Irreducible characters which are zero on only one conjugacy class

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Abstract

Suppose that G is a finite solvable group which has an irreducible character χ which vanishes on exactly one conjugacy class. Then we show that G has a homomorphic image which is a nontrivial 2-transitive permutation group. The latter groups have been classified by Huppert. We can also say more about the structure of G depending on whether χ is primitive or not.

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1 Introduction

Let χ be an irreducible character of a finite group G. A well-known theorem of Burnside [9, page 40] shows that when χ is nonlinear it takes the value 0 on at least one conjugacy class of G. Groups having an irreducible character that vanishes on exactly one class were studied by Zhmud' in [11] (see also [1]). Chillag [2, Lemma 2.4] has proved that if the restriction of χ to the derived group G' is reducible and χ vanishes on exactly one class of G, then G is a Frobenius group with a complement of order 2 and an abelian odd-order kernel.

Our purpose in this paper is to show that, if an irreducible character χ of a finite solvable group G vanishes on exactly one conjugacy class, then G has a homomorphic image which is a nontrivial 2-transitive permutation group. The latter groups have been classified by Huppert: they have degree p^d where p is prime, and are subgroups of the extended affine group $A\Gamma L(1, p^d)$ except for six exceptional degrees (see Remark 8 below). We can also say more about the structure of G depending on whether χ is primitive or not.

2 Main results

We shall initially assume that our character is faithful, and make the following assumptions:

(*) G is a finite group with a faithful irreducible character χ which is 0 on only one class which we denote by $\mathcal{C}$. Furthermore, G has a chief factor K/L which is an elementary abelian p-group of order p^d such that the restriction χ_K is irreducible, but χ_L is not.

Since χ must be nonlinear, the latter condition clearly holds whenever G is solvable, but for the present we shall not assume solvability.

Proposition 1 *Suppose (*) holds. Then $\mathcal{C} = K \setminus L$, $K = \langle \mathcal{C} \rangle$ and L is equal to $L_0 := \{u \in G \mid u\mathcal{C} = \mathcal{C}\}$. In particular, $\mathcal{C}$ consists of p-elements (since L does not contain a Sylow p-subgroup of K). Moreover, either:*

(i) d is even, $\chi_L = p^{d/2}\phi$ for some nontrivial linear character ϕ of L, and $L = Z(G)$; or

(ii) $\chi_L = \phi_1 + ... + \phi_{p^d}$ is the sum of p^d distinct G-conjugate irreducible characters of L, and so χ is imprimitive and $p^d \mid \chi(1)$.

Proof. Since χ_K is irreducible, the theorem of Burnside quoted above shows that $\mathcal{C} \subseteq K$. On the other hand, since χ_L is reducible, [1, Theorem 21.1] shows that $\mathcal{C} \cap L = \emptyset$. Hence $\mathcal{C} \subseteq K \setminus L$.

Now since K/L is an abelian chief factor, and χ_K is irreducible, it follows from [9, (6.18)] that either (i) d is even, $\chi_L = p^{d/2}\phi$ for some $\phi \in \mathrm{Irr}(L)$; or (ii) $\chi_L = \phi_1 + ... + \phi_{p^d}$ is the sum of p^d distinct G-conjugate irreducible characters ϕ_i. We shall consider these two cases separately.

In case (i) we note that, since $\mathcal{C} \cap L = \emptyset$, the irreducible character ϕ does not take the value 0. Thus Burnside's theorem implies that $\phi(1) = 1$. This implies that if ρ is a representation affording χ, then ρ is scalar on L. Since χ is assumed to be faithful, L is contained in the centre $Z(G)$ of G. On the other hand, for each $z \in Z(G)$, $\rho(z)$ is a scalar of the form $\zeta 1$. Thus for each $x \in \mathcal{C}$ we have $\chi(zx) = \operatorname{trace} \rho(zx) = \operatorname{trace} \zeta\rho(x) = \zeta\chi(x) = 0$ and so $zx \in \mathcal{C}$. Therefore $z = (zx)x^{-1} \in \langle \mathcal{C} \rangle \leq K$ for all $z \in Z(G)$. This shows that $Z(G)$ is a normal subgroup of G satisfying $L \leq Z(G) \leq K$. Since K/L is a chief factor and χ_K is a nonlinear irreducible character of K, we conclude that $Z(G) = L$. Finally, since $K/Z(G)$ is abelian, [9, (2.30)] shows that $K \setminus L = \mathcal{C}$.

In case (ii) χ_K is an irreducible constituent of $(\phi_1)^K$ and so comparison of degrees shows that $\chi_K = (\phi_1)^K$. Thus χ_K is 0 everywhere outside of the normal subgroup L, and so $K \setminus L = \mathcal{C}$ in this case as well.

Finally since $|\mathcal{C} \cup \{1\}| > \frac{1}{2}|K|$, therefore $K = \langle \mathcal{C} \rangle$. Finally, it is easily seen that L_0 is a normal subgroup of G, and that $L_0 \subseteq \mathcal{C}\mathcal{C}^{-1}$ and so $L_0 \leq K$. Since $\mathcal{C} = K \setminus L$ is a union of cosets of L, we see that $L \leq L_0$. On the other hand, $\mathcal{C} \nsubseteq L_0$ since $\mathcal{C}$ is not a subgroup. Therefore $L_0 \lhd G$ and $L \leq L_0 < K$; hence $L_0 = L$ as claimed. ■

Corollary 2 *Under the hypothesis (*) every normal subgroup N of G either contains K (when χ_N is irreducible) or is contained in L (when χ_N is reducible). In particular, K/L is the unique chief factor such that χ_K is irreducible and χ_L is reducible and K/L is the socle of G/L. Since K has a nonlinear irreducible character, K is not abelian and so $L \neq 1$.*

Remark 3 *Both cases (i) and (ii) in Proposition 1 can actually occur. The group* $\mathrm{SL}(2,3)$ *has three primitive characters of degree* 2 *which satisfy (*) (case (i) with $|K| = 8$ and $|L| = 2$ for each character), and S_4 has an imprimitive character of degree* 3 *which satisfies (*) (case (ii) with $|K| = 12$ and $|L| = 4$).*

Proposition 4 *Suppose that the hypothesis (*) and case (i) of Proposition 1 hold. Then $L = Z(G)$ has order p, K is an extraspecial p-group and χ is primitive.*

Proof. Let $z \in L$. Then for any $x \in \mathcal{C}$ we have $zx \in \mathcal{C}$ and so $zx = y^{-1}xy$ for some $y \in G$. Since K/L is an elementary abelian p-group, $z^p x^p = (zx)^p = y^{-1}x^p y = x^p$, and so $z^p = 1$. Thus L is of exponent p. Since $L = Z(G)$ is represented faithfully as a group of scalar matrices by a representation affording χ, it follows that L is cyclic and hence $|L| = p$. Because K is nonabelian, $K' = \Phi(K) = L = Z(K)$ and so K is an extraspecial p-group.

We finally show that χ is primitive. Indeed, otherwise there is a maximal subgroup H in G and $\psi \in Irr(H)$ such that $\chi = \psi^G$. The formula for an induced character shows that ψ^G is 0 on each conjugacy class disjoint from H. As is well-known every proper subgroup of a finite group is disjoint from some conjugacy class, and so we conclude that $\mathcal{C}$ is the unique class such that $\mathcal{C} \cap H = \emptyset$. By Proposition 1 this implies that $H \cap K \leq L$. Thus $K \nleq H$, and so $G = HK$ by the maximality of H. Hence

$$\chi(1) = \psi^G(1) \geq |G : H| = |K : H \cap K| \geq |K : L| = p^d.$$

Since $\chi(1) = p^{d/2}$, we obtain a contradiction. Thus χ is primitive. ■

Proposition 5 *Suppose that the hypothesis (*) and case (ii) of Proposition 1 hold (so χ is imprimitive). Then there exists a subgroup M of index p^d in G such that $\chi = \psi^G$ for some $\psi \in \mathrm{Irr}(M)$, $G = MK$ and $M \cap K = L = \mathrm{core}_G(M)$.*

Proof. As noted in the proof of Proposition 1 χ_L is a sum of p^d distinct irreducible constituents ϕ_i. Because χ_K is irreducible, these constituents are K-conjugates (as well as G-conjugates). Let $M := I_G(\phi_1)$ be the inertial subgroup fixing the constituent ϕ_1. Then $|G : M| = p^d$ and $G = MK$ because K acts transitively on the set of ϕ_i. Clearly $L \leq M$. Since $|K : L| = p^d = |G : M| = |K : M \cap K|$, we conclude that $M \cap K = L$. On the other hand, since ψ^G is 0 on any class which does not intersect M, the hypothesis on χ shows that $\mathcal{C} = G \setminus \bigcup_{y \in G} y^{-1}My$. Now $u \in \mathrm{core}_G(M) = \bigcap_{y \in G} y^{-1}My$ and $x \in \mathcal{C}$ implies that ux does not lie in any $y^{-1}My$, and hence $ux \in \mathcal{C}$. Thus with the notation of Proposition 1, $\mathrm{core}_G(M) \leq L_0 = L$. Since L is a normal subgroup contained in M, the reverse inequality is also true and so $\mathrm{core}_G(M) = L$. ■

The proof of the next result requires a theorem of Isaacs [10, Theorem 2] which states:

> Let H be a finite group with centre Z and K be a normal subgroup of H with $Z = Z(K)$. Suppose that H centralizes K/Z and $|Hom(K/Z, Z)| \leq |K/Z|$. Then $H/Z = K/Z \times C_H(K)/Z$.

Proposition 6 *Under the hypothesis (*) the centralizer $C_G(K/L)$ equals K.*

Proof. If χ is primitive, then Proposition 4 shows that the hypotheses of Isaacs' theorem are satisfied for $H := C_G(K/L)$ (the condition $|Hom(K/Z, Z)| \leq |K/Z|$ is trivial since the irreducibility of χ_H implies that Z is cyclic). Also, since χ_K is irreducible, $C_G(K) = Z(G) = L$, and so Isaacs' theorem shows that $H/L = K/L \times C_H(K)/L = K/L$ as required.

If χ is imprimitive, then using the notation of Proposition 5 we can show that $M \cap H = L$ where $H := C_G(K/L)$. Indeed, it is clear from Proposition 5 that $L \leq M \cap H$. To prove the reverse inequality suppose that $u \in M \cap H$. Then for each $x \in K$ we have $xu = yux$ for some $y \in L$. Choose i such that $\phi_1^x = \phi_i$. Then $\phi_i^u = \phi_1^{xu} = \phi_1^{yux} = \phi_1^x = \phi_i$. Hence u fixes ϕ_1^x and hence lies in $x^{-1}Mx$. Since this is true for all $x \in K$, it follows from Proposition 5 that $u \in \mathrm{core}_G(M) = L$. Thus $M \cap H = L$ as claimed. Finally $H = H \cap MK = (H \cap M)K = LK = K$ as required. ■

Corollary 7 *Under the hypothesis (*) G acts transitively by conjugation on the nontrivial elements of the vector space K/L and the kernel of this action is K. Thus G/K is isomorphic to a subgroup of $GL(d, p)$ which is transitive on the nonzero elements of the underlying vector space.*

Remark 8 *Huppert [8, Chapter XII Theorem 7.3] has classified all solvable subgroups S of $GL(d, p)$ which are transitive on the nonzero vectors of the underlying vector space. Apart from six exceptional cases (where $p^d = 3^2, 5^2, 7^2, 11^2, 23^2$*

or 3^4), the underlying vector space can be identified with the Galois field $GF(p^d)$ in such a way that S is a subgroup of the group $\Gamma L(1,p^d)$ consisting of all transformations of the form $\xi \mapsto \alpha\xi^t$ where α is a nonzero element of the field and t is an automorphism of the field. The group $\Gamma L(1,p^d)$ is metacyclic of order $(p^d-1)d$. A classification for nonsolvable groups has been carried out by Hering [5], [6]. It is considerably more complicated to state and prove, but among other things it shows that such groups have only a single nonsolvable composition factor (a summary is given in [8, page 386]).

Since the latter half of hypothesis (*) is certainly satisfied in a solvable group, we can specialize to solvable groups and drop the condition that χ is faithful to obtain the following theorem .

Theorem 9 *Let G be a finite solvable group which has an irreducible character χ which takes the value 0 on only one conjugacy class $\mathcal{C}$. Let $K := \langle \mathcal{C} \rangle$. Then:*

(a) $K = G^{(k)}$ for some $k \geq 0$.

(b) There is a unique normal subgroup L of G such that K/L is a chief factor of G and $K \setminus L = \mathcal{C}$ (we set $|K : L| = p^d$).

(c) G/K acts transitively on the set $(K/L)^{\#}$ of nontrivial elements of the vector space K/L and so is one of the groups classified by Huppert.

(d) If χ is primitive, then $K/\ker\chi$ is an extraspecial group of order p^{d+1} with centre $L/\ker\chi$.

(e) If χ is imprimitive, then G/L is a 2-transitive Frobenius group of degree p^d.

Remark 10 *We also note that (c) and Huppert's classification show that the integer k in (a) is bounded. Indeed, since $\Gamma L(1,p^d)$ is metacyclic, $k = 1$ or 2 except in the six exceptional cases. Computations using GAP [4] show that in the remaining cases $k \leq 4$.*

Proof. (a) Let k be the largest integer such that $K \leq G^{(k)}$. By Corollary 2 we know that the restriction $\chi_{G^{(k+1)}}$ is reducible, and so $G^{(k+1)} \leq L$. Therefore $G^{(k+1)} \leq L < K \leq G^{(k)}$, and so $G^{(k)} \leq C_G(K/L)$. Hence $K = G^{(k)}$ by Proposition 6.

(b), (c) and (d) follow from Proposition 1, Corollary 7 and Proposition 4.

(e) Let M be the subgroup defined in Proposition 5. Since χ is induced from a character on M, its restriction χ_M must be reducible, and so [1, page 145] shows that

$$2 \leq [\chi_M, \chi_M] \leq 1 + \frac{|\mathcal{C} \setminus M|}{|M|} = 1 + \frac{|\mathcal{C}|}{|M|}$$

Hence $|\mathcal{C}| \geq |M|$. Since $G = MK$ and G/K acts transitively on $(K/L)^{\#}$ we conclude using Proposition 5 that

$$\frac{|K|}{|L|} - 1 = p^d - 1 \leq |G : K| = |M : M \cap K| = |M : L| \leq \frac{|\mathcal{C}|}{|L|}$$

However, $|\mathcal{C}| / |L| = p^d - 1$ by Proposition 1, so equality must hold throughout. Thus $|M : L| = p^d - 1$. Hence M/L acts regularly on $(K/L)^{\#}$ and so $G/L = (M/L)(K/L)$ is a 2-transitive Frobenius group. ■

Remark 11 *Not all groups having an irreducible character which takes* 0 *on a single conjugacy class satisfy the second half of hypothesis (*). For example, the Atlas [3] shows that* A_5 *has three characters with this property and its central cover* $2 \cdot A_5$ *also has three. The group* $L_2(7)$ *has two characters with the required property and each of the groups* $L_2(2^k)$ $(k = 3, 4, ...)$ *appears to have one such character (of degree* 2^k*). It would be interesting to know if these were the only simple groups with this property, or whether a group with such a character can have more than one nonabelian composition factor (see Remark 8). Another question which can be asked is what can be said about the kernel of such a character; evidently this kernel is contained in the normal subgroup* $L_0 := \{u \in G \,|\, u\mathcal{C} = \mathcal{C}\}$.

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