(2023.12)

Errata for Dixon and Mortimer "PERMUTATION GROUPS" (Springer 1996)

Chapter 1

10:11 read "stabilizer $(K \times K)_1$."

11:-10 read "on each of its orbits of length > 1,"

12: 21 read "$\{1, 4, 6, 7\}$ and $\{2, 3, 5, 8\}$ are also minimal blocks. Show that there is only one other set of nontrivial blocks and these are also minimal."

13:13-15 read "Suppose that G is a group acting primitively on a set Ω and that Δ is a proper subset of Ω containing at least two points. Show that for each pair of distinct points ..."

13:18 add "[Hint: Show that the relation $\alpha \approx \beta \iff$ (for all $x \in G$, $\{\alpha, \beta\} \cap \Delta^x = \{\alpha, \beta\}$ or $\emptyset$) is a G-congruence.]"

17:-3 read "If $\Delta, \Delta' \in \Sigma$ are fixed setwise by H, then "

19:21 read "If $fix(G_\alpha)$ is finite, show it is a block for G."

22:2 read "... $= (\beta^{\sigma(a)})^{\sigma(x)} = (\lambda(\alpha^{\rho(a)}))^{\sigma(x)} = \lambda(\gamma)^{\sigma(x)}$ "

22:-13 read "Thus Lemma 1.6B shows ..."

23:9 read "and let $\alpha \in \Omega$. "

23:-2 *replace this incorrrect exercise by*

1.6.19 If x, y are distinct elements of order 2 in a finite group G, show that $\langle xy \rangle \lhd \langle x, y \rangle$ and hence that $\langle x, y \rangle$ is a dihedral group. Hence show that every primitive subgroup of order $2n$ in S_n is dihedral.

27:-10 read "acting transitively on a set Ω"

Chapter 2

30:14 read "–at least in principle–"

34:10 read "Suppose that G is a permutation group of degree at least 5. If G is k-transitive for some $k \geq 3$, show that every nontrivial normal subgroup N of G is $(k-2)$-transitive. In particular, ..."

35:12 read "(see Exercise 2.1.7)"

35:20 read "Hence show that S_n acts ..."

39:-3 replace Exercise 2.3.7 by: "Let $n \geq 3$. Consider the graph ... when they commute. Show that: (i) if $n = 3$ or $n \geq 5$ then $Aut(\mathcal{G}) \cong S_n$, and (ii) if $n = 4$ then $Aut(\mathcal{G})$ is imprimitive and isomorphic to $C_2 \times S_4$."

43:14 amend Exercise 2.4.5 by adding the hypothesis "G is of finite exponent" to parts (ii) and (iv). [The following example shows that the statements of (ii) and (iv) as they stand are incorrect. Let $\{\Gamma_i\}_i$ be a partition of Ω and for each i let x_i be a cycle with support Γ_i. Then $G := \langle x_1, x_2, ... \rangle$ is an abelian group. Define $z \in Sym(\Omega)$ such that the restriction $z_{\Gamma_i} = x_i$ for each i. We claim that $z \in G_0$. Indeed the 2-relations on Ω are just the subsets of $\Omega \times \Omega$, and the G-invariant 2-relations are the unions of G-orbits on $\Omega \times \Omega$. Thus $x \in G_0 \iff (\alpha, \beta)^x$ and $(\alpha, \beta)^{x^{-1}}$ lie in $(\alpha, \beta)^G$ for all $\alpha, \beta \in \Omega$. In our case $(\alpha, \beta)^z = (\alpha^z, \beta^z) = (\alpha^{x_i}, \beta^{x_j})$ when $\alpha \in \Gamma_i$ and $\beta \in \Gamma_j$. If $i \neq j$ then $(\alpha^{x_i}, \beta^{x_j}) = (\alpha, \beta)^{x_i x_j} \in (\alpha, \beta)^G$ because G is abelian and $\alpha^{x_j} = \alpha$ and $\beta^{x_i} = \beta$

by the construction of G. On the other hand, if $i = j$ then $(\alpha^{x_i}, \beta^{x_j}) = (\alpha, \beta)^{x_i} \in (\alpha, \beta)^G$. A similar argument shows that $(\alpha, \beta)^{z^{-1}} \in (\alpha, \beta)^G$ and so $z \in G_0$ as claimed. Taking Ω infinite and $|\Gamma_i| = 3^i$ $(i = 1, 2, ...)$ we obtain a group G which contains an element z of infinite order and satisfies the hypotheses of both (ii) and (iv).]

46:-9 read "when $x^{-1} =$"

48:-5 read "S_{p^m} $(\cong Sym(\Delta^m))$ "

51:12-14 read "a constant function in $Fun(\Gamma, \Delta)$ whose value lies in Π cannot be mapped under W to a constant function whose value lies in $\Delta \setminus \Pi$; thus W is intransitive. In the case ..."

51:-9 read "Define $g \in Fun(\Gamma, K)$"

51:-8 read "$[f(\gamma_0), u) \in K \setminus K_\delta$"

51:-4 read "$(1, x)B(\gamma_0)(1, x)^{-1} = B(\gamma_0^x)$

51:-3 read "$B(\gamma) \leq M$ for all $\gamma \in \Gamma$"

52:4 read "Show that a primitive group G of degree > 1 is not regular if and only if ..."

53:15 read "and $G_{\infty 0}$ transitive on the nonzero elements of F."

57:8 read "Put $G := PGL_d(F)$ and define $\Delta := ...$ "

57:-18 delete one copy of "points"

58:11 read "Artin (1957)"

60:7 read "(1234), (13)"

60:14 read "(12)(34)(56), (153)(246)"

60:16 read "(123)(456), (12)(45), (14)

63:-16 read "$PGL_2(5) \cong S_5$"

Chapter 3

66:–9 read "the *diagonal* orbit $\Delta_1 := \{(\alpha, \alpha) \mid \alpha \in \Omega\}$; the other orbitals are called *nondiagonal*."

68:9 read "$H := \langle t \rangle$"

70:-18 read "Theorem 1.5A"

70:-16 delete "that G is finite,"

71:13 read "Then $|(\Sigma \circ \Lambda)(\alpha)| \leq ...$"

71:-10 read "in some $\Phi(s)$ because..."

75:-19 read "A_3 is a composition factor"

77:25 read "Theorem 3.3D"

78:-7 and -5 read "then $y = (\beta\delta\varepsilon) \in N$"

84:-1 and 85:1,2 replace "2-cycle" by "3-cycle" and "$p \neq 2$" by "$p \neq 3$"

93:-8 read "$(a_1 + ... + a_k)^p = a_1^p + ... + a_k^p$ "

96:21 add "for $p > 2$". For elementary abelian 2-groups the situation is more complicated. A wreath product construction enables us to construct a primitive group isomorphic to S_{2^k} wr C_l which contains a regular elementary abelian subgroup of order 2^{kl}, and this group is not 2-transitive when $k, l > 1$. So a regular elementary abelian subgroup of order 2^n is not a B-group when n is composite. A theorem of Cai-Heng Li ("The finite primitive permutation groups containing an abelian regular subgroup", Proc. London Math. Soc. (3) 87 (2003) 725–747) shows that in the remaining cases (n is prime) a primitive group which contains a regular elementary abelian subgroup of order 2^n must be a subgroup of $AGL(n, 2)$ (the 2-transitive subgroups of $AGL(n, 2)$ are discussed in Section 7.7).

102:20 read "$w \in W$"

Chapter 4

109:-5 read "Show that $C \cong C_0 \, wr_\Sigma \, Sym(\Sigma)$ where ..."

110:4 read "each point stabilizer of H is its own normalizer in H, "

113:-2 read "p-group of order p^n"

114:20 read "$K \times C_G(K)$"

119:-10 read "with a finite nontrivial suborbit whose paired suborbit is also finite, show that "

{David Evans, Suborbits in infinite primitive permutation groups, *Bull. London Math. Soc.* **33** (2001) 583–590 gives a construction of an infinite primitive permutation group of arbitrary infinite cardinality with a finite nontrivial suborbit whose paired suborbit is infinite.}

124:-4 read "H is a transitive normal subgroup"

132:-2 read "for all p and m except $(p, m) = (2^b - 1, 2)$ for some integer b or $(p, m) = (2, 6)$; see for example ..."

Chapter 5

163:4 read "5.5.2 Using the fact that $\lambda(s + 1) \geq (2s - 4)/3$ ..."

170:1-4 read "5.7.3 Show that A_6 is isomorphic to $SL_2(9)$ modulo its centre. Hence $\lambda(6) = 2$."

170:5-6 read "5.7.4 Show that there is no field F for which $SL_2(F)$ contains a finite preimage G of A_7. (However, A_7 is isomorphic to a section of $SL_3(25)$, and so $\lambda(7) = 3$.)"

170:13 read "For all $k \geq 5$, $\lambda(k) \geq (2k - 6)/3$."

172:8 read "... Since $k \geq 8$, we have $d \geq 3$"

172:21-22 read "... shows that $d - 2 \geq \{2(k - 3) - 6\}/3$ and hence $d \geq (2k - 6)/3$ as required. ..."

172:after the last line add the following paragraph:

"Note that if $d = 3$ then the Jordan form for x cannot consist of a single block. Indeed, the centralizer of such a block is a group of upper triangular matrices and hence solvable, but we know that $C_G(x)$ is not solvable."

173:2 delete "(since $d \geq 4$)"

Chapter 6

181:3-5 The off-diagonal entries of AA^T should be λ_2. Then read: "The determinant of the $v \times v$ matrix AA^T is $(r + (v - 1)\lambda_2)(r - \lambda_2)^{v-1}$ (see Exercise 6.2.2 below). This determinant is nonzero since $r > \lambda_2$ by the formulae above and our general assumption that $k < v$."

183:-6 read "and that $\mu_{ij} = \mu_{i-1,j} - \mu_{i,j+1}$"

188:18 Yervand Yeghiazarian points out that there is a fourth triple of three quadrangles which covers the 6 triangles with base $00, 01$, namely $\{\Xi_3, \Xi_5, \Xi_6\}$, and there are no others. However, unlike the other three possibilities listed in on page 188, there is no i such that this fourth triple lies in $\mathcal{S}_i$. This leaves a potential gap in the proof.

Write this triple as: (*) 00 01: 10 21; 11 20; 12 22. We shall show that (*) cannot be extended to a set $\mathcal{S}$ satisfying (6.1) and so we can eliminate (*) as a possible value of $\mathcal{S} \cap \mathcal{Q}(00, 01)$. Then the rest of the proof on page 188 follows unchanged.

Consider the list of triples of quadrangles which give a covering of the 6 triangles with base $00, 10$ (obtained by switching coordinates of each point in the corresponding list for $00, 01$):

(1) 00 10: 01 11; 21 22; 02 12
(2) 00 10: 01 21; 11 02; 12 22
(3) 00 10: 01 12; 21 11; 02 22
(4) 00 10: 01 12; 11 02; 21 22

Each of the triples (1)–(4) contains a quadrangle which intersects one of the quadrangles of (*) in a triangle: for example, for the triples (1), (3) and (4) take the triangle 00 01 10, and for the triple (2) take the triangle 00 12 22. This is contrary to (6.1) so none of these possibilities for $\mathcal{S} \cap \mathcal{Q}(00, 10)$ is consistent with $\mathcal{S} \cap \mathcal{Q}(00, 01) = (*)$. Hence the only possibilities for $\mathcal{S} \cap \mathcal{Q}(00, 01)$ are those three listed on p. 188.

207:2 read "Lemma 6.8B"

Chapter 7

210:-2 read "the stabilizers $G_{\alpha_1\alpha_2\ldots\alpha_k}$ of k points"
217:10 read "Theorem 7.2C shows"
217:13 read "finite Frobenius"
237:13 read "and $4 \nmid n$ if $q \equiv 3 \pmod 4$"
239: Table 7.1 for each of the seven groups the generator a should read $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
244:-18 read "If $n > 8$, show"
245:-18 read "that $|Sp_{2m}(F)| = q^{m^2} \prod_{i=1}^{m} (q^{2i} - 1)$ [see Taylor (1992)]."
246:17 read "$t_a = t_a^{-1}$, and that"
248:-14 read "$G = Sp_4(2) \cong S_6$ and $H = G$."
251:19 read "σ^2 is the Frobenius automorphism $\xi \mapsto \xi^3$"
251:24 read "$\lambda_3 = \eta_1 \eta_3^{\sigma} - \eta_1^{\sigma+1} \eta_2^{\sigma} + \eta_1^{\sigma+3} \eta_2 +$ "
251:-3 read "$(\eta_1, \eta_2, \eta_3, \lambda_1, \lambda_2, \lambda_3) \leftrightarrow (\lambda_2/\lambda_3, \lambda_1/\lambda_3, \eta_3/\lambda_3, \eta_2/\lambda_3, \eta_1/\lambda_3, 1/\lambda_3)$

{The permutation representation of $R(q)$ on p. 251 can be deduced, for example, from [KLM] G. Kemper, F. Luebeck and K. Magaard, "Matrix generators for the Ree groups ${}^2G_2(q)$", *Comm. Algebra* **29** (2001) 407-413 where the authors give explicit 7×7 matrices over $GF(q)$ generating $R(q)$. The 2-transitive permutation action of degree $q^3 + 1$ comes from right multiplication by $R(q)$ on the set of right cosets of the subgroup H consisting of all lower traingular matrices. If we define Q as the Sylow 3-subgroup consisting of the matrices $x_S(t, u, v)$ in [KLM], and use w to denote the involution denoted by n in [KLM], then $Q \cup \{w\}$ is a set of coset representatives of H. Using the parametrization $(\eta_1, \eta_2, \eta_3) = (t^\theta, -u^\theta, v^\theta - u^\theta t^\theta)$ for the coset with representative $x_S(t, u, v)$, and ∞ for Hw, we obtain the permutation representation on page 251 with $f_1 = \lambda_1$, $f_2 = \lambda_2$ and $f_3 = \lambda_3$.). Note that θ in [KLM] is the reciprocal of our σ.}

Chapter 8

256:15 read "has order $|\Omega|$ for $\mathbf{c} = \aleph_0$ and order at most $|\Omega|^{\mathbf{c}}$ for $\aleph_0 < \mathbf{c} \leq |\Omega|$."

262:12 read "Theorem 3.3C shows "

263: *replace the second paragraph by:*

Let $G \leq FSym(\Omega)$ be residually finite. We have to show that every orbit of G is finite. Suppose the contrary and let Σ be the union of the infinite G-orbits. Put $K := G_{(\Omega\setminus\Sigma)}$.

First note that if $H \leq G$ has finite index in G, then Σ is a union of infinite H-orbits. Indeed, if $\gamma \in \Sigma$, then $\left|\gamma^H\right| = |H : H_\gamma| \geq |G : G_\gamma| \,/\, |G : H|$.

We next show that K must be transitive on each infinite G-orbit Γ. Fix $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$ and choose $x \in G$ such that $\alpha^x = \beta$; we must show that $\alpha^z = \beta$ for some $z \in K$. Put $\Delta := \mathrm{supp}(x) \cap \Sigma$ and $\Phi := \mathrm{supp}(x) \setminus \Delta$. Since each point in the finite set Φ lies in a finite G-orbit, $G_{(\Phi)}$ has finite index in G, and so all the $G_{(\Phi)}$-orbits in Σ are infinite. Thus Theorem 3.3C shows that there exists $y \in G_{(\Phi)}$ such that the finite subset $\Delta \subseteq \Sigma$ satisfies $\Delta^y \cap \Delta = \emptyset$. Since the supports of x and y on the invariant subset $\Omega \setminus \Sigma$ are disjoint, $z := xyx^{-1}y^{-1}$ leaves all points in $\Omega \setminus \Sigma$ fixed, and so z lies in K. On the other hand, $\beta^y \in \Delta^y \subseteq \Sigma \setminus \Delta$ and so $\beta^y \notin \mathrm{supp}(x)$. Therefore $\alpha^z = (\beta^y)^{x^{-1}y^{-1}} = \beta$ as required. This proves the transitivity of K on each infinite G-orbit.

Finally, note that for each subgroup H of finite index in K, Lemma 8.3C(i) shows that $(K^\Sigma)' \leq H^\Sigma$ and so $K' \leq H$. Since K is a subgroup of a residually finite group G, K is also residually finite, and so the intersection of all subgroups of finite index in K must be 1. Thus $K' = 1$ and so K is abelian. However, if Γ is an infinite K-orbit, then Lemma 8.3C(ii) applied to K^Γshows that $Z(K^\Gamma) = 1$. Thus $K^\Gamma = 1$ contradicting the transitivity of K on Γ. This completes the proof.

Remark 1 *This proof is based on P.M. Neumann, "The structure of finitary permutation groups", Archiv Math. 27 (1976) 3-17.*

Appendix B *(These corrections are due to Heiko Theissen and Colva Roney-Dougal)*

In Table B.2 the ranks of the normalizers of the following groups should be corrected:

A_9 (degree 840): rank 9; $L_2(5^2)$ (degree 325): rank 10; $L_3(2^2).3$ (degree 960): rank 10; $L_3(2^2).2$ (degree 336): rank 6; $U_3(2^2)$ (degree 208): rank 4 and (degree 416): rank 5; $S_4(2^2).4$ (degree 425): rank 5; $Sz(2^3)$ (degree 560): rank 7; M_{12} (degree 495): both of rank 8.

Also the normalizer for $H = L_2(p^2)$ (degree $p^2 + 1$ with p prime) should be $H.2^2$ and for $H = S_4(2^3)$ (degree 585) should be $H.3$.

In Table C.2 the normalizer for $H = L_3(3)$ (degree 13) should be H and for $L_3(4)$ (degree 21) should be $H.S_3$. Also under $L_3(q)$ (degree q^2+q+1) the lower bound should be 11 (see below).

In Table B.4 the following counts should be corrected:

Degree 91: there is only one cohort of type C ($L_3(9)$ is incorrectly listed twice)

Degree 244: there is only one cohort of type B

Degree 585: there is only one cohort of type E

Degree 364: there is a cohort of type C

Degree 384: there is no cohort of type C.

In Tables B.2 and B.4 for degree 574 there is a cohort missing for the group $L_2(41)$. It has stabilizer A_5, rank 16 and is its own normalizer in S_{574} (Colva Roney-Dougal 2004.06)

Both GAP and MAGMA include extended lists of primitive groups up to degree 2499.

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