2ia}$ as opposed to $e^{ia} - e^{-ia}$. The second difference lies in the ambiguity of sign in (36). Differences in Weyl denominators and sign also occurred in the integral case of section 2.4.2. The distinction between the numerators in the integral case of section 2.4.2 is not present here. This is because (33) is a bijection, whereas (23) is not.

In closing, one may wonder how special these examples of twisted spectral transfer are. There are two serious hurdles we have not encountered in our examples. The first has been alluded to in mentioning choices of sign. Such choices become complicated when more than one representation in an L-packet is equivalent to itself under twisting. In that case one must choose an appropriate coefficient for each twisted character so that the resulting linear combination of twisted characters is comparable to the sum of the characters in the L-packet(s) of the endoscopic group. This is the subject of spectral transfer factors ([She], 6.3 [Mez13]).

The other hurdle has to do with the maps from conjugacy classes of an endoscopic group to twisted conjugacy classes. In our examples these maps $\mathcal{A}_{H/G}$ and $\mathcal{A}_{H'/G'}$ sent real conjugacy classes to real conjugacy classes. Whenever the initial group is quasisplit this can always be arranged without much trouble. In general, twisting must be introduced on the endoscopic group (5.4 [KS99]). Then real twisted conjugacy classes of the endoscopic group are mapped to twisted conjugacy classes, and sums of twisted characters of the endoscopic group must be compared with linear combinations of twisted characters of the initial group. Spectral transfer in this generality is an open problem at present.

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