Revisiting a short proof of Cauchy's polygonal number theorem and formalizing it in Lean 4

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Abstract. Melvyn B. Nathanson's proof of a a stronger version of Cauchy's polygonal number theorem is revisited. A tighter analysis of the proof is presented and a formalization of the proof in the Lean 4 theorem prover is described.

Keywords: polygonal number · Lean 4 · formal proof

1 Introduction

The motivation for the discussion in this paper stemmed from our attempt to formalize in Lean 4 a short piece of real mathematics. We landed on Cauchy's polygonal number theorem, which states that for every integer $m \geq 1$, every nonnegative integer is the sum of $m+2$ polygonal numbers of order $m+2$, where polygonal numbers of order $m+2$ are the integers $p_m(k) := \frac{m}{2}\left(k^2 - k\right) + k$ for $k = 0, 1, 2, \ldots$. The short proof of the theorem by Nathanson [7] appeared to fit our purpose. In fact, Nathanson proved the following strengthened version of the result, deferring the cases for the original result when $n < 120m$ to tables by Pepin [9] and Dickson [2].

Theorem 1 (Theorem 1 in [7]). *Let $m \geq 3$ and $n \geq 120m$. Then n is the sum of $m+1$ polygonal numbers of order $m+2$, at most four of which are different from 0 or 1.*

Nathanson also gave short a proof of a result of Legendre:

Theorem 2 (Theorem 2 in [7]). *Let $m \geq 3$. If m is odd, then every sufficiently large integer is the sum of four polygonal numbers of order $m+2$. If m is even, then every sufficiently large integer is the sum of five polygonal numbers of order $m+2$, one of which is either 0 or 1.*

Nathanson gave the following updated versions in his book [8] published nearly a decade later:

Theorem 3 (Theorem 1.9 in [8]). *If $m \geq 4$ and $N \geq 108m$, then N can be written as the sum of $m+1$ polygonal numbers of order $m+2$, at most four of which are different from 0 or 1. If $N \geq 324$, then N can be written as the sum of five pentagonal numbers, at least one of which is 0 or 1.*

Theorem 4 (Theorem 1.10 in [8]). *Let $m \geq 3$ and $N \geq 28m^3$. If m is odd, then N is the sum of four polygonal numbers of order $m+2$. If m is even, then N is the sum of five polygonal numbers of order $m+2$, at least one of which is 0 or 1.*

As these updated versions were formalized in Isabelle quite recently by Lee *et al.* [4], we decided to formalize the proof of the older Theorem 1 instead.

It was not immediately clear why the weaker Theorem 1, albeit with a better constant in the inequality for the case when $m \geq 4$, appeared in Nathanson's book. Even though the book does include [7] in the bibliography, the results in the paper are not cited in the body of the text. Incidentally, the same proof for Theorem 1 also appears in [1].

Our direct attempt at formalizing the proof of Theorem 1 was impeded by a gap in the beginning of the proof:

> Let b_1 and b_2 be consecutive odd integers. The set of numbers of the form $b + r$, where $b \in \{b_1, b_2\}$ and $r \in \{0, 1, \ldots, m-3\}$, contains a complete set of residue classes modulo m.

Note that the statement fails to hold for $m = 3$. Since the rest of the proof requires an odd integer b and an integer $r \in \{0, \ldots, m-3\}$ so that m divides $n - b - r$, an apparent fix is to establish the following for $m = 3$:

> Let b_1, b_2, b_3 be consecutive odd integers. The set $\{b_1, b_2, b_3\}$ contains a complete set of residue classes modulo 3.

In the process of implementing this fix, we decided to perform tighter analyses in some of the technical lemmas, thus obtaining the following:

Theorem 5. *Let n and m be positive integers. If either*

(a) $m \geq 4$ and $n \geq 53m$; or

(b) $m = 3$ and $n \geq 159m$,

then n is the sum of $m+1$ polygonal numbers of order $m+2$, at most four of which are different from 0 or 1.

From this, the next two results can be derived:

Theorem 6. *Every positive integer $n \notin \{9, 21, 31, 43, 55, 89\}$ can be expressed as the sum of at most four positive pentagonal numbers.*

Proof. From Theorem 5 part (b), we obtain that if $n \geq 477$, then n is the sum of four polygonal numbers of order five (i.e. pentagonal numbers). For $n < 476$ and $n \notin \{9, 21, 31, 43, 55, 89\}$, see Table 1 and Table 2, noting that the only pentagonal numbers between 1 to 476, inclusive, are 1, 5, 12, 22, 35, 51, 70, 92, 117, 145, 176, 210, 247, 287, 330, 376, and 425.

Theorem 7. *Every positive integer $n \notin \{11, 26\}$ can be expressed as the sum of at most five positive hexagonal numbers.*

Proof. From Theorem 5 part (a) with $m = 4$, we obtain that if $n \geq 212$, then n is the sum of five polygonal numbers of order six (i.e. pentagonal numbers). For $n < 211$ and $n \notin \{11, 26\}$, see Table 3, noting that the only hexagonal numbers between 1 to 211, inclusive, are 1, 6, 15, 28, 45, 66, 91, 120, 153, and 190.

Our formalization in Lean 4 of the last three results can be found in [5]. In our formalization, we take the following theorem as an axiom since it has not yet been formalized in Lean 4 and formalizing it is expected to be a huge undertaking.

Theorem 8 (Gauss' Triangular Number Theorem). *Let n be a positive integer. If $n \equiv 3 \pmod 8$, then there exist odd integers $x \geq y \geq z > 0$ such that*

$$n = x^2 + y^2 + z^2.$$

In the rest of the paper, we provide a detailed informal proof of Theorem 5 and a brief description of our formalization in the final section of the paper.

Some historical remarks

In light of Theorem 4, the assertions of Theorems 6 and 7 are certainly not new. Nevertheless, our proofs involved manually checking far fewer cases and the theorems are stated explicitly here to address some uncertainties that appeared as recently as October 2022.[3]

For instance, on the On-line Encyclopedia of Integer Sequences website, there is the following comment for sequence A133929 (`https://oeis.org/A133929`):

> Equivalently, integers m such that the smallest number of pentagonal numbers (A000326) which sum to m is exactly five, that is, A100878(a(n)) = 5. Richard Blecksmith & John Selfridge found these six integers among the first million, they believe that they have found them all (Richard K. Guy reference). - Bernard Schott, Jul 22 2022

The relevant passage in Guy [3] appears on p. 222:

> Richard Blecksmith & John Selfridge found six numbers among the first million, namely 9, 21, 31, 43, 55 and 89, which require five pentagonal numbers of positive rank, and two hundred and four others, the largest of which is 33066, which require four. They believe that they have found them all.

We were unable to locate the reference for Blecksmith by Selfridge as there appears to be no entry for it in the bibliography of [3].

The paragraph that immediately follows concerns representation as hexagonal numbers:

> Many numbers (what fraction of the whole, or are they of zero density?) require four hexagonal numbers of positive rank; several, e.g.,
>
> $$5, 10, 20, 25, 38, 39, 54, 65, 70, 114, 130, \ldots,$$
>
> require five, and 11 and 26 require six. Which numbers require five?

Theorem 7 certainly does not quite answer this question—it only asserts that every positive integer other than 11 and 26 is the sum of at most (but not necessarily exactly) five hexagonal numbers.

[3] `https://math.stackexchange.com/q/4560516`

1 = 1	2 = 1 + 1	3 = 1 + 1 + 1	4 = 1 + 1 + 1 + 1
5 = 5	6 = 1 + 5	7 = 1 + 1 + 5	8 = 1 + 1 + 1 + 5
9 = 1 + 1 + 1 + 1 + 5	10 = 5 + 5	11 = 1 + 5 + 5	12 = 12
13 = 1 + 12	14 = 1 + 1 + 12	15 = 5 + 5 + 5	16 = 1 + 5 + 5 + 5
17 = 5 + 12	18 = 1 + 5 + 12	19 = 1 + 1 + 5 + 12	20 = 5 + 5 + 5 + 5
21 = 1 + 5 + 5 + 5 + 5	22 = 22	23 = 1 + 22	24 = 12 + 12
25 = 1 + 12 + 12	26 = 1 + 1 + 12 + 12	27 = 5 + 22	28 = 1 + 5 + 22
29 = 5 + 12 + 12	30 = 1 + 5 + 12 + 12	31 = 1 + 1 + 5 + 12 + 12	32 = 5 + 5 + 22
33 = 1 + 5 + 5 + 22	34 = 12 + 22	35 = 35	36 = 1 + 35
37 = 1 + 1 + 35	38 = 1 + 1 + 1 + 35	39 = 5 + 12 + 22	40 = 5 + 35
41 = 1 + 5 + 35	42 = 1 + 1 + 5 + 35	43 = 1 + 1 + 1 + 5 + 35	44 = 22 + 22
45 = 1 + 22 + 22	46 = 12 + 12 + 22	47 = 12 + 35	48 = 1 + 12 + 35
49 = 5 + 22 + 22	50 = 5 + 5 + 5 + 35	51 = 51	52 = 1 + 51
53 = 1 + 1 + 51	54 = 1 + 1 + 1 + 51	55 = 1 + 1 + 1 + 1 + 51	56 = 5 + 51
57 = 22 + 35	58 = 1 + 22 + 35	59 = 12 + 12 + 35	60 = 1 + 12 + 12 + 35
61 = 5 + 5 + 51	62 = 5 + 22 + 35	63 = 12 + 51	64 = 1 + 12 + 51
65 = 1 + 1 + 12 + 51	66 = 22 + 22 + 22	67 = 5 + 5 + 22 + 35	68 = 5 + 12 + 51
69 = 12 + 22 + 35	70 = 70	71 = 1 + 70	72 = 1 + 1 + 70
73 = 22 + 51	74 = 1 + 22 + 51	75 = 5 + 70	76 = 1 + 5 + 70
77 = 1 + 1 + 5 + 70	78 = 5 + 22 + 51	79 = 22 + 22 + 35	80 = 5 + 5 + 70
81 = 1 + 5 + 5 + 70	82 = 12 + 70	83 = 1 + 12 + 70	84 = 1 + 1 + 12 + 70
85 = 12 + 22 + 51	86 = 35 + 51	87 = 5 + 12 + 70	88 = 1 + 5 + 12 + 70
89 = 5 + 5 + 22 + 22 + 35	90 = 5 + 12 + 22 + 51	91 = 5 + 35 + 51	92 = 92
93 = 1 + 92	94 = 1 + 1 + 92	95 = 22 + 22 + 51	96 = 5 + 5 + 35 + 51
97 = 5 + 92	98 = 1 + 5 + 92	99 = 1 + 1 + 5 + 92	100 = 5 + 22 + 22 + 51
101 = 22 + 22 + 22 + 35	102 = 51 + 51	103 = 1 + 51 + 51	104 = 12 + 92
105 = 35 + 70	106 = 1 + 35 + 70	107 = 5 + 51 + 51	108 = 22 + 35 + 51
109 = 5 + 12 + 92	110 = 5 + 35 + 70	111 = 1 + 5 + 35 + 70	112 = 5 + 5 + 51 + 51
113 = 5 + 22 + 35 + 51	114 = 22 + 92	115 = 1 + 22 + 92	116 = 12 + 12 + 92
117 = 117	118 = 1 + 117	119 = 1 + 1 + 117	120 = 1 + 1 + 1 + 117
121 = 51 + 70	122 = 5 + 117	123 = 1 + 5 + 117	124 = 22 + 51 + 51
125 = 1 + 22 + 51 + 51	126 = 12 + 22 + 92	127 = 35 + 92	128 = 1 + 35 + 92
129 = 12 + 117	130 = 1 + 12 + 117	131 = 1 + 1 + 12 + 117	132 = 5 + 35 + 92
133 = 12 + 51 + 70	134 = 5 + 12 + 117	135 = 1 + 5 + 12 + 117	136 = 22 + 22 + 92
137 = 35 + 51 + 51	138 = 1 + 35 + 51 + 51	139 = 22 + 117	140 = 70 + 70
141 = 12 + 12 + 117	142 = 1 + 1 + 70 + 70	143 = 51 + 92	144 = 5 + 22 + 117
145 = 145	146 = 1 + 145	147 = 1 + 1 + 145	148 = 5 + 51 + 92
149 = 22 + 35 + 92	150 = 5 + 145	151 = 1 + 5 + 145	152 = 35 + 117
153 = 51 + 51 + 51	154 = 5 + 22 + 35 + 92	155 = 12 + 51 + 92	156 = 35 + 51 + 70
157 = 12 + 145	158 = 1 + 12 + 145	159 = 1 + 1 + 12 + 145	160 = 5 + 12 + 51 + 92
161 = 22 + 22 + 117	162 = 70 + 92	163 = 1 + 70 + 92	164 = 12 + 35 + 117
165 = 22 + 51 + 92	166 = 5 + 22 + 22 + 117	167 = 22 + 145	168 = 51 + 117
169 = 12 + 12 + 145	170 = 1 + 1 + 51 + 117	171 = 22 + 22 + 35 + 92	172 = 51 + 51 + 70
173 = 5 + 51 + 117	174 = 22 + 35 + 117	175 = 35 + 70 + 70	176 = 176
177 = 1 + 176	178 = 1 + 1 + 176	179 = 12 + 22 + 145	180 = 35 + 145
181 = 5 + 176	182 = 1 + 5 + 176	183 = 1 + 1 + 5 + 176	184 = 92 + 92
185 = 1 + 92 + 92	186 = 5 + 5 + 176	187 = 70 + 117	188 = 12 + 176
189 = 1 + 12 + 176	190 = 22 + 51 + 117	191 = 51 + 70 + 70	192 = 5 + 70 + 117
193 = 5 + 12 + 176	194 = 51 + 51 + 92	195 = 5 + 22 + 51 + 117	196 = 51 + 145
197 = 35 + 70 + 92	198 = 22 + 176	199 = 12 + 70 + 117	200 = 12 + 12 + 176
201 = 5 + 51 + 145	202 = 22 + 35 + 145	203 = 5 + 22 + 176	204 = 51 + 51 + 51 + 51
205 = 5 + 12 + 12 + 176	206 = 22 + 92 + 92	207 = 5 + 22 + 35 + 145	208 = 12 + 51 + 145
209 = 92 + 117	210 = 210	211 = 1 + 210	212 = 1 + 1 + 210
213 = 51 + 70 + 92	214 = 5 + 92 + 117	215 = 5 + 210	216 = 1 + 5 + 210
217 = 1 + 1 + 5 + 210	218 = 22 + 51 + 145	219 = 51 + 51 + 117	220 = 5 + 5 + 210
221 = 12 + 92 + 117	222 = 12 + 210	223 = 1 + 12 + 210	224 = 1 + 12 + 35 + 176
225 = 5 + 22 + 22 + 176	226 = 35 + 51 + 70 + 70	227 = 51 + 176	228 = 1 + 51 + 176
229 = 1 + 1 + 51 + 176	230 = 12 + 22 + 51 + 145	231 = 35 + 51 + 145	232 = 22 + 210
233 = 22 + 35 + 176	234 = 117 + 117	235 = 51 + 92 + 92	236 = 1 + 51 + 92 + 92
237 = 92 + 145	238 = 51 + 70 + 117	239 = 12 + 51 + 176	240 = 1 + 5 + 117 + 117

Table 1. Representations as sum of pentagonal numbers (1 – 240)

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241 = 22 + 35 + 92 + 92	242 = 5 + 92 + 145	243 = 1 + 5 + 92 + 145	244 = 12 + 22 + 210
245 = 35 + 210	246 = 70 + 176	247 = 247	248 = 1 + 247
249 = 1 + 1 + 247	250 = 5 + 35 + 210	251 = 5 + 70 + 176	252 = 5 + 247
253 = 1 + 5 + 247	254 = 70 + 92 + 92	255 = 22 + 22 + 35 + 176	256 = 22 + 117 + 117
257 = 5 + 5 + 247	258 = 12 + 70 + 176	259 = 12 + 247	260 = 1 + 12 + 247
261 = 51 + 210	262 = 117 + 145	263 = 1 + 117 + 145	264 = 5 + 12 + 247
265 = 1 + 5 + 12 + 247	266 = 51 + 70 + 145	267 = 22 + 35 + 210	268 = 92 + 176
269 = 22 + 247	270 = 1 + 22 + 247	271 = 12 + 12 + 247	272 = 35 + 92 + 145
273 = 12 + 51 + 210	274 = 5 + 22 + 247	275 = 1 + 12 + 117 + 145	276 = 92 + 92 + 92
277 = 1 + 92 + 92 + 92	278 = 51 + 51 + 176	279 = 70 + 92 + 117	280 = 70 + 210
281 = 35 + 70 + 176	282 = 35 + 247	283 = 1 + 35 + 247	284 = 22 + 117 + 145
285 = 5 + 70 + 210	286 = 1 + 5 + 70 + 210	287 = 287	288 = 1 + 287
289 = 1 + 1 + 287	290 = 145 + 145	291 = 22 + 22 + 247	292 = 5 + 287
293 = 117 + 176	294 = 1 + 117 + 176	295 = 5 + 145 + 145	296 = 35 + 51 + 210
297 = 5 + 5 + 287	298 = 51 + 247	299 = 12 + 287	300 = 1 + 12 + 287
301 = 92 + 92 + 117	302 = 92 + 210	303 = 5 + 51 + 247	304 = 70 + 117 + 117
305 = 12 + 117 + 176	306 = 22 + 22 + 117 + 145	307 = 70 + 92 + 145	308 = 1 + 5 + 92 + 210
309 = 22 + 287	310 = 12 + 51 + 247	311 = 12 + 12 + 287	312 = 51 + 51 + 210
313 = 51 + 117 + 145	314 = 5 + 22 + 287	315 = 22 + 117 + 176	316 = 70 + 70 + 176
317 = 70 + 247	318 = 1 + 70 + 247	319 = 51 + 92 + 176	320 = 22 + 51 + 247
321 = 145 + 176	322 = 35 + 287	323 = 1 + 35 + 287	324 = 22 + 92 + 210
325 = 35 + 145 + 145	326 = 92 + 117 + 117	327 = 117 + 210	328 = 1 + 117 + 210
329 = 12 + 70 + 247	330 = 330	331 = 1 + 330	332 = 1 + 1 + 330
333 = 12 + 145 + 176	334 = 12 + 35 + 287	335 = 5 + 330	336 = 1 + 5 + 330
337 = 35 + 92 + 210	338 = 51 + 287	339 = 92 + 247	340 = 1 + 92 + 247
341 = 51 + 145 + 145	342 = 12 + 330	343 = 1 + 12 + 330	344 = 5 + 92 + 247
345 = 12 + 12 + 145 + 176	346 = 70 + 92 + 92 + 92	347 = 5 + 12 + 330	348 = 1 + 5 + 12 + 330
349 = 51 + 51 + 247	350 = 70 + 70 + 210	351 = 12 + 92 + 247	352 = 22 + 330
353 = 1 + 22 + 330	354 = 92 + 117 + 145	355 = 145 + 210	356 = 1 + 145 + 210
357 = 70 + 287	358 = 1 + 70 + 287	359 = 1 + 1 + 70 + 287	360 = 70 + 145 + 145
361 = 22 + 92 + 247	362 = 35 + 117 + 210	363 = 70 + 117 + 176	364 = 117 + 247
365 = 35 + 330	366 = 1 + 35 + 330	367 = 12 + 145 + 210	368 = 51 + 70 + 247
369 = 12 + 70 + 287	370 = 5 + 35 + 330	371 = 70 + 92 + 92 + 117	372 = 70 + 92 + 210
373 = 35 + 51 + 287	374 = 35 + 92 + 247	375 = 5 + 5 + 35 + 330	376 = 376
377 = 1 + 376	378 = 1 + 1 + 376	379 = 92 + 287	380 = 1 + 92 + 287
381 = 5 + 376	382 = 1 + 5 + 376	383 = 1 + 1 + 5 + 376	384 = 5 + 92 + 287
385 = 92 + 117 + 176	386 = 176 + 210	387 = 35 + 176 + 176	388 = 12 + 376
389 = 1 + 12 + 376	390 = 35 + 145 + 210	391 = 5 + 176 + 210	392 = 145 + 247
393 = 12 + 51 + 330	394 = 92 + 92 + 210	395 = 5 + 51 + 92 + 247	396 = 22 + 22 + 22 + 330
397 = 70 + 117 + 210	398 = 22 + 376	399 = 1 + 22 + 376	400 = 70 + 330
401 = 22 + 92 + 287	402 = 70 + 70 + 117 + 145	403 = 51 + 176 + 176	404 = 117 + 287
405 = 1 + 117 + 287	406 = 51 + 145 + 210	407 = 117 + 145 + 145	408 = 51 + 70 + 287
409 = 5 + 117 + 287	410 = 12 + 22 + 376	411 = 35 + 376	412 = 1 + 35 + 376
413 = 92 + 145 + 176	414 = 35 + 92 + 287	415 = 51 + 117 + 247	416 = 35 + 51 + 330
417 = 1 + 12 + 117 + 287	418 = 92 + 92 + 117 + 117	419 = 92 + 117 + 210	420 = 210 + 210
421 = 1 + 210 + 210	422 = 92 + 330	423 = 176 + 247	424 = 1 + 176 + 247
425 = 425	426 = 1 + 425	427 = 51 + 376	428 = 5 + 176 + 247
429 = 1 + 1 + 51 + 376	430 = 5 + 425	431 = 1 + 5 + 425	432 = 145 + 287
433 = 22 + 35 + 376	434 = 12 + 92 + 330	435 = 145 + 145 + 145	436 = 1 + 5 + 5 + 425
437 = 12 + 425	438 = 1 + 12 + 425	439 = 35 + 117 + 287	440 = 1 + 35 + 117 + 287
441 = 22 + 92 + 117 + 210	442 = 5 + 12 + 425	443 = 51 + 145 + 247	444 = 22 + 92 + 330
445 = 22 + 176 + 247	446 = 70 + 376	447 = 22 + 425	448 = 1 + 117 + 330
449 = 70 + 92 + 287	450 = 35 + 51 + 117 + 247	451 = 5 + 70 + 376	452 = 5 + 22 + 425
453 = 1 + 5 + 117 + 330	454 = 22 + 145 + 287	455 = 35 + 210 + 210	456 = 92 + 117 + 247
457 = 210 + 247	458 = 1 + 210 + 247	459 = 12 + 117 + 330	460 = 35 + 425
461 = 1 + 35 + 425	462 = 35 + 51 + 376	463 = 176 + 287	464 = 1 + 176 + 287
465 = 5 + 35 + 425	466 = 145 + 145 + 176	467 = 35 + 145 + 287	468 = 92 + 376
469 = 1 + 92 + 376	470 = 70 + 70 + 330	471 = 51 + 210 + 210	472 = 117 + 145 + 210
473 = 51 + 92 + 330	474 = 51 + 176 + 247	475 = 145 + 330	476 = 51 + 425

Table 2. Representations as sum of pentagonal numbers (241 – 476)

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1 = 1	2 = 1 + 1	3 = 1 + 1 + 1	4 = 1 + 1 + 1 + 1
5 = 1 + 1 + 1 + 1 + 1	6 = 6	7 = 1 + 6	8 = 1 + 1 + 6
9 = 1 + 1 + 1 + 6	10 = 1 + 1 + 1 + 1 + 6	11 = 1 + 1 + 1 + 1 + 1 + 6	12 = 6 + 6
13 = 1 + 6 + 6	14 = 1 + 1 + 6 + 6	15 = 15	16 = 1 + 15
17 = 1 + 1 + 15	18 = 6 + 6 + 6	19 = 1 + 6 + 6 + 6	20 = 1 + 1 + 6 + 6 + 6
21 = 6 + 15	22 = 1 + 6 + 15	23 = 1 + 1 + 6 + 15	24 = 6 + 6 + 6 + 6
25 = 1 + 6 + 6 + 6 + 6	26 = 1 + 1 + 6 + 6 + 6 + 6	27 = 6 + 6 + 15	28 = 28
29 = 1 + 28	30 = 15 + 15	31 = 1 + 15 + 15	32 = 1 + 1 + 15 + 15
33 = 6 + 6 + 6 + 15	34 = 6 + 28	35 = 1 + 6 + 28	36 = 6 + 15 + 15
37 = 1 + 6 + 15 + 15	38 = 1 + 1 + 6 + 15 + 15	39 = 6 + 6 + 6 + 6 + 15	40 = 6 + 6 + 28
41 = 1 + 6 + 6 + 28	42 = 6 + 6 + 15 + 15	43 = 15 + 28	44 = 1 + 15 + 28
45 = 45	46 = 1 + 45	47 = 1 + 1 + 45	48 = 1 + 1 + 1 + 45
49 = 6 + 15 + 28	50 = 1 + 6 + 15 + 28	51 = 6 + 45	52 = 1 + 6 + 45
53 = 1 + 1 + 6 + 45	54 = 1 + 1 + 1 + 6 + 45	55 = 6 + 6 + 15 + 28	56 = 28 + 28
57 = 1 + 28 + 28	58 = 15 + 15 + 28	59 = 1 + 15 + 15 + 28	60 = 15 + 45
61 = 1 + 15 + 45	62 = 6 + 28 + 28	63 = 6 + 6 + 6 + 45	64 = 6 + 15 + 15 + 28
65 = 1 + 6 + 15 + 15 + 28	66 = 66	67 = 1 + 66	68 = 1 + 1 + 66
69 = 1 + 1 + 1 + 66	70 = 1 + 1 + 1 + 1 + 66	71 = 15 + 28 + 28	72 = 6 + 66
73 = 28 + 45	74 = 1 + 28 + 45	75 = 15 + 15 + 45	76 = 1 + 15 + 15 + 45
77 = 6 + 15 + 28 + 28	78 = 6 + 6 + 66	79 = 6 + 28 + 45	80 = 1 + 6 + 28 + 45
81 = 15 + 66	82 = 1 + 15 + 66	83 = 1 + 1 + 15 + 66	84 = 28 + 28 + 28
85 = 6 + 6 + 28 + 45	86 = 15 + 15 + 28 + 28	87 = 6 + 15 + 66	88 = 15 + 28 + 45
89 = 1 + 15 + 28 + 45	90 = 45 + 45	91 = 91	92 = 1 + 91
93 = 1 + 1 + 91	94 = 28 + 66	95 = 1 + 28 + 66	96 = 6 + 45 + 45
97 = 6 + 91	98 = 1 + 6 + 91	99 = 1 + 1 + 6 + 91	100 = 6 + 28 + 66
101 = 28 + 28 + 45	102 = 6 + 15 + 15 + 66	103 = 6 + 6 + 91	104 = 1 + 6 + 6 + 91
105 = 15 + 45 + 45	106 = 15 + 91	107 = 1 + 15 + 91	108 = 1 + 1 + 15 + 91
109 = 15 + 28 + 66	110 = 1 + 15 + 28 + 66	111 = 45 + 66	112 = 1 + 45 + 66
113 = 1 + 6 + 15 + 91	114 = 1 + 1 + 1 + 45 + 66	115 = 6 + 15 + 28 + 66	116 = 15 + 28 + 28 + 45
117 = 6 + 45 + 66	118 = 28 + 45 + 45	119 = 28 + 91	120 = 120
121 = 1 + 120	122 = 1 + 1 + 120	123 = 1 + 1 + 1 + 120	124 = 6 + 28 + 45 + 45
125 = 6 + 28 + 91	126 = 6 + 120	127 = 1 + 6 + 120	128 = 1 + 1 + 6 + 120
129 = 28 + 28 + 28 + 45	130 = 6 + 6 + 28 + 45 + 45	131 = 6 + 6 + 28 + 91	132 = 66 + 66
133 = 1 + 66 + 66	134 = 15 + 28 + 91	135 = 15 + 120	136 = 45 + 91
137 = 1 + 45 + 91	138 = 6 + 66 + 66	139 = 28 + 45 + 66	140 = 6 + 15 + 28 + 91
141 = 6 + 15 + 120	142 = 6 + 45 + 91	143 = 1 + 6 + 45 + 91	144 = 6 + 6 + 66 + 66
145 = 6 + 28 + 45 + 66	146 = 28 + 28 + 45 + 45	147 = 15 + 66 + 66	148 = 28 + 120
149 = 1 + 28 + 120	150 = 15 + 15 + 120	151 = 15 + 45 + 91	152 = 1 + 15 + 45 + 91
153 = 153	154 = 1 + 153	155 = 1 + 1 + 153	156 = 45 + 45 + 66
157 = 66 + 91	158 = 1 + 66 + 91	159 = 6 + 153	160 = 1 + 6 + 153
161 = 1 + 1 + 6 + 153	162 = 6 + 45 + 45 + 66	163 = 6 + 66 + 91	164 = 28 + 45 + 91
165 = 45 + 120	166 = 1 + 45 + 120	167 = 28 + 28 + 45 + 66	168 = 15 + 153
169 = 1 + 15 + 153	170 = 1 + 1 + 15 + 153	171 = 6 + 45 + 120	172 = 15 + 66 + 91
173 = 1 + 15 + 66 + 91	174 = 6 + 15 + 153	175 = 28 + 28 + 28 + 91	176 = 28 + 28 + 120
177 = 45 + 66 + 66	178 = 15 + 15 + 28 + 120	179 = 15 + 28 + 45 + 91	180 = 15 + 45 + 120
181 = 28 + 153	182 = 91 + 91	183 = 15 + 15 + 153	184 = 1 + 15 + 15 + 153
185 = 28 + 66 + 91	186 = 66 + 120	187 = 6 + 28 + 153	188 = 6 + 91 + 91
189 = 6 + 15 + 15 + 153	190 = 190	191 = 1 + 190	192 = 1 + 1 + 190
193 = 28 + 45 + 120	194 = 6 + 6 + 91 + 91	195 = 15 + 15 + 45 + 120	196 = 6 + 190
197 = 1 + 6 + 190	198 = 45 + 153	199 = 1 + 45 + 153	200 = 1 + 1 + 45 + 153
201 = 15 + 66 + 120	202 = 6 + 6 + 190	203 = 1 + 45 + 66 + 91	204 = 6 + 45 + 153
205 = 15 + 190	206 = 1 + 15 + 190	207 = 1 + 1 + 15 + 190	208 = 6 + 6 + 6 + 190
209 = 28 + 28 + 153	210 = 28 + 91 + 91	211 = 91 + 120	

Table 3. Representations as sum of hexagonal numbers

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2 Proof of Theorem 5

We follow the structure of the proof of Thereom 1 in [7]. The original argument is reorganized and considerably expanded so that it is more straightforward to formalize.

Lemma 1 (Cauchy's Lemma). *Let a and b be odd positive integers such that $b^2 < 4a$ and $3a < b^2 + 2b + 4$. Then there exist nonnegative integers s, t, u, v such that*

$$a = s^2 + t^2 + u^2 + v^2,$$
$$b = s + t + u + v.$$

Proof. Since a and b are odd, there exist nonnegative integers p and q such that $a = 2p + 1$ and $b = 2q + 1$. Then $4a - b^2 = 8p + 4 - 4q^2 - 4q - 1 = 8p + 4q(q + 1) + 3 \equiv 3 \pmod 8$. By Theorem 8, there exist odd integers $x \geq y \geq z > 0$ such that

$$4a - b^2 = x^2 + y^2 + z^2.$$

Claim: $x + y + z < b + 4$. Indeed, by the Cauchy-Schwarz inequality, we have

$$(x + y + z)^2 \leq (x^2 + y^2 + z^2)(1^2 + 1^2 + 1^2)$$

Hence,

$$x + y + z \leq \sqrt{3(x^2 + y^2 + z^2)} = \sqrt{12a - 3b^2} < \sqrt{4(b^2 + 2b + 4) - 3b^2} = b + 4.$$

Writing x, y, z as $2\alpha + 1$, $2\beta + 1$, $2\gamma + 1$ gives

$$a = (q^2 + \alpha^2 + \beta^2 + \gamma^2) + (q + \alpha + \beta + \gamma) + 1$$

and

$$\alpha + \beta + \gamma \leq q. \tag{1}$$

We consider two cases.

Case 1: $q + \alpha + \beta + \gamma$ is even.

Set

$$s = \frac{q + \alpha + \beta + \gamma}{2} + 1,$$
$$t = q + \alpha + 1 - s,$$
$$u = q + \beta + 1 - s,$$
$$v = q + \gamma + 1 - s.$$

Then s, t, u, v are integers satisfying

$$a = s^2 + t^2 + u^2 + v^2$$
$$b = s + t + u + v,$$

and $s \geq t \geq u \geq v$. It remains to show that $v \geq 0$. Note that

$$\begin{aligned} v &= q + \gamma + 1 - \left(\frac{q + \alpha + \beta + \gamma}{2} + 1 \right) \\ &= \frac{q - \alpha - \beta + \gamma}{2} \\ &\geq 0 \end{aligned}$$

by (1).

Case 2: $q + \alpha + \beta + \gamma$ is odd.

Hence, $q + \alpha + \beta - \gamma + 1$ is even. Set

$$\begin{aligned} s &= \frac{q + \alpha + \beta - \gamma + 1}{2}, \\ t &= q + \alpha + 1 - s, \\ u &= q + \beta + 1 - s, \\ v &= q - \gamma - s. \end{aligned}$$

Then s, t, u, v are integers satisfying

$$\begin{aligned} a &= s^2 + t^2 + u^2 + v^2 \\ b &= s + t + u + v, \end{aligned}$$

and $s \geq t \geq u \geq v$. It remains to show that $v \geq 0$. Note that

$$\begin{aligned} v &= q - \gamma - \frac{q + \alpha + \beta - \gamma + 1}{2} \\ &= \frac{q - \alpha - \beta - \gamma - 1}{2} \\ &\geq \frac{-1}{2} \end{aligned}$$

by (1). Since v is an integer at least $-\frac{1}{2}$, it must be at least 0.

We now establish a series of technical lemmas from which Theorem 5 readily follows.

Define

$$u(m, n) := 2\left(1 - \frac{2}{m}\right) + \sqrt{4\left(1 - \frac{2}{m}\right)^2 + 8\left(\frac{n - (m-3)}{m}\right)} - 0.001$$

and

$$\ell(m, n) := \left(\frac{1}{2} - \frac{3}{m}\right) + \sqrt{\left(\frac{1}{2} - \frac{3}{m}\right)^2 + 6\left(\frac{n}{m}\right) - 4} + 0.001.$$

Lemma 2. *Let n and m be positive integers. If $m \geq 4 \wedge n \geq 53m$ or $m = 3 \wedge n \geq 159m$, then there exist integers b and r such that b is odd, $\ell(n, m) \leq b \leq u(n, m)$, $0 \leq r \leq m - 3$, and m divides $n - b - r$.*

Lemma 3. *Let* $n, m, b, r \in \mathbb{Z}$. *If* $m \geq 3$, $n \geq 2m$, $0 \leq r \leq m-3$, $\ell(n,m) \leq b \leq u(n,m)$ *and* $m \mid n-b-r$, *then* $a := 2\left(\frac{n-b-r}{m}\right) + b$ *satisfies* $b^2 - 4a < 0$ *and* $b^2 + 2b + 4 - 3a > 0$.

We postpone the proofs of these lemmas to the next section.

Proof (of Theorem 5). By Lemma 2, there exist integers b and r such that b is odd, $\ell(n,m) \leq b \leq u(n,m)$, $0 \leq r \leq m-3$, and m divides $n-b-r$.

By Lemma 3, $a := 2\left(\frac{n-b-r}{m}\right) + b$ is an integer such that $b^2 - 4a < 0$ and $b^2 + 2b + 4 - 3a > 0$.

By Lemma 1, there exist nonnegative integers s, t, u, v such that

$$\begin{aligned} a &= s^2 + t^2 + u^2 + v^2, \\ b &= s + t + u + v. \end{aligned}$$

Hence,

$$\begin{aligned} n &= \frac{m}{2}(a-b) + b + r \\ &= \frac{m}{2}(s^2 - s) + s + \frac{m}{2}(t^2 - t) + t + \frac{m}{2}(u^2 - u) + u + \frac{m}{2}(v^2 - v) + v + r \\ &= p_m(s) + p_m(t) + p_m(u) + p_m(v) + r. \end{aligned}$$

The result now follows.

3 Proofs of technical lemmas

In this section, we give proofs of Lemma 2 and Lemma 3.

We first address Lemma 3. The following is straightforward to show:

Lemma 4. *Let* $x, p, c \in \mathbb{R}$ *with* $c > 0$.

(a) If $0 \leq x < \frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 + c}$, *then* $x^2 - px - c < 0$.

(b) If $x > \frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 + c}$, *then* $x^2 - px - c > 0$.

Proof. Since $c > 0$, we have $\pm\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 + c} > \pm\frac{p}{2} + \left|\frac{p}{2}\right| \geq 0$.

(a) The statement holds trivially when $x = 0$.

Assume that $x > 0$. Since $x < \frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 + c}$, we have $x - p < -\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 + c}$. Thus,

$$\begin{aligned} x^2 - px - c &= x(x-p) - c \\ &< x\left(-\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 + c)}\right) - c \\ &< \left(\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 + c}\right)\left(-\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 + c)}\right) - c \\ &= 0. \end{aligned}$$

(b) Since $x > \frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 + c} > 0$, we have $x - p > -\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 + c} > -\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 + c} > 0$. Hence,

$$\begin{aligned} x^2 - px - c &= x(x-p) - c \\ &> \left(\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 + c}\right)(x-p) - c \\ &> \left(\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 + c}\right)\left(-\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 + c)}\right) - c \\ &= 0. \end{aligned}$$

Proof (of Lemma 3). Note that

$$\begin{aligned} b \geq \ell(n,m) &= \left(\frac{1}{2} - \frac{3}{m}\right) + \sqrt{\left(\frac{1}{2} - \frac{3}{m}\right)^2 + 6\left(\frac{n}{m}\right) - 4} + 0.001 \\ &> \left(1 - \frac{6}{m}\right)/2 + \sqrt{\left(\left(1 - \frac{6}{m}\right)/2\right)^2 + 6\left(\frac{n-r}{m}\right) - 4} \end{aligned}$$

Note that $n - r \geq 2m - (m-3) = m + 3$. Setting $p := 1 - \frac{6}{m}$ and $c := 6\left(\frac{n-r}{m}\right) - 4$, we have $c > 0$ and so, by Lemma 4 part (b), we obtain that $b^2 + 2b + 4 - 3a = b^2 - \left(1 - \frac{6}{m}\right)b - \left(6\left(\frac{n-r}{m}\right) - 4\right) > 0$.

We can also see from the above derivation that $b > 0$.

Now,

$$\begin{aligned} b \leq u(n,m) &= 2\left(1 - \frac{2}{m}\right) + \sqrt{4\left(1 - \frac{2}{m}\right)^2 + 8\left(\frac{n-(m-3)}{m}\right)} - 0.001 \\ &< \left(4\left(1 - \frac{2}{m}\right)/2\right) + \sqrt{\left(4\left(1 - \frac{2}{m}\right)/2\right)^2 + 8\left(\frac{n-r}{m}\right)}. \end{aligned}$$

Setting $p := 4\left(1 - \frac{2}{m}\right)$ and $c := 8\left(\frac{n-r}{m}\right)$, we have $c > 0$ and so, by Lemma 4 part (a), we obtain that $b^2 - 4a = b^2 - 4\left(1 - \frac{2}{m}\right)b - \frac{8n-r}{m} < 0$.

Our proof of Lemma 2 relies on the next two lemmas:

Lemma 5. *Let $p, q \in \mathbb{R}$. Let k be a positive integer such that $q - p \geq 2k$. Then there exists an integer m such that for $i = 0, \ldots, k-1$, if $b_i = 2(m+i)+1$, then $p \leq b_i \leq q$.*

Proof. Let $\ell = \lceil p \rceil$. Note that $p > \ell - 1$. We can take m to be the least integer such that $2m+1 \geq \ell$. Indeed, for all $i = 0, \ldots, k-1$, $b_i \geq b_0 = 2m+1 \geq p$ and $b_i \leq b_{k-1} = 2(m + (k-1)) + 1 = 2m + 1 + 2(k-1)$.

If ℓ is even, then $2m+1 = \ell+1$. Hence, $2m+1+2(k-1) = \ell+1+2(k-1) = \ell - 1 + 2k < p + 2k \leq p + q - p = q$.

If ℓ is odd, then $2m+1=\ell$. Hence, $2m+1+2(k-1)=\ell+2(k-1)=\ell-1+2k-1<p+2k-1\leq p+q-p-1<q$.

Lemma 6. *Let m and n be positive integers.*

(a) If $m\geq 4$ and $n\geq 53m$, then $u(n,m)-\ell(n,m)\geq 4$.

(b) If $m=3$ and $n\geq 159m$, then $u(n,m)-\ell(n,m)\geq 6$.

Before we prove this, we first establish a technical result to obtain two key inequalities which allow us to obtain a tighter analysis of what was in Nathanson's original proof.

Lemma 7. *Let $a,b,p,q\in\mathbb{R}$ such that $a>b>0$. Define $f(t):=\sqrt{at+p}-\sqrt{bt+q}$. Then for all x and y such that $x\geq y\geq \dfrac{b^2p-a^2q}{ab(a-b)}$, $ay+p\geq 0$ and $by+q\geq 0$,*

$$f(x)\geq f(y).$$

Proof. Let x and y be such that $x\geq y\geq \dfrac{b^2p-a^2q}{ab(a-b)}$. If $x=y$, there is nothing to prove.

Assume that $x>y$. Then there exist δ and γ, where $\delta>\gamma\geq 0$, such that $x=\dfrac{b^2p-a^2q}{ab(a-b)}+\delta$, and $y=\dfrac{b^2p-a^2q}{ab(a-b)}+\gamma$. Let $\theta=\dfrac{bp-aq}{a-b}$. Then

$$\begin{aligned}
ax+p &= a\left(\frac{b^2p-a^2q}{ab(a-b)}+\delta\right)+p\\
&= \frac{b^2p-a^2q+bap-b^2p}{b(a-b)}+a\delta\\
&= \frac{a(bp-aq)}{b(a-b)}+a\delta\\
&= \frac{a}{b}\theta+a\delta,
\end{aligned}$$

and

$$\begin{aligned}
bx+q &= b\left(\frac{b^2p-a^2q}{ab(a-b)}+\delta\right)+q\\
&= \frac{b^2p-a^2q+a^2q-abq}{a(a-b)}+b\delta\\
&= \frac{b(bp-aq)}{a(a-b)}+b\delta\\
&= \frac{b}{a}\theta+b\delta.
\end{aligned}$$

Similarly, $ay+p=\dfrac{a}{b}\theta+a\gamma$ and $by+q=\dfrac{b}{a}\theta+b\gamma$.

Then

$$\begin{aligned}\sqrt{\frac{a}{b}\theta + a\delta} - \sqrt{\frac{a}{b}\theta + a\gamma} &= \frac{\left(\sqrt{\frac{a}{b}\theta + a\delta}\right)^2 - \left(\sqrt{\frac{a}{b}\theta + a\gamma}\right)^2}{\sqrt{\frac{a}{b}\theta + a\delta} + \sqrt{\frac{a}{b}\theta + a\gamma}} \\ &= \frac{a(\delta - \gamma)}{\sqrt{\frac{a}{b}\theta + a\delta} + \sqrt{\frac{a}{b}\theta + a\gamma}} \\ &= \frac{a(\delta - \gamma)}{\sqrt{\frac{a}{b}}\left(\sqrt{\theta + b\delta} + \sqrt{\theta + b\gamma}\right)} \\ &= \frac{b(\delta - \gamma)}{\sqrt{\frac{b}{a}}\left(\sqrt{\theta + b\delta} + \sqrt{\theta + b\gamma}\right)} \\ &\geq \frac{b(\delta - \gamma)}{\sqrt{\frac{b}{a}}\left(\sqrt{\theta + a\delta} + \sqrt{\theta + a\gamma}\right)} \\ &= \frac{\left(\sqrt{\frac{b}{a}\theta + b\delta}\right)^2 - \left(\sqrt{\frac{b}{a}\theta + b\gamma}\right)^2}{\sqrt{\frac{b}{a}\theta + b\delta} + \sqrt{\frac{b}{a}\theta + b\gamma}} \\ &= \sqrt{\frac{b}{a}\theta + b\delta} - \sqrt{\frac{b}{a}\theta + b\gamma}.\end{aligned}$$

Hence,

$$\sqrt{\frac{a}{b}\theta + a\delta} - \sqrt{\frac{b}{a}\theta + b\delta} \geq \sqrt{\frac{a}{b}\theta + a\gamma} - \sqrt{\frac{b}{a}\theta + b\gamma},$$

giving

$$f(x) \geq f(y).$$

Corollary 1. *If* $x \geq 53$, *then* $\frac{5}{4} + \sqrt{8x - 4} - \sqrt{6x - \frac{15}{4}} - 0.002 \geq 4$.

Proof. By Lemma 7 with $a = 8$, $b = 6$, $p = -4$, and $q = -\frac{15}{4}$, we have

$$\begin{aligned}\frac{5}{4} + \sqrt{8x - 4} - \sqrt{6x - \frac{15}{4}} - 0.002 &\geq \frac{5}{4} + \sqrt{8(53) - 4} - \sqrt{6(53) - \frac{15}{4}} - 0.002 \\ &\geq 4.\end{aligned}$$

Corollary 2. *If* $x \geq 159$, *then* $\frac{7}{6} + \sqrt{8x + \frac{4}{9}} - \sqrt{6x - \frac{15}{4}} - 0.002 \geq 6$.

Proof. By Lemma 7 with $a = 8$, $b = 6$, $p = \frac{4}{9}$, and $q = -\frac{15}{4}$, we have

$$\begin{aligned}\frac{7}{6} + \sqrt{8x + \frac{4}{9}} - \sqrt{6x - \frac{15}{4}} - 0.002 &\geq \frac{7}{6} + \sqrt{8(159) + \frac{4}{9}} - \sqrt{6(159) - \frac{15}{4}} - 0.002 \\ &\geq 6.\end{aligned}$$

Proof (of Lemma 6). With $m \geq 4$, we have

$$\begin{aligned} u(n,m) - \ell(n,m) &= \frac{3}{2} - \frac{1}{m} + \sqrt{8\left(\frac{n}{m}\right) + \frac{16}{m^2} + \frac{8}{m} - 4} - \sqrt{6\left(\frac{n}{m}\right) - \frac{3}{m}\left(1 - \frac{3}{m}\right) - \frac{15}{4}} - 0.002 \\ &\geq \frac{3}{2} - \frac{1}{4} + \sqrt{8\left(\frac{n}{m}\right) - 4} - \sqrt{6\left(\frac{n}{m}\right) - \frac{15}{4}} - 0.002 \\ &= \frac{5}{4} + \sqrt{8\left(\frac{n}{m}\right) - 4} - \sqrt{6\left(\frac{n}{m}\right) - \frac{15}{4}} - 0.002 \\ &\geq 4 \end{aligned}$$

by Corollary 1 with $x = \frac{n}{m}$.

When $m = 3$, we have

$$\begin{aligned} u(n,m) - \ell(n,m) &= \frac{7}{6} + \sqrt{8\left(\frac{n}{m}\right) + \frac{4}{9}} - \sqrt{6\left(\frac{n}{m}\right) - \frac{15}{4}} - 0.002 \\ &\geq 6 \end{aligned}$$

by Corollary 2 with $x = \frac{n}{m}$.

Proof (of Lemma 2). First, consider the case when $m \geq 4$ and $n \geq 53m$. By Lemma 6 part (a), we have $u(n,m) - \ell(n,m) \geq 4$. It follows from Lemma 5 that there exist odd integers b_0, b_1 in the interval $[\ell(n,m), u(n,m)]$ such that $b_1 = b_0 + 2$.

Let r' be the remainder when $n - b_0$ is divided by m. Note that $r' \leq m - 1$ and $n - b_0 - r' \equiv 0 \pmod{m}$.

If $r' \geq m - 2$, set r to $r' - 2$. Since $r' \leq m - 1$, we have that $r = r' - 2 \leq m - 3$. Also, $r = r' - 2 \geq m - 2 - 2 = m - 4 \geq 4 - 4 = 0$. Then setting b to b_1, we have that $n - b - r = n - b_1 - (r' - 2) = n - b_0 - r' \equiv 0 \pmod{m}$. Hence, m divides $n - b - r$.

Otherwise, we have $r' \leq m - 3$. Setting r to r' and b to b_0, we have that $n - b - r = n - b_0 - r' \equiv 0 \pmod{m}$. Hence, m divides $n - b - r$.

Next, consider the case when $m = 3$ and $n \geq 159m$. We set r to 0. By Lemma 6 part (b), we have $u(n,m) - \ell(n,m) \geq 6$. It follows from Lemma 5 that there exist odd integers b_0, b_1, b_2 in the interval $[\ell(n,m), u(n,m)]$ such that $b_1 = b_0 + 2$ and $b_2 = b_1 + 2$.

Since $b_1 \equiv b_0 + 2 \pmod{3}$ and $b_2 \equiv b_1 + 2 \equiv b_0 + 4 \equiv b_0 + 1 \pmod{3}$, it follows that for some $b \in \{b_0, b_1, b_2\}$, we have $n - b - r \equiv n - b \equiv 0 \pmod{3}$.

4 On our formalization in Lean 4

We formalized the proofs of Theorem 6 and Theorem 7 in the Lean 4 Theorem Prover [6], asserting Theorem 8 without proof. In the following, we outline the essential details. The full proof can be found in the Lean files [5].

We defined the proposition

```
def IsnPolygonal (s : ℤ) (_ : s ≥ 3) (n : ℕ) := n = 0
  ∨ ∃ (k : ℕ), (((s : ℚ) - 2) / 2) * (k * (k - 1)) + k = n
```

for stating if `n` is a polygonal number of order `s`. (The letter s is sometimes used in the extant literature to denote the order, i.e. $s = m + 2$ with $m \geq 1$, as it corresponds more clearly to the number of sides. For example, a triangular number is a polygonal number of order 3.)

We chose `ℤ` instead of `ℕ` for the type of `s` for two reasons. The first is to accommodate future extensions to polygonal numbers of negative orders (which do exist). The second is that subtraction of natural numbers in Lean is truncated. For example, $2 - 4 = 0$. This means that something like $a - b + b$ cannot be rewritten as a unless one has a proof that $a \geq b$.

In addition, we could have avoided an explicit requirement of a proof that `s ≥ 3` by defining a subtype for the argument `s`. However, it is rather inconvenient to work with such a subtype and we decided that it was not worth the trouble for having a cleaner interface.

With the above definition, we can establish that 13 is a triangular number as follows:

```
example : IsnPolygonal 3 (by show 3 ≥ 3; simp) 36 := by right; use 8; norm_num
```

However, proving that a number is not polygonal of some particular order is not necessarily trivial as it might involve a detailed case analysis:

```
example : ¬IsnPolygonal 3 (by show 3 ≥ 3; simp) 2 := by
dsimp [IsnPolygonal]
push_neg
constructor
. norm_num
. intro k
  by_cases hk : k ≤ 2
  . interval_cases k <;> norm_num
  . qify at hk; nlinarith
```

To facilitate automated proof generation via the `decide` tactic, we used the following equivalent definition:

```
def IsnPolygonal₀ (s : ℤ) (_ : s ≥ 3) (n : ℕ) :=
  n = 0 ∨  (IsSquare (8 * (s - 2) * n + (s - 4) ^ 2)
        ∧ (Int.sqrt (8 * (s - 2) * n + (s - 4) ^ 2) + (s - 4)) % (2 * (s - 2)) = 0)
```

Since in Mathlib, there is already a decidable instance for `IsSquare`, it is straightforward to define a decidable instance for `IsnPolygonal₀`:

```
instance : Decidable (IsnPolygonal₀ s n h) := by
dsimp [IsnPolygonal₀]
exact instDecidableOr
```

```
example : IsnPolygonal₀ 5 (by show 5 ≥ 3; simp) 5 := by decide +kernel
example : ¬IsnPolygonal₀ 3 (by show 3 ≥ 3; simp) 2 := by decide +kernel
```

Note that `+kernel` is needed since `decide` alone does not work for `IsSquare`. The reason is technical and is beyond the scope of this paper. Nevertheless, the reduction is performed in the kernel and does not reduce the trustworthiness of the result.

A decidable instance for `IsnPolygonal` can then be obtained as follows:

```
instance : Decidable (IsnPolygonal s n h) := by
  apply decidable_of_iff (IsnPolygonal₀ s n h)
  refine Eq.to_iff ?_
  -- Equivalence proof omitted.
```

The proof that `IsnPolygonal` and `IsnPolygonal₀` are equivalent is rather involved. Readers interested in the details are referred to the Lean files [5].

Unfortunately, proving by `decide` turned out to be quite slow. The bottleneck was the decidable instance for `IsSquare`. Therefore, in the case analyses for our formalization of the proofs of Theorem 6 and Theorem 7, we avoided using `decide`.

We also defined the following proposition

```
def IsNKPolygonal (s : ℤ) (hs : s ≥ 3) (k : ℕ) (n : ℕ) :=
  ∃ S : List ℕ, S.all (IsnPolygonal s hs) ∧ S.length = k ∧ S.sum = n
```

With this definition, the statement of Theorem 6 can be formalized as

```
def pentaExceptions : Finset ℕ := {9, 21, 31, 43, 55, 89}

theorem SumOfFourPentagonalNumber : ∀ n : ℕ, ¬ (n ∈ pentaExceptions)
  → IsNKPolygonal 5 (by norm_num) 4 n := by sorry
```

For efficiency, we first defined all the pentagonal numbers less than 477:

```
def p0 : IsnPolygonal 5 (by norm_num) 0 := by simp [IsnPolygonal];
def p1 : IsnPolygonal 5 (by norm_num) 1 := by simp [IsnPolygonal]; use 1; ring
def p5 : IsnPolygonal 5 (by norm_num) 5 := by simp [IsnPolygonal]; use 2; ring
def p12 : IsnPolygonal 5 (by norm_num) 12 := by simp [IsnPolygonal]; use 3; ring
def p22 : IsnPolygonal 5 (by norm_num) 22 := by simp [IsnPolygonal]; use 4; ring
def p35 : IsnPolygonal 5 (by norm_num) 35 := by simp [IsnPolygonal]; use 5; ring
def p51 : IsnPolygonal 5 (by norm_num) 51 := by simp [IsnPolygonal]; use 6; ring
def p70 : IsnPolygonal 5 (by norm_num) 70 := by simp [IsnPolygonal]; use 7; ring
def p92 : IsnPolygonal 5 (by norm_num) 92 := by simp [IsnPolygonal]; use 8; ring
def p117 : IsnPolygonal 5 (by norm_num) 117 := by simp [IsnPolygonal]; use 9; ring
def p145 : IsnPolygonal 5 (by norm_num) 145 := by simp [IsnPolygonal]; use 10; ring
def p176 : IsnPolygonal 5 (by norm_num) 176 := by simp [IsnPolygonal]; use 11; ring
def p210 : IsnPolygonal 5 (by norm_num) 210 := by simp [IsnPolygonal]; use 12; ring
def p247 : IsnPolygonal 5 (by norm_num) 247 := by simp [IsnPolygonal]; use 13; ring
def p287 : IsnPolygonal 5 (by norm_num) 287 := by simp [IsnPolygonal]; use 14; ring
def p330 : IsnPolygonal 5 (by norm_num) 330 := by simp [IsnPolygonal]; use 15; ring
def p376 : IsnPolygonal 5 (by norm_num) 376 := by simp [IsnPolygonal]; use 16; ring
def p425 : IsnPolygonal 5 (by norm_num) 425 := by simp [IsnPolygonal]; use 17; ring
```

One can then handle each number less than 477 by directly making use of these definitions. For instance, we can prove that 113 is the sum of four pentagonal numbers as follows:

```
example : IsNKPolygonal 3 (by norm_num) 4 113 := by
  use [5, 22, 35, 51]
  simp [p5, p22, p35, p51]
```

Finally, the statement of Theorem 7 is formalized as

```
def hexaExceptions : Finset ℕ := {11, 26}

theorem SumOfFiveHexagonalNumber : ∀ n : ℕ, ¬ (n ∈ hexaExceptions)
  → IsNKPolygonal 6 (by norm_num) 5 n := by sorry
```

We employed a similar strategy as for Theorem 6 to improve efficiency. Both theorems could be type-checked by Lean within minutes.

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