Assignment 6–Algebra I
Due at the beginning of tutorial Nov. 10

1. Suppose $v, w \in \mathbb{C}^n$.

   (a) Suppose $a \in \mathbb{C}$. Prove /3

   $$\|v - aw\|^2 = \langle v, v\rangle - \bar{a}\langle v, w\rangle - a\langle w, v\rangle + a\bar{a}\langle w, w\rangle.$$

   (b) Suppose $w \neq 0$, and prove that /4

   $$0 \leq \|v\|^2 - \frac{|\langle v, w\rangle|^2}{\|w\|^2}$$

   by setting $a = \langle v, w\rangle / \|w\|^2$ in (a).

   (c) Prove the *Cauchy-Schwarz inequality* /3

   $$|\langle v, w\rangle| \leq \|v\| \, \|w\|$$

   for any $v, w \in \mathbb{C}^n$. (Hint: Be sure to include the case $w = 0$.)

2. Suppose $v, w \in \mathbb{C}^n$.

   (a) Prove that /3

   $$\|v + w\|^2 = \|v\|^2 + 2\mathrm{Re}(\langle v, w\rangle) + \|w\|^2,$$

   where $\mathrm{Re}(a + bi) = a$ for $a, b \in \mathbb{R}$.

   (b) Prove that $\mathrm{Re}(a + bi) \leq |a + bi|$ for every $a, b \in \mathbb{R}$. (Hint: First prove that the square of the LHS is less than or equal to the square of the RHS.) /3

   (c) Prove the *triangle inequality* /4

   $$\|v + w\| \leq \|v\| + \|w\|,$$

   (Hint: Apply 2 (b) to 2 (a) and then use the Cauchy-Schwartz inequality to prove that the square of the LHS is less than or equal to the square of the RHS.)

3. Suppose $v \in \mathbb{C}^n$. Prove that $\langle v, e_j\rangle = [v]_j$, where e_j is the jth standard vector. What is $\langle e_j, v\rangle$ equal to? /5

SUGGESTED EXERCISES

- Exercises C20 - T20 on page 164 (subsection O.EXC).
- Use exercise 3 to prove that $v = \sum_{j=1}^{n} \langle v, e_j\rangle \, e_j$. This proves that $\mathbb{C}^n = \mathrm{span}\{e_1, \ldots e_n\}$.