MATH 3705 Final Examination Solutions
April 2004

1. $\mathcal{L}\{e^{2t}\cos(3t)\}$ =(a)

   (a) $\dfrac{s-2}{(s-2)^2+9}$

   (b) $\dfrac{s}{(s-2)^2+9}$

   (c) $\dfrac{s-2}{s^2+9}$

   (d) $\dfrac{e^{-2s}}{s^2+9}$

   (e) None of the above.

2. $\mathcal{L}\{t\sin(2t)\}$ =(d)

   (a) $\dfrac{s}{(s^2+4)^2}$

   (b) $\dfrac{-s}{(s^2+4)^2}$

   (c) $\dfrac{2}{(s-1)^2+4}$

   (d) $\dfrac{4s}{(s^2+4)^2}$

   (e) None of the above.

3. $\mathcal{L}^{-1}\left\{\dfrac{3e^{-2s}}{s^2+s-2}\right\}$ =(b)

   (a) $u(t-2)\left[e^{t}-e^{-2t}\right]$

   (b) $u(t-2)\left[e^{t-2}-e^{-2t+4}\right]$

   (c) $e^{t-2}-e^{-2(t-2)}$

   (d) $u(t)e^{t}-u(t-2)e^{t-2}$

   (e) None of the above.

4. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+10}\right\}$ =(d)

(a) $e^{-t}\cos(3t)$

(b) $e^{-t}\sin(3t)$

(c) $e^{-t}\cos(3t) - e^{-t}\sin(3t)$

(d) $e^{-t}[\cos(3t) - \frac{1}{3}\sin(3t)]$

(e) None of the above.

5. The general solution of the differential equation $2x^2y'' - 5xy' + 3y = 0$, valid for $x \neq 0$, is given by (a)

(a) $c_1|x|^3 + c_2|x|^{\frac{1}{2}}$

(b) $|x|^3[c_1 + c_2\ln|x|]$

(c) $|x|^3\left[c_1\cos\left(\frac{1}{2}\ln|x|\right) + c_2\sin\left(\frac{1}{2}\ln|x|\right)\right]$

(d) $c_1|x|^{\frac{1}{2}} + c_2|x|^3\ln|x|$

(e) None of the above.

6. The general solution of the differential equation $x^2y'' + 2xy' + \frac{1}{4}y = 0$, valid for $x \neq 0$, is given by (d)

(a) $|x|^{-1}\left[c_1\cos\left(\frac{\sqrt{3}}{2}\ln|x|\right) + c_2\sin\left(\frac{\sqrt{3}}{2}\ln|x|\right)\right]$

(b) $c_1|x|^{-1} + c_2|x|^{\frac{\sqrt{3}}{2}}$

(c) $c_1|x|^{-\frac{1}{2}} + c_2|x|^{-\frac{1}{2}}$

(d) $|x|^{-\frac{1}{2}}(c_1 + c_2\ln|x|)$

(e) None of the above.

7. The general solution of the differential equation $x^2y'' + xy' + (7x^2 - 4)y = 0$, valid for $x > 0$, is given by (c)

   (a) $c_1 J_2(\sqrt{7}\,x) + c_2 J_{-2}(\sqrt{7}\,x)$

   (b) $c_1 J_{\sqrt{7}}(2x) + c_2 J_{-\sqrt{7}}(2x)$

   (c) $c_1 J_2(\sqrt{7}\,x) + c_2 Y_2(\sqrt{7}\,x)$

   (d) $c_1 J_{\sqrt{7}}(2x) + c_2 Y_{\sqrt{7}}(2x)$

   (e) None of the above.

8. Let $f(x) = \left\{ \begin{array}{ll} 1, & 0 \le x < 1 \\ 2, & 1 \le x \le 2 \end{array} \right\}$ on $[0, 2]$. At $x = 79$, the Fourier sine series of f converges to (d)

   (a) 2

   (b) $\frac{3}{2}$

   (c) 0

   (d) $-\frac{3}{2}$

   (e) None of the above.

9. The differential equation $y'' - 2y' + xy + \lambda y = 0$, when placed in the Sturm-Liouville form $(py')' - qy + \lambda ry = 0$, has the weight function $r(x) =$ (c)

   (a) $-xe^{-2x}$

   (b) xe^{-2x}

   (c) e^{-2x}

   (d) x

   (e) None of the above.

10. Given the Bessel identity $\frac{1}{\alpha}\frac{d}{dx}\left[x^\nu J_\nu(\alpha x)\right] = x^\nu J_{\nu-1}(\alpha x),\ \nu > 0,\ \alpha \neq 0,\ \int_0^3 x^4 J_1(2x)\,dx =$ (a)

(a) $\frac{1}{2}[81J_2(6) - 27J_3(6)]$

(b) $81J_2(6) - 27J_3(6)$

(c) $27J_3(6) - 81J_2(6)$

(d) $\frac{243}{5}J_2(6)$

(e) None of the above.

11. $\mathcal{F}\{e^{-2ix-|x+3|}\} =$(c)

(a) $\frac{2e^{-3i\lambda}}{1+(\lambda-2)^2}$

(b) $\frac{2e^{-3i(\lambda-2)}}{1+\lambda^2}$

(c) $\frac{2e^{-3i(\lambda-2)}}{1+(\lambda-2)^2}$

(d) $\frac{2e^{-3i\lambda}}{1+(\lambda+2)^2}$

(e) None of the above.

12. $\mathcal{F}\{xe^{-3x^2}\} =$(b)

(a) $\frac{\sqrt{\pi}}{3}\lambda e^{-\frac{\lambda^2}{12}}$

(b) $\frac{i\sqrt{\pi}}{6\sqrt{3}}\lambda e^{-\frac{\lambda^2}{12}}$

(c) $\frac{\sqrt{\pi}}{3}e^{-3\lambda^2}$

(d) $\frac{1}{2\pi}\lambda e^{-3\lambda^2}$

(e) None of the above.

13. $\mathcal{F}^{-1}\left\{\frac{e^{2i\lambda}}{1+(\lambda+3)^2}\right\}$ =(d)

(a) $\frac{1}{2}e^{3ix-|x-2|}$

(b) $\frac{1}{2}e^{3i(x-2)-|x|}$

(c) $\frac{1}{2}e^{3i(x+2)-|x+2|}$

(d) $\frac{1}{2}e^{3i(x-2)-|x-2|}$

(e) None of the above.

14. $\mathcal{F}^{-1}\{\lambda e^{-|\lambda|}\}$ =(b)

(a) $\frac{2ix}{\pi(1+x^2)^2}$

(b) $\frac{-2ix}{\pi(1+x^2)^2}$

(c) $\frac{x}{\pi(1+x^2)}$

(d) $\frac{-x}{\pi(1+x^2)}$

(e) None of the above.

15. Employ the Laplace transform to solve the initial-value problem
$y'' - 4y' + 13y = \delta(t-3),\ y(0) = 1,\ y'(0) = 5.$
Solution:
$s^2Y(s) - sy(0) - y'(0) - 4[sY(s) - y(0)] + 13Y(s) = e^{-3s}$
$\Rightarrow\ (s^2 - 4s + 13)Y(s) - s - 1 = e^{-3s}$
$\Rightarrow\ Y(s) = \frac{s+1}{s^2-4s+13} + \frac{e^{-3s}}{s^2-4s+13} = \frac{(s-2)+3}{(s-2)^2+9} + \frac{e^{-3s}}{(s-2)^2+9}$
$\Rightarrow\ y(t) = e^{2t}\cos(3t) + e^{2t}\sin(3t) + \frac{1}{3}u(t-3)e^{2(t-3)}\sin[3(t-3)].$

16. Find one (non-zero) series solution y_1 of the differential equation $xy'' + 2xy' + 2y = 0$, valid for $x > 0$ near $x_0 = 0$. Express the solution as an elementary function.
Solution:

$$xp(x) = 2x,\ p_0 = 0,\ x^2q(x) = 2x,\ q_0 = 0 \Rightarrow r^2+(p_0-1)r+q_0 = r^2-r = r(r-1) = 0 \Rightarrow r_1 = 1$$

$$\Rightarrow y = \sum_{n=0}^{\infty} a_n x^{n+1},\ y' = \sum_{n=0}^{\infty}(n+1)a_n x^n,\ y'' = \sum_{n=0}^{\infty} n(n+1)a_n x^{n-1}$$

$$\Rightarrow \sum_{n=0}^{\infty} n(n+1)a_n x^n + \sum_{n=0}^{\infty} 2(n+1)a_n x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

$$\Rightarrow (n+1)(n+2)a_{n+1} + [2(n+1)+2]a_n = 0,\ n \ge 0,\ \Rightarrow a_{n+1} = -\frac{2a_n}{n+1},\ n \ge 0.$$

$$n = 0 \Rightarrow a_1 = -\frac{2a_0}{1},\ n = 1 \Rightarrow a_2 = -\frac{2a_1}{2} = \frac{2^2a_0}{1\cdot 2},$$

$$n = 2 \Rightarrow a_3 = -\frac{2a_2}{3} = -\frac{2^3a_0}{1\cdot 2\cdot 3},\ \text{etc., so}$$

$$a_n = \frac{(-1)^n 2^n a_0}{n!},\ n \ge 0,\ \Rightarrow y = \sum_{n=0}^{\infty}\frac{(-1)^n 2^n a_0}{n!}x^{n+1} = a_0 x\sum_{n=0}^{\infty}\frac{(-2x)^n}{n!} = a_0 x e^{-2x}.$$

17. Find the Fourier cosine series of $f(x) = \left\{\begin{array}{ll} x, & 0 \le x < 1 \\ 0, & 1 \le x \le 2 \end{array}\right\}$ on $[0, 2]$.

Solution:

$$\text{The series is } \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right),\ \text{with}$$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^1 x\,dx = \frac{1}{2},\ \text{and for } n \ge 1,$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \int_0^1 x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{2}{n\pi}x\sin\left(\frac{n\pi x}{2}\right)\Bigg|_0^1 - \frac{2}{n\pi}\int_0^1 \sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\Bigg|_0^1$$

$$= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right]$$

Thus, the cosine series of f is

$$\frac{1}{4} + \sum_{n=1}^{\infty}\left\{\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right]\right\}\cos\left(\frac{n\pi x}{2}\right).$$

18. The solution of the heat equation $u_{xx} = \frac{1}{\alpha^2}u_t$, $0 < x < L$, which satisfies the boundary conditions $u_x(0,t) = u_x(L,t) = 0$, has the form

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{L^2}t}.$$

Find the solution of $u_{xx} = \frac{1}{9}u_t$, $0 < x < \pi$, which satisfies the boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$ and the initial condition $u(x,0) = 3\cos(2x) - 2\cos(5x)$. Write down the complete solution $u(x,t)$.

Solution:

$$\alpha = 3,\ L = \pi \ \Rightarrow\ u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) e^{-9n^2 t}.$$

$$3\cos(2x) - 2\cos(5x) = u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \ \Rightarrow\ a_2 = 3,\ a_5 = -2,\ a_n = 0 \text{ otherwise.}$$

Thus, $u(x,t) = 3\cos(2x)e^{-36t} - 2\cos(5x)e^{-225t}$.

19. The solution of Laplace's equation $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ inside the circle $r = a$ has the form

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n [a_n \cos(n\theta) + b_n \sin(n\theta)].$$

Find the solution of Laplace's equation inside the circle $r = 2$, which satisfies the boundary condition $u(2,\theta) = 2 + \sin(2\theta) - \cos(\theta)$. Write down the complete solution $u(r,\theta)$.

Solution:

$$2 + \sin(2\theta) - \cos(\theta) = u(2,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} 2^n [a_n \cos(n\theta) + b_n \sin(n\theta)]$$

$$\Rightarrow\ \frac{a_0}{2} = 2,\ 2a_1 = -1,\ 2^2 b_2 = 1,\ a_n = b_n = 0 \text{ otherwise. Thus,}$$

$$a_0 = 4,\ a_1 = -\frac{1}{2},\ b_2 = \frac{1}{4}\ \Rightarrow\ u(r,\theta) = 2 - \frac{1}{2}r\cos(\theta) + \frac{1}{4}r^2\sin(2\theta).$$

20. Find all eigenvalues and corresponding eigenfunctions of the Sturm-Liouville problem $y'' + \lambda y = 0,\ y(0) = 0,\ y'(1) = 0.$

Solution:

$$y = e^{rx} \ \Rightarrow\ r^2 + \lambda = 0 \ \Rightarrow\ r = \pm\sqrt{-\lambda}.$$

$$\lambda = -\mu^2 < 0 \ \Rightarrow\ r = \pm\mu \ \Rightarrow\ y = Ae^{\mu x} + Be^{-\mu x},\ y(0) = 0 \ \Rightarrow\ A + B = 0$$

$$\Rightarrow\ y = A\left(e^{\mu x} - e^{-\mu x}\right),\ y' = A\mu\left(e^{\mu x} + e^{-\mu x}\right),\ y'(1) = 0 \ \Rightarrow\ A\mu\left(e^{\mu} + e^{-\mu}\right) = 0$$

$$\Rightarrow\ A = 0 \ \Rightarrow\ B = 0 \ \Rightarrow\ y \equiv 0.$$

$$\lambda = 0 \ \Rightarrow\ y = Ax+B,\ y(0) = 0 \ \Rightarrow\ B = 0 \ \Rightarrow\ y = Ax,\ y' = A,\ y'(1) = 0 \ \Rightarrow\ A = 0 \ \Rightarrow\ y \equiv 0.$$

$$\lambda = \mu^2 > 0 \ \Rightarrow\ y = A\cos(\mu x) + B\sin(\mu x),\ y(0) = 0 \ \Rightarrow\ A = 0 \ \Rightarrow\ y = B\sin(\mu x),$$

$$y' = B\mu\cos(\mu x),\ y'(1) = 0 \ \Rightarrow\ B\mu\cos(\mu) = 0 \ \Rightarrow\ \mu_n = (2n+1)\frac{\pi}{2},\ n \geq 0$$

$$\Rightarrow\ \lambda_n = \frac{(2n+1)^2\pi^2}{4},\ y_n = B_n \sin\left[\frac{(2n+1)\pi x}{2}\right],\ n \geq 0.$$