MATH 3705 Final Examination Solutions
April 2003

1. $\mathcal{L}\{t^3 e^{2t}\} =$ (d)

   (a) $\dfrac{6}{(s+2)^3}$

   (b) $\dfrac{6e^{-2s}}{s^3}$

   (c) $\dfrac{6}{(s-2)^3}$

   (d) $\dfrac{6}{(s-2)^4}$

   (e) None of the above.

2. $\mathcal{L}\{e^{-3t}\cos(4t)\} =$ (b)

   (a) $\dfrac{s}{(s+3)^2+16}$

   (b) $\dfrac{s+3}{(s+3)^2+16}$

   (c) $\dfrac{e^{-3s}}{s^2+16}$

   (d) $\dfrac{se^{-3s}}{s^2+16}$

   (e) None of the above.

3. $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s^2-2s+5}\right\} =$ (a)

   (a) $\frac{1}{2}u(t-3)e^{t-3}\sin(2t-6)$

   (b) $\frac{1}{2}u(t-3)e^{t}\sin(2t-6)$

   (c) $\frac{1}{2}u(t-3)e^{t}\sin(2t)$

   (d) $\frac{1}{2}u(t-3)e^{-3t}\sin(2t)$

   (e) None of the above.

4. $\mathcal{L}^{-1}\left\{\dfrac{3s}{(s^2+9)^2}\right\} =$ (c)

(a) $t\sin(3t)$

(b) $-t\sin(3t)$

(c) $\dfrac{1}{2}t\sin(3t)$

(d) $-\dfrac{1}{2}t\sin(3t)$

(e) None of the above.

5. The general solution of $4x^2y'' - 8xy' + 9y = 0$, valid for $x \neq 0$, is given by (d)

(a) $c_1|x|^{3/2} + c_2|x|^{3/2}$

(b) $|x|\left[c_1\cos\left(\dfrac{\sqrt{5}}{2}\ln|x|\right) + c_2\sin\left(\dfrac{\sqrt{5}}{2}\ln|x|\right)\right]$

(c) $c_1|x| + c_2|x|^{\sqrt{5}/2}$

(d) $c_1|x|^{3/2} + c_2|x|^{3/2}\ln|x|$

(e) None of the above.

6. The general solution of $x^2y'' + xy' + (5x^2 - 9)y = 0$ near $x_0 = 0$, valid for $x > 0$, is given by (b)

(a) $c_1J_3(\sqrt{5}x) + c_2J_{-3}(\sqrt{5}x)$

(b) $c_1J_3(\sqrt{5}x) + c_2Y_3(\sqrt{5}x)$

(c) $c_1J_{\sqrt{5}}(3x) + c_2J_{-\sqrt{5}}(3x)$

(d) $c_1J_{\sqrt{5}}(3x) + c_2Y_{\sqrt{5}}(3x)$

(e) None of the above.

7. At $x = 999$, the Fourier sine series of $f(x) = x$ on $[0, 1]$ converges to (c)

(a) 1

(b) -1

(c) 0

(d) $\frac{1}{2}$

(e) None of the above.

8. The differential equation $4x^2y'' - 8xy' + 9\lambda y = 0$, when placed in the Sturm-Liouville form $(py')' - qy + \lambda ry = 0$, has the weight function $r(x)$ given by (d)

(a) $\dfrac{9x^2}{4}$

(b) $-\dfrac{9x^2}{4}$

(c) $\dfrac{9x}{4}$

(d) $\dfrac{9}{4x^4}$

(e) None of the above.

9. $\mathcal{F}\{e^{3ix-|x-2|}\} =$ (a)

(a) $\dfrac{2e^{2i(\lambda+3)}}{1+(\lambda+3)^2}$

(b) $\dfrac{2e^{2i(\lambda-3)}}{1+(\lambda-3)^2}$

(c) $\dfrac{2e^{-2i(\lambda+3)}}{1+(\lambda+3)^2}$

(d) $\dfrac{2e^{-2i(\lambda-3)}}{1+(\lambda-3)^2}$

(e) None of the above.

10. $\mathcal{F}\{(x-3)e^{-(x-3)^2}\} =$ (d)

(a) $\dfrac{i\sqrt{\pi}}{2}(\lambda-3)e^{-\frac{(\lambda-3)^2}{4}}$

(b) $\dfrac{i\sqrt{\pi}}{2}(\lambda+3)e^{-\frac{(\lambda+3)^2}{4}}$

(c) $\dfrac{i\sqrt{\pi}}{2}\lambda e^{-3i\lambda-\frac{\lambda^2}{4}}$

(d) $\dfrac{i\sqrt{\pi}}{2}\lambda e^{3i\lambda-\frac{\lambda^2}{4}}$

(e) None of the above.

11. $\mathcal{F}^{-1}\{\lambda e^{-\lambda^2}\} =$ (d)

(a) xe^{-x^2}

(b) $\dfrac{1}{2\sqrt{\pi}}xe^{-\frac{x^2}{4}}$

(c) $\dfrac{i}{4\sqrt{\pi}}xe^{-4x^2}$

(d) $-\dfrac{i}{4\sqrt{\pi}}xe^{-\frac{x^2}{4}}$

(e) None of the above.

12. $\mathcal{F}^{-1}\{e^{-|\lambda|}\} =$ (a)

(a) $\dfrac{1}{\pi}\dfrac{1}{1+x^2}$

(b) $e^{-|x|}$

(c) $\dfrac{2}{1+x^2}$

(d) $\dfrac{1}{\sqrt{\pi}}e^{-x}$

(e) None of the above.

13. Solve the initial-value problem $y'' + y' - 2y = \delta(t-2), \quad y(0) = 1, \quad y'(0) = -2.$
<u>Solution:</u>
$s^2Y(s) - sy(0) - y'(0) + [sY(s) - y(0)] - 2Y(s) = e^{-2s}$
$\Rightarrow \; (s^2 + s - 2)Y(s) - s + 1 = e^{-2s}$
$\Rightarrow \; Y(s) = \dfrac{s-1}{s^2+s-2} + \dfrac{e^{-2s}}{s^2+s-2} = \dfrac{1}{s+2} + \dfrac{e^{-2s}}{(s+2)(s-1)}.$
$\dfrac{1}{(s+2)(s-1)} = \dfrac{A}{s+2} + \dfrac{B}{s-1} \;\Rightarrow\; A = -\dfrac{1}{3},\; B = \dfrac{1}{3}$
$\Rightarrow \; \mathcal{L}^{-1}\left\{\dfrac{1}{(s+2)(s-1)}\right\} = \dfrac{1}{3}\mathcal{L}^{-1}\left\{\dfrac{1}{s-1} - \dfrac{1}{s+2}\right\} = \dfrac{1}{3}\left(e^t - e^{-2t}\right)$
$\Rightarrow \; y(t) = e^{-2t} + \dfrac{1}{3}u(t-2)\left[e^{t-2} - e^{-2(t-2)}\right].$

14. Consider the differential equation $xy'' - y = 0$, $x > 0$. For one (non-zero) series solution y_1 about $x_0 = 0$,

(a) find the coefficient recursion relation,

(b) solve the recursion relation.

Solution:
(a) $p(x)=0,\ q(x)=-\frac{1}{x},\ xp(x)=0,\ x^2q(x)=-x\ \Rightarrow\ p_0=q_0=0\ \Rightarrow\ r(r-1)+p_0r+q_0=r(r-1)=0\ \Rightarrow\ r_1=1,\ r_2=0$. One solution is $y=\sum\limits_{n=0}^{\infty}a_nx^{n+1}$, so
$y'=\sum\limits_{n=0}^{\infty}(n+1)a_nx^n,\ y''=\sum\limits_{n=0}^{\infty}n(n+1)a_nx^{n-1}$, and $xy''-y=0\ \Rightarrow$
$\sum\limits_{n=0}^{\infty}n(n+1)a_nx^n-\sum\limits_{n=0}^{\infty}a_nx^{n+1}=0\ \Rightarrow\ (n+1)(n+2)a_{n+1}-a_n=0$,
i.e., $a_{n+1}=\dfrac{a_n}{(n+1)(n+2)},\ n\geq 0$.
(b) $n=0\ \Rightarrow\ a_1=\dfrac{a_0}{1\cdot 2},\ n=1\ \Rightarrow\ a_2=\dfrac{a_1}{2\cdot 3}=\dfrac{a_0}{1\cdot 2^2\cdot 3}$,
$n=2\ \Rightarrow\ a_3=\dfrac{a_2}{3\cdot 4}=\dfrac{a_0}{1\cdot 2^2\cdot 3^2\cdot 4}$, etc., so $a_n=\dfrac{a_0}{(n!)^2(n+1)},\ n\geq 0$.

15. Let $f(x)=\left\{\begin{array}{ll}1, & 0\leq x\leq 1\\ 0, & 1<x\leq 2\end{array}\right\}$. Find the full Fourier series of f on $[0,2]$.
Solution:
The series is $\frac{a_0}{2}+\sum\limits_{n=1}^{\infty}[a_n\cos(n\pi x)+b_n\sin(n\pi x)]$, with
$b_n=\dfrac{1}{1}\displaystyle\int_0^2 f(x)\sin(n\pi x)\,dx=\int_0^1\sin(n\pi x)\,dx=-\dfrac{1}{n\pi}\cos(n\pi x)\Big|_0^1$
$=\dfrac{1}{n\pi}\left[1-(-1)^n\right],\ n\geq 1$,
$a_n=\dfrac{1}{1}\displaystyle\int_0^2 f(x)\cos(n\pi x)\,dx=\int_0^1\cos(n\pi x)\,dx=\dfrac{1}{n\pi}\sin(n\pi x)\Big|_0^1=0,\ n\geq 1$,
$a_0=\dfrac{1}{1}\displaystyle\int_0^2 f(x)dx=\int_0^1 dx=1$. Thus, the series is
$\dfrac{1}{2}+\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n\pi}\left[1-(-1)^n\right]\sin(n\pi x)$.

16. The solution of the wave equation $u_{xx}=\dfrac{1}{c^2}u_{tt},\ 0<x<L$, satisfying the boundary conditions $u(0,t)=u(L,t)=0$, has the form

$$u(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left[a_n\cos\left(\frac{n\pi ct}{L}\right)+b_n\sin\left(\frac{n\pi ct}{L}\right)\right].$$

Find the solution of $u_{xx}=u_{tt},\ 0<x<\pi$, which also satisfies the initial conditions $u(x,0)=0,\ u_t(x,0)=x(\pi-x)$.
Solution:
$L=\pi,\ c=1\ \Rightarrow\ u(x,t)=\sum\limits_{n=1}^{\infty}\sin(nx)[a_n\cos(nt)+b_n\sin(nt)]$.
$u(x,0)=0\ \Rightarrow\ \sum\limits_{n=1}^{\infty}a_n\sin(nx)=0\ \Rightarrow\ a_n=0,\ n\geq 1$.
$u_t(x,0)=x(\pi-x)\ \Rightarrow\ x(\pi-x)=\sum\limits_{n=1}^{\infty}nb_n\sin(nx)\ \Rightarrow$

$$
\begin{aligned}
nb_n &= \frac{2}{\pi}\int_0^\pi (\pi x - x^2)\sin(nx)dx \\
&= -\frac{2}{n\pi}(\pi x - x^2)\cos(nx)\Big|_0^\pi + \frac{2}{n\pi}\int_0^\pi (\pi - 2x)\cos(nx)dx \\
&= \frac{2}{n^2\pi}(\pi - 2x)\sin(nx)\Big|_0^\pi + \frac{4}{n^2\pi}\int_0^\pi \sin(nx)dx \\
&= -\frac{4}{n^3\pi}\cos(nx)\Big|_0^\pi = \frac{4}{\pi n^3}[1-(-1)^n]. \text{ Thus,}
\end{aligned}
$$

$$
u(x,t) = \sum_{n=1}^{\infty} \frac{4}{\pi n^4}[1-(-1)^n]\sin(nx)\sin(nt) = \frac{8}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}\sin[(2k+1)x]\sin[(2k+1)t].
$$

17. The bounded solution of Laplace's equation $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ outside the circle $r = a$ has the form

$$
u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n}\left[a_n\cos(n\theta) + b_n\sin(n\theta)\right].
$$

Find the bounded solution of Laplace's equation outside the circle $r = 1$, subject to the boundary condition $u(1,\theta) = 1 - \cos(2\theta) + \sin\theta$.

Solution:

$$
1 - \cos(2\theta) + \sin(\theta) = u(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(n\theta) + b_n\sin(n\theta)\right]
$$

$\Rightarrow \dfrac{a_0}{2} = 1,\ a_2 = -1,\ b_1 = 1,$ and $a_n = b_n = 0$ otherwise. Thus,
$u(r,\theta) = 1 - r^{-2}\cos(2\theta) + r^{-1}\sin(\theta)$.

18. Find all eigenvalues and corresponding eigenfunctions of the Sturm-Liouville problem

$y'' + \lambda y = 0,\ y'(0) = 0,\ y(1) = 0.$

Solution:

$\lambda = -\mu^2 < 0 \Rightarrow y = Ae^{\mu x} + Be^{-\mu x},\ y' = A\mu e^{\mu x} - B\mu e^{-\mu x},\ y'(0) = 0 \Rightarrow (A-B)\mu = 0 \Rightarrow B = A \Rightarrow y = A\left(e^{\mu x} + e^{-\mu x}\right),\ y(1) = 0 \Rightarrow A\left(e^{\mu} + e^{-\mu}\right) = 0 \Rightarrow A = 0 \Rightarrow B = 0 \Rightarrow y \equiv 0$. Thus, there are no eigenvalues $\lambda < 0$.

$\lambda = 0 \Rightarrow y = Ax+B,\ y' = A,\ y'(0) = 0 \Rightarrow A = 0 \Rightarrow y = B,\ y(1) = 0 \Rightarrow B = 0 \Rightarrow y \equiv 0$. Thus, $\lambda = 0$ is not an eigenvalue.

$\lambda = \mu^2 > 0 \Rightarrow y = A\cos(\mu x) + B\sin(\mu x),\ y' = -A\mu\sin(\mu x) + B\mu\cos(\mu x),\ y'(0) = 0 \Rightarrow B = 0 \Rightarrow y = A\cos(\mu x),\ y(1) = 0 \Rightarrow A\cos(\mu) = 0 \Rightarrow \mu = \mu_n = (2n+1)\frac{\pi}{2},\ n \geq 0,$

so the eigenvalues are $\lambda_n = \mu_n^2 = \dfrac{(2n+1)^2\pi^2}{4},\ n \geq 0$, with the corresponding eigenfunctions

$$
y_n = A_n\cos(\mu_n x) = A_n\cos\left[\frac{(2n+1)\pi x}{2}\right].
$$