(1988.07 + minor revisions)

THE TITS ALTERNATIVE

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1 Introduction

We shall say that the Tits Alternative holds for a class of groups if each group in the class is either solvable by finite (that is, contains a solvable normal subgroup of finite index) or contains a free subgroup of rank at least 2. Free groups of rank 2 contain free subgroups of countable rank, so in the second case there will be of free subgroups of all finite ranks.

The Tits Alternative is so called because it arises in a major theorem published by J. Tits [20] in 1972:

TITS' THEOREM Let G be a finitely generated linear group over a (commutative) field. Then either G is solvable by finite or G contains a noncyclic free subgroup. $\square$

The object of the present paper is to give an exposition of a simplified version of Tits' proof, and to discuss some of the ramifications of Tits' Theorem. In what follows F will denote an arbitrary (commutative) field, and a linear group of degree n over F will refer to a subgroup of $GL(n, F)$, or a group of linear transformations of an n-dimensional vector space over F.

Acknowledgements The present paper is a major revision of [5] published in 1972, which purports to give a proof of Tits' Theorem but contains some very serious flaws. I am indebted to several mathematicians, including B. Wehrfritz and Ju.I. Merzljakov, who pointed out what now appear as embarrassingly elementary errors, and I am particularly indebted to J. Hulse who in his correspondence in 1974 insisted that I get it right. Why then should I publish this revised version at this late date? The reasons are two-fold. Firstly, as far as I know, except for the original paper of Tits the only places where the proof of his theorem appears are in [14] and [23] both of which follow the original proof quite closely. (Actually, only an outline of the proof appears in [23]). The proof given below in Sects. 3-4 seems (at least to me) to be technically simpler, partly because Tits is interested in proving a more general result and some of the technicalities can be avoided for that reason. Since the theorem remains of major interest, it seems worthwhile having an alternative approach to its proof. Secondly, it seems worthwhile putting on record in Sect. 2 some of the work which has arisen as a consequence of the theorem - its applications and some generalizations which it has stimulated. The reference journal *Citation Index* records over 60 citations to Tits' original paper up to the end of 1987.

2 Consequences of Tits' Theorem

We shall describe some of the consequences of Tits' Theorem and some attempts to generalize it. Theorem 1 is in fact the original form in which Tits stated the theorem in [20] and Theorems 2-4 are listed there as corollaries. (See Sect. 3 below for the definition of $Solv(G)$ and a proof of Theorem 1.)

THEOREM 1 (Tits [20]) For any linear group G, the factor group $G/Solv(G)$ is locally finite. If the characteristic of the underlying field is 0, then $G/Solv(G)$ is finite, and so G is solvable by finite. Moreover, in the latter case there is a bound on the index $|G : Solv(G)|$ depending only on the degree of G. □

The second theorem was announced in [25] but the proof there is faulty. A group is *noetherian* if it satisfies the ascending chain condition on subgroups. Theorems 3 and 4 are due to V.P. Platonov who gave earlier proofs independent of Tits' results.

THEOREM 2 Every noetherian linear group contains a polycyclic subgroup of finite index. □

THEOREM 3 (Platonov [18]) If‘ G is a linear group such that for some d each finitely generated subgroup can be generated by d elements, then G is solvable by finite. □

THEOREM 4 (Platonov [17]) A linear group is either solvable by finite or it generates the variety of all groups. □

For more details about the proofs of Platonov's theorems see [23] Theorems 10.9 and 10.15.

Let G be a group which is generated by a finite set S of generators, and let $\gamma(m)$ denote the number of elements in G which can be written as a product of at most m elements from $S \cup S^{-1}$. Then we say G has polynomial growth if, for some constant $C > 0$, $\gamma(m) < m^C$ for all sufficiently large m; and G has exponential growth if, for some $C > 1$, $\gamma(m) > m^C$ for all sufficiently large m. It is easy to show that these definitions do not depend on the choice of the generating set S. Then a result of J. Milnor and J.A. Wolf ([24] p. 421) together with Tits' Theorem shows that each linear group has either polynomial or exponential growth. It is easily shown that a finitely generated group which is nilpotent by finite has polynomial growth. Using methods which generalize those of [20], M.L. Gromov proved the much deeper converse which had been conjectured by Wolf.

THEOREM 5 (Gromov [7]) Every finitely generated group with polynomial growth is nilpotent by finite. □

For related papers on polynomial growth see [16] and [6]. □

G.A. Margulis and G.A. Soifer have also extended the techniques of [20]. One of their most striking results is the following.

THEOREM 6 (Margulis and Soifer [13]) A finitely generated linear group has a maximal subgroup of infinite index if and only if it has a free subgroup of rank 2. In the latter case the group has uncountably many maximal subgroups of infinite index. □

N.V. Ivanov [10] and J.D. McCarthy [15] have independently shown that the Tits Alternative holds for a class of groups which arise in topology. S. Wagon

[21] has shown how Tits' Theorem is related to paradoxical decompositions (the Banach-Tarski Theorem) and to amenable groups. For the latter topic see also [2].

Very soon after Tits' Theorem was published the question arose as to what extent it might be generalized. In its original form, Tits' Theorem states that the Tits Alternative holds for finitely generated subgroups of $Aut(V)$ where V is a finitely generated vector space over a commutative field. S. Bachmuth and H.Y. Mochizuki [1] showed that the theorem remains true when V is replaced by a finitely generated group which it nilpotent by abelian; but B. Hartley [8] gave examples to show that the Tits Alternative does not always hold when V is replaced by a solvable group of derived length 3. A.I. Lichtman has written a series of papers related to this subject for the case where V is a vector space of finite dimension over a skew field. In particular, he has given examples for all possible characteristics to show that the Tits Alternative need not hold over skew fields even in dimension 1 (see [12] and [19]). In [9] B. Hartley and P.F. Pickel consider the case of free subgroups in the unit group of an integral group ring.

3 Some Basic Lemmas

Before turning to the proof of Tits' Theorem we shall isolate some of the general results on which the proof will be based.

LEMMA 1 (Zassenhaus and Mal'cev) If G is a linear group, then each locally solvable subgroup is solvable. Hence G has a unique maximal normal solvable subgroup (the *solvable radical* of G) which we shall denote by $Solv(G)$.

Proof See [4] Theorem 6.2B or [23] Corollary 3.8. □

We shall need some elementary properties of the Zariski topology with reference to linear groups (see, for example, [4] Chapter 8 or [23] Chapters 5 and 14.

LEMMA 2 Let G be a subgroup of $GL(n, F)$. Then the connected component G_0 of 1 in the Zariski topology is a normal subgroup of finite index in G. Moreover, G_0 is contained in every Zariski closed subgroup of finite index in G.

Proof See [4] Theorem 8.5 or [23] Lemmas 5.2 and 5.3. □

Remark If E is a field extension of F, then the topology induced on $G \leq GL(n, F)$ by the Zariski topology relative to E is the same as that induced by the Zariski topology relative to F.

LEMMA 3 Let G be a subgroup of $GL(n, F)$ and H be a normal subgroup of G. If H is closed in the Zariski topology, then for some integer $m > 1$ there is a linear representation $\phi : G \to GL(m, F)$ with kernel H. The mapping ϕ is a rational mapping and so continuous in the Zariski topology.

Proof See [23] Theorem 6.4. □

LEMMA 4 (Wehrfritz) Suppose that G is a linear group over F and each finitely generated subgroup of G is solvable by finite. Then $G/Solv(G)$ is a locally finite linear group over F.

Proof See [23] Lemma 2. Note that every periodic linear group is locally finite. □

LEMMA 5 (Jordan and Schur) Let G be a periodic linear group of degree n over a field F of characteristic 0. Then G has a normal abelian subgroup A whose index $|G : A|$ can be bounded by a function depending only on n.

Proof See [4] Theorem 9.5 or [23] Theorem 9.4. □

REMARK The proof of Theorem 1 in Sect. 2 now follows immediately from Tits' Theorem and Lemmas 3-5 because $Solv(G)$ is a Zariski closed subgroup of G.

The following is a modest generalization of a theorem of Schur. Compare with Sect. 2 of [20].

LEMMA 6 Let G be a finitely generated linear group and let H be a subgroup . If each $x \in H$ has all of its eigenvalues roots of unity (so a suitable power of x is unipotent), then H has a normal subgroup of finite index consisting entirely of unipotent elements. In particular, if H is completely reducible, then H is finite.

Proof The proof of Theorem 9.2 in [4] shows that there exists an integer N which depends only on G such that x^N is unipotent for all $x \in H$. Then the Exercise following Corollary 2.8C of [4] shows that H has a normal subgroup of finite index consisting of unipotent elements. In particular, if H is completely reducible then this subgroup is trivial by Theorem 2.8C of [4] and so H is finite. □

LEMMA 7 ("Pingpong Lemma") Let. G be a group acting on a set Ω and let A and B be two subgroups. Suppose that there exist nonempty sets $\Delta, \Gamma \subseteq \Omega$ such that

1. $\Delta \cap \Gamma = \emptyset$;
2. $\Gamma a \subseteq \Delta$ and $\Delta b \subseteq \Gamma$ for all nontrivial $a \in A$ and $b \in B$;
3. for all $a \in A$, $\Delta a \cap \Delta \neq \emptyset$.

Then the subgroup $\langle A, B \rangle$ generated by A and B is the free product $A * B$.

Proof It is enough to show that no product $g = a_1 b_1 ... a_k b_k$ (with $k \geq 1$ and nontrivial $a_i \in A$ and $b_i \in B$) is equal to 1. Moreover, by 3. there exists $\delta \in \Delta$ such that $\delta a_1 \in \Delta$. Then 2. shows that $\delta g \in \Gamma$, and so 1. proves that $g \neq 1$. □

EXAMPLE. An interesting illustrative example of Lemma 7 is given in the case where $G := GL(2, \mathbb{C})$ acts on $\mathbb{C}^2$ and we define

$$a := \begin{bmatrix} 1 & 0 \\ \theta & 1 \end{bmatrix} \text{ and } b := \begin{bmatrix} 1 & \theta \\ 0 & 1 \end{bmatrix}.$$

Take $A := \langle a \rangle$ and $B := \langle b \rangle$, $\Gamma := \{(\alpha, \beta) \mid |\alpha| < |\beta|\}$ and $\Delta := \{(\alpha, \beta) \mid |\alpha| > |\beta|\}$. If $|\theta| \geq 2$ then the hypotheses of the lemma hold and so $\langle a, b \rangle = A * B$ is a free group of rank 2. (If $|\theta| < 2$ then $\langle a, b \rangle$ need not be free.) □

4 Proof of Tits' Theorem

We now turn to the proof of Tits' Theorem, Through a series of propositions we shall show that a finitely generated linear group which is not solvable by finite must have a free subgroup of rank 2. At the end we shall link these propositions together in a complete proof. In understanding the approach it is helpful to recall that if G is a group with a normal subgroup N and two elements Na and Nb generate a free subgroup of rank 2 in G/N, then the subgroup $\langle a, b\rangle$ in G is also free of rank 2. Thus to show that G has a free subgroup of rank 2 it is enough to prove that some factor group has this property.

PROPOSITION 1 Let $G \leq GL(n, F)$ be a finitely generated and completely reducible group which is connected in the Zariski topology. If G is not solvable by finite then there exists $x \in G'$ with at least one eigenvalue which is not a root of unity.

Proof Since G is completely reducible, so is the derived group G' by Clifford's Theorem. Thus by Lemma 6 it is enough to show that G' is infinite. Put $C := C_G(G')$. Then $C/(C \cap G') \cong CG'/G' \leq G/G'$, and so C is metabelian. Since G/C can be embedded in $Aut(G')$, the hypothesis on G shows that G' must be infinite. □

The first major step in the proof of Tits' Theorem is to construct a suitable normed field over which G has a representation. Let E_0 denote the prime subfield of F and let E be the subfield of F generated by all entries from a set of matrices which generate G; thus $G \leq GL(n, E)$ and E is finitely generated over E_0. The following proposition shows how we can extend E to a locally compact normed field $K, ||$ in an appropriate way.

PROPOSITION 2 Suppose that $\xi \in E$ is not a root of unity. Then there exists an extension K of E with a norm $||$ (= multiplicative valuation) such that $|\xi| \neq 1$ and $K, ||$ is a locally compact normed field.

Proof Since E is finitely generated over E_0 there is a finite transcendence basis T for E over E_0 and E is of finite degree over $E_0(T)$.

Case 1: $\operatorname{char} E = p > 0$.

Put $D := E_0[T]$ and let P be the ideal generated by T in D. Then there is a unique valuation $||_0$ on $E_0(T)$ such that if $\alpha \in D$ then $|\alpha|_0 = 2^{-1}$ when $\alpha \in P^i \setminus P^{i+1}$ for all $i \geq 0$. Since E has finite degree over $E_0(T)$, this norm can be extended to a norm $||_1$ on E, and then to the completion $\hat{E}, ||$ of $E, ||_1$ (see [[3] Chapter 2). The completion is a locally compact normed field since for each β in $\hat{E}$, $U := \left\{\alpha \in \hat{E} \mid |\alpha - \beta| \leq 1\right\}$ is a compact neighbourhood of β (see [3] pp. 49-50).

Case 2: $\operatorname{char} E = 0$.

In this case we may take $E_0 = \mathbb{Q}$, and we shall show that there is an embedding ϕ of E into a locally compact normed field $L, ||$ with $|\phi(\xi)| \neq 1$. This will give an extension $K \cong L$ of E with the required properties.

If ξ is transcendental over E_0 then we can take $\xi \in T$. Since $\mathbb{C}$ has infinite transcendental degree over $\mathbb{Q}$, there exists an embedding ϕ_0 of $E_0(T)$ into $\mathbb{C}$ such that $\phi_0(\xi)$ is a transcendental element of $\mathbb{C}$ of absolute value > 1.

Since $\mathbb{C}$ is algebraically closed, and $[E : E_0(T)] < \infty$, this embedding can be extended to an embedding ϕ of E into $L := \mathbb{C}$ which satisfies the required conditions (with the usual norm).

Thus we may suppose that ξ is algebraic over $E_0 = \mathbb{Q}$ and let $m(X)$ be its (monic) minimal polynomial over $\mathbb{Q}$. Since ξ is not a root of unity, $m(X)$ is not a cyclotomic polynomial. In particular, if all the coefficients of $m(X)$ are integers then Kronecker's Theorem shows that $m(X)$ has a root $\eta \in \mathbb{C}$ of absolute value > 1 (see [3] p.72). In this case, we consider first the $\mathbb{Q}$-embedding ϕ_0 of $E_0(\xi)$ into $\mathbb{C}$ with $\phi_0(\xi) = \eta$; extend this to an embedding ϕ_1 of $E_0(T_1 \cup \{\xi\})$ where T_1 is a transcendence basis for E over $E_0(\xi)$, and finally to an embedding ϕ of E into $\mathbb{C}$ (see [11] Chapter VII §2). We can then take $L = \mathbb{C}$ with the usual absolute value to satisfy our conditions.

This leaves the case where the minimal polynomial $m(X)$ of ξ has a nonintegral coefficient. Let p be a prime dividing the denominator of such a coefficient, and let $\mathbb{Q}_p, ||_0$ be the p-adic completion of the rationals. Let L_1 be a splitting field of $m(X)$ over $\mathbb{Q}_p$, and let $||_1$ be the (unique) extension of the norm $||_0$ to L_1. Since $||_1$ is non-archimedean, and at least one of the elementary symmetric functions of the roots of $m(X)$ in L_1 has norm > 1, therefore $|\eta|_1 > 1$ for at least one root η of $m(X)$. Now let ϕ_0 be the $\mathbb{Q}$-embedding of $E_0(\xi)$ into L_1 with $\phi_0(\xi) = \eta$. Since $\mathbb{Q}_p$ is uncountable it has infinite transcendence degree over $\mathbb{Q}$, and so there is an embedding ϕ_1 of $E_0(T_1 \cup \{\xi\})$ into L_1 where T_1 is a transcendence basis of E over $E_0(\xi)$. This embedding can be extended to an embedding ϕ of E into some finite extension L of L_1, and the norm on L_1 can be extended (uniquely) to a norm $||$ on L. The normed field L is locally compact because it is a finite extension of the locally compact field $\mathbb{Q}_p, ||_0$ (see [3] pp. 49-50). Since $|\phi(\xi)| = |\eta| > 1$, the construction is completed in this case as well. $\square$

From now on K will denote a locally compact field with norm $||$.

PROPOSITION 3 Suppose that $G \leq GL(n, K)$ has an element x whose eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ in K satisfy

$$|\lambda_1| = |\lambda_2| = ... = |\lambda_r| > |\lambda_{r+1}| \geq .. \geq |\lambda_n|$$

Then there is a representation $\rho : G \to GL(\binom{n}{r}, K)$ such that $\rho(x)$ has a unique (simple) eigenvalue of maximum norm. Moreover, if G is connected in the Zariski topology, then so is $\rho(G)$.

Proof Le $e_1 = (1\,0 \ldots .\,0), ..., e_n = (0\,0 \ldots 1)$ be the standard basis of K^n. Without loss in generality we may assume (by replacing G be a conjugate group if necessary) that x is in Jordan form so that for each i, $e_i x = \lambda_i e_i$ or $\lambda_i e_i + e_{i-1}$.

Consider the exterior algebra $E := \bigwedge K^n$. If we define $e_\Delta := e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_s}$ for each subset $\Delta := \{i_1, i_2, ..., i_s\}$ of $\{1, 2, ..., n\}$ with $i_1 < i_2 < ... < i_s$, then the set of all e_Δ forms a K-basis for E. Moreover, G acts linearly on E in a natural way, and the subspace V of E spanned by the e_Δ with $|\Delta| = r$ is a G-invariant subspace of dimension $\binom{n}{r}$. The action of G on this subspace therefore gives a representation ρ of G of this dimension. We now show that x has a unique eigenvalue of maximum norm in its action on V.

We define a partial ordering on the r-subsets of $\{1, 2, .., n\}$ by

$$\Delta \leq \Gamma \text{ when } \Delta = \{i_1, i_2, ..., i_r\} \text{ and } \Gamma = \{j_1, j_2, ..., j_r\} \text{ with } i_t \leq j_t \text{ for all } t$$

Then from the action of x on K^n we see that

$$e_\Delta x = \lambda_\Delta e_\Delta + \sum \alpha_\Gamma e_\Gamma \qquad (1)$$

where $\lambda_\Delta := \prod_{i \in \Delta} \lambda_i$ and the sum in (1) is over certain $\Gamma < \Delta$ with $\alpha_\Gamma \in K$. In particular, if $\Phi := \{1, 2, ..., r\}$ then $e_\Phi x = \lambda_\Phi e_\Phi$ since there are no $\Gamma < \Phi$; thus λ_Φ is an eigenvalue for x. One the other hand, suppose that μ is an eigenvalue for x with an eigenvalue v which is not a scalar multiple of e_Φ. Write $v = \sum \beta_\Gamma e_\Gamma$ and choose Δ maximal in the partial ordering such that $\beta_\Delta \neq 0$. Then (1) shows that $vx = \mu v$ implies that $\mu = \lambda_\Delta$, and so $|\mu| = |\lambda_\Delta| < |\lambda_\Phi|$ by the definition of r. Hence x has λ_Φ as it unique eigenvalue of maximum norm. Finally, since ρ is a rational mapping it is continuous under the Zariski topology, and so maps connected sets onto connected sets. $\square$

If $x \in GL(m, K)$ then the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_m$ of x lie in a finite extension K_1 of K. Since $K, ||$ is locally compact, it is complete ([3] p. 50), and so there is a unique extension $||_1$ of $||$ to K_1, and the norms $|\lambda_i|_1$ $(i = 1, 2, ..., m)$ are independent of the choice of K_1. We say that λ_i is a *dominant* eigenvalue for x if $|\lambda_i|_1 > |\lambda_j|_1$ for all $j \neq i$.

PROPOSITION 4 Suppose that $G \leq GL(m, K)$ is irreducible and connected in the Zariski topology. If G has an element x whose eigenvalues lie in K and with a dominant eigenvalue, then it contains an element x_0 such that both x_0 and x_0^{-1} have dominant eigenvalues.

Proof Let the eigenvalues of x be $\lambda_1, \lambda_2, ..., \lambda_m$ with $|\lambda_1| > |\lambda_2| \geq ... \geq |\lambda_m|$. Without loss in generality we may assume that x is in Jordan form:

$$x = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & \lambda_m \end{bmatrix}$$

with all entries above the main diagonal equal to 0. If $\operatorname{char} K = p > 0$ then x^{p^k} is diagonal for suitably large k, so in this case we may replace x by a power of itself and assume that x is diagonal. We now define r as the integer ≥ 0 such that $r+1$ is the dimension of the largest Jordan block for an eigenvalue λ_i with $|\lambda_i| = |\lambda_m|$ (a minimal eigenvalue); so $r = 0$ when $\operatorname{char} K > 0$.

If $\mu \neq 0$, then for each integer s the sth power of a $k \times k$ Jordan block with eigenvalue μ is given by

$$\begin{bmatrix} \mu & 0 & \cdots & 0 & 0 \\ 1 & \mu & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \mu & 0 \\ 0 & 0 & \cdots & 1 & \mu \end{bmatrix}^s = \begin{bmatrix} \gamma_o & 0 & \cdots & 0 & 0 \\ \gamma_1 & \gamma_0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \gamma_{k-2} & \ddots & \ddots & \gamma_0 & 0 \\ \gamma_{k-1} & \gamma_{k-2} & \cdots & \gamma_1 & \gamma_0 \end{bmatrix}$$

where $\gamma_j = \binom{s}{j}\mu^{s-j}$. Thus, if we define $w_d := d^{-r}\lambda_m^d x^{-d}$ for $d = 1, 2, \ldots$ then the nonzero entries of w_d have the form

$$d^{-r}\lambda_m^d \binom{-d}{j} \lambda_i^{-d-j} \qquad (2)$$

where $j \le r$ when $|\lambda_i| = |\lambda_m|$. By the choice of r there is at least one i with $|\lambda_i| = |\lambda_m|$ where there is an entry in (2) with $j = r$, and the norm of this entry tends to the nonzero limit $|\lambda_m|^{-r}/r!$ as $d \to \infty$. On the other hand, for sufficiently large d, all entries of w_d have norm $\le |\lambda_m|^{-r}$ whenever $j > r$ because $|\lambda_i| \ge |\lambda_m|$ in (2) (remember that $r = 0$ when $\operatorname{char} K > 0$.). Since K is locally compact and complete, this means that there is a subsequence $\{w_{d_k}\}$ which converges to some matrix $w \ne 0$ over K.

Consider the set

$$S := \{(u, v) \in G \times G \,|\, \text{the } (1,1)\text{th entry of } uwv \text{ is } 0\}$$

Clearly S is a closed subset of $G \times G$ in the Zariski topology. We claim that S is a proper subset. Suppose on the contrary that $S = G \times G$, and so $euwve^\top = 0$ for all $u, v \in G$ where $e := (1\,0 \ldots 0) \in K^m$ and $e^\top$ is its transpose. Since $w \ne 0$, there exists $f \in K^m$ such that $fw \ne 0$. Now since G is irreducible, there are K-linear combinations a and b of matrices in G such that $ea = f$ and $(fw)b = e$. But then $0 = eawbe^\top = (fwb)e^\top = ee^\top = 1$ gives a contradiction. Hence S is a proper Zariski closed subset of $G \times G$. Similarly, $S^{-1} := \left\{(u^{-1}, v^{-1}) \,|\, (u, v) \in S\right\}$ is also a proper Zariski closed subset of $G \times G$. Since G is connected in the Zariski topology by hypothesis, $G \times G$ is also connected and so it cannot be the union of two proper closed subsets (see [23] Lemma 14.3). Thus there exist $y, z \in G$ such that $(y^{-1}, z) \notin S \cup S^{-1}$, and so both $y^{-1}wz$ and ywz^{-1} have nonzero entries, say μ_1 and μ_2, in the $(1,1)$th position. Now define

$$f_d := d^{-r}\lambda_m^d \lambda_1^{-d} x^d y^{-1} x^{-d} z = (\lambda_1^{-d} x^d) y^{-1} w_d z$$

$$g_d := d^{-r}\lambda_m^d \lambda_1^{-d} x^d y x^{-d} z^{-1} = (\lambda_1^{-d} x^d) y w_d z^{-1}$$

Since λ_1 is the dominant eigenvalue of x

$$\lambda_1^{-d} x^d \to \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \text{ as } d \to \infty$$

Therefore, since $\{w_{d_k}\}$ converges to w, we conclude that $\{f_{d_k}\}$ and $\{g_{d_k}\}$ converge to matrices whose first rows are of the form $(\mu_1 \, * \ldots *)$ and $(\mu_2 \, * \ldots *)$, respectively, and whose other entries are all 0. Thus for infinitely many integers d, the matrices f_d and g_d will have dominant eigenvalues (approximately μ_1 and μ_2 respectively). Choose such a value of d and put $x_0 := x^d y^{-1} x^{-d} z \in G$.

Then x_0 is a scalar multiple of f_d and x_0^{-1} is a scalar multiple of $z^{-1}g_d z$, so both x_0 and x_0^{-1} have dominant eigenvalues as required. □

PROPOSITION 5 Let $G \leq GL(n, K)$ be an irreducible group of degree $n > 1$ which is connected in the Zariski topology. Suppose that G has an element x such that x and x^{-1} both have dominant eigenvalues. Then for some conjugate $y = b^{-1}xb$ of x in G there exists an integer $k \geq 1$ such that $\langle x^k, y^k \rangle$ is a free subgroup of rank 2.

Proof Let λ and μ be the dominant eigenvalues of x and x^{-1}, respectively, and let u_1 and u_2 be the corresponding eigenvectors; so $u_1 x = \lambda u_1$ and $u_2 x = \mu^{-1} u_2$. Put $U_1 := K u_1$, $U_2 := K u_2$ and let U_3 be the x-invariant subspace such that $K^n = U_1 \oplus U_2 \oplus U_3$. To simplify notation we shall write $U_1' := U_2 + U_3$, $U_2' := U_1 + U_3$ and $U_3' := U_1 + U_2$.

We shall use the "Pingpong Lemma" (Lemma 7) to prove the proposition. In our particular case we shall have $A := \langle x^k \rangle$, $B := \langle b^{-1}x^k b \rangle$ and $\Gamma = \Delta b$. The hypothesis of the lemma then simplify to:

(i) $\Delta \cap \Delta b = \emptyset$;
(ii) $(\Delta b \cup \Delta b^{-1}) x^{kr} \subseteq \Delta$ for all integers $r \neq 0$; and
(iii) $\Delta x^{kr} \cap \Delta \neq \emptyset$ for all integers $r \neq 0$.

In order to satisfy (ii) we shall choose Δ as a neighbourhood of $(U_1 \cup U_2) \backslash \{0\}$, and use the fact that U_1 and U_2 are "attracting" subspaces under positive powers of x and x^{-1}, respectively.

We begin by finding b. First note that for any pair of subspaces V_1 and V_2 or K^n the set

$$L(V_1, V_2) := \{z \in G \,|\, V_1 z \cap V_2 \neq 0\}$$

is Zariski-closed in G. Indeed, fix bases for V_1 and V_2. Then $z \in L(V_1, V_2) \Leftrightarrow$ a certain matrix has rank less than $(\dim V_1 + \dim V_2)$; and this latter condition is equivalent to the entries in z satisfying a set of (determinantal) polynomial conditions. In particular, the sets $L(U_i, U_i')$, $L(U_i', U_i)$ and $L(U_i, U_i)$ for $i = 1, 2$ are closed subsets of G and are all proper subsets of G (in the latter cases because of the irreducibility of G). Since G is connected in the Zariski topology, it is not a union of a finite number of proper closed subsets (see [23] Lemma 14.3). Thus there exists $b \in G$ which lies in none of these subsets and hence satisfies

$$(U_i b \cup U_i b^{-1}) \cap (U_i \cup U_i') = \emptyset \text{ for } i = 1, 2 \qquad (3)$$

We now define Δ. Since K is a locally compact normed field we can define a norm $\| \|$ on K^n (take $\|(\alpha_1\, \alpha_2 \ldots \alpha_n)\| := \max(|\alpha_1|, |\alpha_2|, \ldots, |\alpha_n|)$, for example) to make $K^n, \| \|$ a locally compact normed space over K. The conditions (3) show that the set $\{u_1, u_2\}$ is disjoint from $S := U_3 \cup U_1' b \cup U_1' b^{-1} \cup U_2' b \cup U_2' b^{-1}$. Every subspace of K^n is closed in the norm topology, so there exists a compact neighbourhood N_0 of $\{u_1, u_2\}$ with $N_0 \cap S = \emptyset$. Put $W_0 := \{\alpha u \,|\, \alpha \in K, u \in N_0\}$, so $W_0 \cap S = \emptyset$. We claim that W_0 is also closed in the norm topology. Indeed, suppose that $\{\alpha_i w_i\}$ is a sequence in W_0 (with $\alpha_i \in K$ and $w_i \in N_0$) converging to a point v in K^n; we must show that $v \in W_0$.

Since N_0 is compact, we can assume (by dropping to a subsequence if necessary) that $\{w_i\}$ converges to some point $w \in N_0$. Now $w \neq 0$ because $0 \notin N_0$, and by considering a component where w is nonzero we can see that $\{\alpha_i\}$ also converges to some limit $\alpha \in K$. Hence $v = \alpha w \in W_0$ as required, and so we have shown that W_0 is closed. Since $W_0 \cap \{u_1, u_2\}\, b \subseteq W_0 \cap S = \emptyset$, there exists a compact neighbourhood N of $\{u_1, u_2\}$ such that $N \subseteq N_0$ and $W_0 \cap Nb = \emptyset$. Finally put

$$\Delta := \{\alpha u \mid \alpha \in K, u \in N \text{ and } \alpha \neq 0\}$$

Since $\Delta \subseteq W_0$ we have $\Delta \cap \Delta b = \emptyset$ which is condition (i) above, and condition (iii) is satisfied because $u_1 \in \Delta$.

It remains to choose $k \geq 1$ so that condition (ii) is satisfied. First note that since U_1 is the eigenspace for the dominant eigenvalue λ of x, therefore $\lambda^{-k} u x^k \to 0$ as $k \to \infty$ for all $u \in U_1'$. Thus if $v \in K^n$ is written in the form

$$v = \alpha_1 u_1 + \alpha_2 u_2 + u_3 \text{ with } \alpha_1, \alpha_2 \in K \text{ and } u_3 \in U_3$$

then $\lambda^{-k} v x^k \to \alpha_1 u_1$ as $k \to \infty$. Similarly, $\mu^{-k} v x^{-k} \to \alpha_2 u_2$ as $k \to \infty$. In particular, α_1 and α_2 are nonzero if $v \in Nb \cup Nb^{-1}$ because the latter is disjoint from $U_1' \cup U_2'$. Now Δ is a neighbourhood of $\alpha_1 u_1$ and $\alpha_2 u_2$ when α_1 and α_2 are nonzero, and $Nb \cup Nb^{-1}$ is compact. Thus there exists an integer $k \geq 1$ such that $l \geq k$ implies that $\lambda^{-l} v x^l \in \Delta$ and $\mu^{-l} v x^{-l} \in \Delta$ for all $v \in Nb \cup Nb^{-1}$. Then $(\Delta b \cup \Delta b^{-1}) x^{kr} \subseteq \Delta$ for all integers $r \neq 0$ which shows that condition (ii) is satisfied. Hence for these choices of b, Δ and k the hypotheses of Lemma 7 are satisfied and so $\langle x^k, b^{-1} x^k b \rangle$ is a free subgroup of rank 2 as required. □

The proof of Tits' Theorem is now completed as follows. Let G be a subgroup of $GL(n, F)$ which is finitely generated and is not solvable by finite. Without loss in generality we may assume that the matrices in G are lower triangular block matrices of the form

$$x = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ * & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & x_h \end{bmatrix}$$

where the representations $x \mapsto x_i$ are irreducible over F. Since G is not solvable by finite, at least one of the irreducible constituents G_1 of G is not solvable by finite, and so neither is the connected component $G_2 := G_1^0$ by Lemma 2. Since G_1 is a homomorphic image of G, the remark at the beginning of this section shows that it is enough to prove that G_2 contains a free subgroup of rank 2. By Clifford's Theorem G_2 is a completely reducible subgroup of the irreducible group G_1. Moreover, G_2 is finitely generated since G_1 is finitely generated and $|G_1 : G_2|$ is finite by Lemma 2. Since G_2 is connected in the Zariski topology, Proposition 1 shows that there exists $x \in G_2'$ with at least one eigenvalue not a root of unity. Thus Propositions 2 and 3 show that there is a locally compact normed field K, and a representation $\rho : G_2 \to GL(m, K)$ such that $\rho(x)$ has a dominant eigenvalue, and that $G_3 := \rho(G_2)$ is Zariski connected. Without loss in generality we may suppose that G_3 is in lower block triangular form where the diagonal blocks correspond to the irreducible constituents of G_3. Let G_4 denote the irreducible constituent in which $\rho(x)$ has its dominant eigenvalue. Then G_4 is a continuous homomorphic image of G_3 under the Zariski topology, and so it is also Zariski connected, as well as being irreducible as a linear group. Moreover the degree of G_4 is > 1 since the image of $\rho(x)$ in G_4 lies in G_4' and is nontrivial. Now Proposition 4 shows that G_4 contains an element x_0 such that both x_0 and x_0^{-1} have dominant eigenvalues. Finally Proposition 5 shows that G_4 (and hence G) contains a free subgroup of rank 2.

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