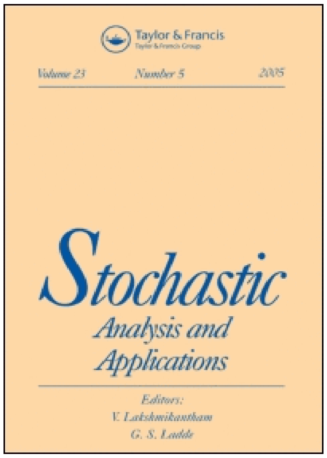

Stochastic Analysis and Applications
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597300

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Online Publication Date: 01 November 2007
To cite this Article: Luo, Jiaowan and Shaikhet, Leonid (2007) 'Stability in Probability of Nonlinear Stochastic Volterra Difference Equations with Continuous Variable', Stochastic Analysis and Applications, 25:6, 1151 - 1165
To link to this article: DOI: 10.1080/07362990701567256
URL: http://dx.doi.org/10.1080/07362990701567256

Stochastic Analysis and Applications, 25: 1151–1165, 2007

ISSN 0736-2994 print/1532-9356 online
DOI: 10.1080/07362990701567256

Stability in Probability of Nonlinear Stochastic Volterra Difference Equations with Continuous Variable

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Abstract: The general method of Lyapunov functionals construction, that was proposed by Kolmanovskii and Shaikhet and successfully used already for functional-differential equations, difference equations with discrete time, difference equations with continuous time, and is used here to investigate the stability in probability of nonlinear stochastic Volterra difference equations with continuous time. It is shown that the investigation of the stability in probability of nonlinear stochastic difference equation with order of nonlinearity more than one can be reduced to investigation of the asymptotic mean square stability of the linear part of this equation.

Keywords: Asymptotic mean square stability; Continuous variable; Method of Lyapunov functionals construction; Nonlinear stochastic Volterra difference equations; Stability in probability.

Mathematics Subject Classification (2000): 39A11; 37H10.

Received March 18, 2006; Accepted September 29, 2006

Jiaowan Luo has been partially supported by NNSF of China (Grant No. 10671043).

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1. INTRODUCTION

Difference equations with continuous variable are difference equations in which the unknown function is a function of a continuous variable. (The term "difference equations" is usually used for difference equations with discrete variable.) In practice, time often is involved as the independent variable in difference equations with continuous variable. In view of this fact, we may also refer to them as *difference equations with continuous time*. Difference equations with continuous variable appear as natural descriptions of observed evolution phenomena in many branches of the natural sciences (see, e.g., [5, 12]).

Deterministic and stochastic difference equations with continuous variable are enough popular with researchers [1, 4, 5, 7, 9–11]. For example, Korenevskii [4] made a first attempt to study spectral and algebraic coefficient criteria (necessary and sufficient conditions) as well as sufficient algebraic coefficient conditions for the mean-square asymptotic stability of solutions to systems of linear stochastic delay difference equations with continuous time under white noise coefficient perturbations for the case in which all delay ratios are rational. For some results on the oscillation of ordinary difference equations with continuous variable, we refer to Domshlak [1]. For the asymptotic results of ordinary difference equations with continuous variable, we refer to Philos and Purnaras [7], stability conditions for linear and nonlinear stochastic Volterra difference equations with continuous time were obtained by Shaikhet [9–11].

Motivated by the results in Shaikhet [8], concerning the stability in probability of solutions of nonlinear stochastic differential equations, and in Paternoster and Shaikhet [6], concerning the stability in probability of solutions of nonlinear stochastic Volterra difference equations with discrete time, in the present article, we will be interested in the stability in probability of nonlinear stochastic Volterra difference equations with continuous variable. It is noticed that the above articles [4, 9–11] deal with the mean-square stability which is different from the stability in probability we are concerned in this article. Similar to [6, 8], here it will be shown that investigation of the stability in probability of nonlinear stochastic difference equation with order of nonlinearity more than one can be reduced to investigation of the asymptotic mean square stability of the linear part of this equation.

The conditions of the stability in probability are obtained here via the general method of Lyapunov functionals construction, that was proposed by Kolmanovskii and Shaikhet and successfully used already for functional-differential equations, for difference equations with discrete time and for difference equations with continuous time [2, 3, 6, 8–11].

2. PRELIMINARIES AND GENERAL THEOREMS

Let $\{\Omega, \mathcal{F}, \mathbf{P}\}$ be a probability space and $\{\mathcal{F}_t\}_{t\geq t_0}$ be a nondecreasing family of sub-σ-algebras of $\mathcal{F}$, i.e., $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$ for $t_1 < t_2$, $\mathbf{E}$ be the mathematical expectation with respect to $\mathbf{P}$, $\mathbf{P}_t\{\cdot\} = \mathbf{P}\{\cdot/\mathcal{F}_t\}$ and $\mathbf{E}_t\{\cdot\} = \mathbf{E}\{\cdot/\mathcal{F}_t\}$ be the conditional probability and the conditional expectation.

Consider a stochastic Volterra difference equation with unbounded delay

$$x(t+\tau) = \sum_{i=0}^{m(t)} a_i x(t-h_i) + \sum_{i=0}^{m(t)} b_i x(t-h_i)\xi(t+\tau) + g(t, x(t), x(t-h_1), \ldots, x(t-h_{m(t)})), \quad t > t_0 - \tau \quad (2.1)$$

with the initial condition

$$x(\theta) = \phi(\theta), \quad \theta \in \Theta := [t_0 - \tau - h_m, t_0]. \quad (2.2)$$

Here

$$m(t) = m + \left[\frac{t-t_0}{\tau}\right], \quad (2.3)$$

m is a given nonnegative integer, $[t]$ denotes the largest integer less than or equal to t when $t \geq 0$, and $[t]$ is the smallest integer greater than or equal to t when $t < 0$. The time increment τ per iteration step is a positive constant, the delays h_i, $i = 0, 1, \ldots$, are constants satisfying the rational relation $h_i = i\tau$, a_i, b_i, $i = 0, 1, \ldots$ are known constants, the functional g satisfies the condition

$$\left|g(t, x(t), x(t-h_1), \ldots, x(t-h_{m(t)}))\right| \leq \sum_{i=0}^{m(t)} \gamma_i |x(t-h_i)|^{\nu_i},$$
$$t > t_0 - \tau, \ \ \gamma_i \geq 0, \ \ \nu_i > 1, \ \ i = 0, 1, \ldots, \quad (2.4)$$

$\phi(\theta)$, $\theta \in \Theta$, is a $\mathcal{F}_{t_0}$-measurable function, the perturbation $\xi(t) \in R$ is a $\mathcal{F}_t$-measurable stationary stochastic process with conditions

$$\mathbf{E}_t \xi(t+\tau) = 0, \quad \mathbf{E}_t \xi^2(t+\tau) = 1, \ \ t > t_0 - \tau. \quad (2.5)$$

A solution of problem (2.1)–(2.2) is a $\mathcal{F}_t$-measurable process $x(t) = x(t; t_0, \phi)$, which is equal to the initial function $\phi(t)$ from (2.2) for $t \leq t_0$ and with probability 1 is defined by Equation (2.1) for $t > t_0$.

Definition 2.1. The trivial solution of Equations (2.1)–(2.2) is called stable in probability if for any $\epsilon > 0$, $\epsilon_1 > 0$ and $t_0 \geq 0$, there exists a $\delta = \delta(\epsilon, \epsilon_1, t_0) > 0$ such that

$$\mathbf{P}_{t_0}\left\{\sup_{t\geq t_0} |x(t)| > \epsilon\right\} < \epsilon_1 \quad (2.6)$$

for any initial function ϕ which are less than δ with probability 1, that is,

$$\mathbf{P}\{\|\phi\|_0 < \delta\} = 1, \tag{2.7}$$

where $\|\phi\|_0 := \sup_{\theta\in\Theta} |\phi(\theta)|$.

Definition 2.2. The trivial solution of Equations (2.1)–(2.2) is called mean square stable if for any $\epsilon > 0$ and $t_0 \geq 0$ there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that $\mathrm{E}|x(t)|^2 < \epsilon$ for all $t \geq t_0$ if $\|\phi\|_1^2 := \sup_{\theta\in\Theta} \mathrm{E}|\phi(\theta)|^2 < \delta$. If besides, $\lim_{t\to\infty} \mathrm{E}|x(t)|^2 = 0$ for all initial functions, then the trivial solution of Equations (2.1)–(2.2) is called asymptotically mean square stable.

Theorem 2.1. *Let there exist a nonnegative functional*

$$V(t) = V(t, x(t), x(t-h_1), \dots, x(t-h_{m(t)})), \quad t > t_0 - \tau,$$

and positive numbers c_1, c_2, such that

$$V(t) \geq c_1 x^2(t), \quad t \geq t_0, \tag{2.8}$$

$$V(s) \leq c_2 \sup_{\theta\leq s} x^2(\theta), \quad s \in (t_0 - \tau, t_0], \tag{2.9}$$

and

$$\mathbf{E}_t \Delta V(t) \leq 0, \quad t > t_0 - \tau, \tag{2.10}$$

if

$$x(s) \in U_\epsilon = \{x : |x| \leq \epsilon\}, \quad s \leq t,$$

where

$$\Delta V(t) = V(t+\tau) - V(t). \tag{2.11}$$

Then the trivial solution of Equations (2.1)–(2.2) *is stable in probability.*

Proof. We will show that for any positive numbers ϵ and ϵ_1, there exists a positive number δ such that the solution of Equation (2.1) satisfies (2.6) with the initial condition (2.2) satisfies condition (2.7).

Let $x(t)$, $t \geq t_0$, be a solution of Equation (2.1). Consider the random variable T such that

$$T = \inf\{t \geq t_0 : |x(t)| > \epsilon\} \tag{2.12}$$

and two events: $\{\sup_{t\geq t_0} |x(t)| > \epsilon\}$ and $\{|x(T)| \geq \epsilon\}$. It is easy to see that

$$\left\{\sup_{t\geq t_0} |x(t)| > \epsilon\right\} \subset \{|x(T)| \geq \epsilon\}. \tag{2.13}$$

Put

$$q = \begin{cases} \left[\dfrac{T-t_0}{\tau}\right], & \text{if } \left[\dfrac{T-t_0}{\tau}\right] = \dfrac{T-t_0}{\tau}, \\ \left[\dfrac{T-t_0}{\tau}\right] + 1, & \text{if } \left[\dfrac{T-t_0}{\tau}\right] < \dfrac{T-t_0}{\tau}. \end{cases} \tag{2.14}$$

From (2.10) we have

$$\begin{aligned} \mathbf{E}_{t_0} \sum_{i=1}^{q} \mathbf{E}_{T-i\tau} \Delta V(T-i\tau) &= \sum_{i=1}^{q} \mathbf{E}_{t_0} (V(T-(i-1)\tau) - V(T-i\tau)) \\ &= \mathbf{E}_{t_0} V(T) - V(s) \le 0, \end{aligned} \tag{2.15}$$

where $s = T - q\tau \in (t_0 - \tau, t_0]$.

So, using (2.13), Chebyshev inequality, (2.7)–(2.9), and (2.15), we get

$$\begin{aligned} \mathbf{P}_{t_0}\left\{ \sup_{t \ge t_0} |x(t)| > \epsilon \right\} &\le \mathbf{P}_{t_0}\{|x(T)| > \epsilon\} \le \frac{\mathbf{E}_{t_0} x^2(T)}{\epsilon^2} \le \frac{\mathbf{E}_{t_0} V(T)}{c_1 \epsilon^2} \\ &\le \frac{V(s)}{c_1 \epsilon^2} \le \frac{c_2 \|\phi\|_0^2}{c_1 \epsilon^2} \le \frac{c_2 \delta^2}{c_1 \epsilon^2}. \end{aligned} \tag{2.16}$$

Choosing δ such that $\delta = \epsilon\sqrt{\epsilon_1 c_1 / c_2}$, we get (2.6). □

Remark 2.1. It is easy to see that if $\epsilon \ge \epsilon_0$, then $\mathrm{P}\{\sup_{t \ge t_0} |x(t)| > \epsilon\} \le \mathrm{P}\{\sup_{t \ge t_0} |x(t)| > \epsilon_0\}$. It means that if condition (2.6) holds for enough small $\epsilon > 0$, then it holds for every $\epsilon > 0$. Thus, for the stability in probability of the trivial solution of Equations (2.1)–(2.2), it is sufficient to prove condition (2.6) for enough small $\epsilon > 0$.

From Theorem 2.1, it follows that an investigation of the stability in probability of the trivial solution of Equation (2.1) can be reduced to construction of appropriate Lyapunov functionals.

3. CONDITIONS OF STABILITY IN PROBABILITY

Denote

$$a = \sum_{i=0}^{\infty} |a_i|, \quad b = \sum_{i=0}^{\infty} |b_i|, \quad \gamma = \sum_{i=0}^{\infty} \gamma_i. \tag{3.1}$$

Theorem 3.1. *Let $\gamma < \infty$ and*

$$a^2 + b^2 < 1. \tag{3.2}$$

Then the trivial solution of Equations (2.1)–(2.2) *is stable in probability.*

Proof. Using the general method of Lyapunov functionals construction for Equations (2.1)–(2.2) we will construct Lyapunov functional in the form $V(t) = V_1(t) + V_2(t)$, where $V_1(t) = x^2(t)$. Via (2.1), (2.5) for $t > t_0 - \tau$ we have

$$\begin{aligned}\mathbf{E}_t \Delta V_1(t) &= \mathbf{E}_t x^2(t+\tau) - x^2(t)\\ &= \mathbf{E}_t\Bigg(g(t, x(t), x(t-h_1), \ldots, x(t-h_{m(t)}))\\ &\qquad + \sum_{i=0}^{m(t)} a_i x(t-h_i) + \sum_{i=0}^{m(t)} b_i x(t-h_i)\xi(t+\tau)\Bigg)^2 - x^2(t)\\ &= \left(\sum_{i=0}^{m(t)} a_i x(t-h_i)\right)^2 + \left(\sum_{i=0}^{m(t)} b_i x(t-h_i)\right)^2\\ &\quad + g^2(t, x(t), x(t-h_1), \ldots, x(t-h_{m(t)}))\\ &\quad + 2\sum_{i=0}^{m(t)} a_i x(t-h_i) g(t, x(t), x(t-h_1), \ldots, x(t-h_{m(t)})) - x^2(t).\end{aligned} \tag{3.3}$$

Using (3.1) we obtain

$$\left(\sum_{i=0}^{m(t)} a_i x(t-h_i)\right)^2 \leq a \sum_{i=0}^{m(t)} |a_i| x^2(t-h_i), \tag{3.4}$$

$$\left(\sum_{i=0}^{m(t)} b_i x(t-h_i)\right)^2 \leq b \sum_{i=0}^{m(t)} |b_i| x^2(t-h_i). \tag{3.5}$$

Via (2.4) we have

$$\begin{aligned}g^2(t, x(t), x(t-h_1), \ldots, x(t-h_{m(t)})) &\leq \gamma \sum_{i=0}^{m(t)} \gamma_i |x(t-h_i)|^{2\nu_i}\\ &\leq \gamma \sum_{i=0}^{m(t)} \gamma_i \epsilon^{2(\nu_i - 1)} x^2(t-h_i)\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}&2\sum_{i=0}^{m(t)} a_i x(t-h_i) g(t, x(t), x(t-h_1), \ldots, x(t-h_{m(t)}))\\ &\quad \leq 2\sum_{i=0}^{m(t)} |a_i||x(t-h_i)| \sum_{k=0}^{m(t)} \gamma_k \big|x(t-h_k)\big|^{\nu_k}\end{aligned}$$

$$= \sum_{i=0}^{m(t)} |a_i| \sum_{k=0}^{m(t)} \gamma_k \big|x(t-h_k)\big|^{\nu_k-1} \big(2|x(t-h_k)||x(t-h_i)|\big)$$

$$\leq \sum_{i=0}^{m(t)} |a_i| \sum_{k=0}^{m(t)} \gamma_k \big|x(t-h_k)\big|^{\nu_k-1} \big(|x(t-h_k)|^2 + |x(t-h_i)|^2\big)$$

$$= \sum_{i=0}^{m(t)} |a_i| \sum_{k=0}^{m(t)} \gamma_k \big|x(t-h_k)\big|^{\nu_k-1} |x(t-h_k)|^2$$

$$+ \sum_{i=0}^{m(t)} |a_i| \sum_{k=0}^{m(t)} \gamma_k \big|x(t-h_k)\big|^{\nu_k-1} |x(t-h_i)|^2. \tag{3.7}$$

Assume that $x(s) \in U_\epsilon$ for $s \leq t$ and put

$$\mu_k(\epsilon) = \sum_{i=0}^{\infty} \gamma_i \epsilon^{k(\nu_i-1)}, \quad k = 1, 2. \tag{3.8}$$

If $\epsilon \leq 1$ then $\mu_k(\epsilon) \leq \gamma < \infty$.

So,

$$2\sum_{i=0}^{m(t)} a_i x(t-h_i) g(t, x(t), x(t-h_1), \ldots, x(t-h_{m(t)}))$$

$$\leq a \sum_{k=0}^{m(t)} \gamma_k \epsilon^{\nu_k-1} |x(t-h_k)|^2 + \mu_1(\epsilon) \sum_{i=0}^{m(t)} |a_i||x(t-h_i)|^2$$

$$\leq \sum_{i=0}^{m(t)} (a\gamma_i \epsilon^{\nu_i-1} + \mu_1(\epsilon)|a_i|) x^2(t-h_i). \tag{3.9}$$

As a result we obtain

$$\mathbf{E}_t \Delta V_1(t) \leq (A_0 - 1)x^2(t) + \sum_{i=1}^{m(t)} A_i x^2(t-h_i), \tag{3.10}$$

where

$$A_i = (a + \mu_1(\epsilon))|a_i| + b|b_i| + (a + \gamma\epsilon^{\nu_i-1})\gamma_i \epsilon^{\nu_i-1}, \quad i = 0, 1, \ldots. \tag{3.11}$$

Choosing now the functional V_2 in the form

$$V_2(t) = \sum_{i=1}^{m(t)} \left(x^2(t-h_i) \sum_{l=i}^{\infty} A_l \right), \tag{3.12}$$

we have

$$\begin{aligned}\Delta V_2(t) &= \sum_{i=1}^{m(t)+1}\left(x^2(t+\tau-h_i)\sum_{l=i}^{\infty}A_l\right) - \sum_{i=1}^{m(t)}\left(x^2(t-h_i)\sum_{l=i}^{\infty}A_l\right)\\ &= \sum_{i=0}^{m(t)}\left(x^2(t-h_i)\sum_{l=i+1}^{\infty}A_l\right) - \sum_{i=1}^{m(t)}\left(x^2(t-h_i)\sum_{l=i}^{\infty}A_l\right)\\ &= x^2(t)\sum_{i=1}^{\infty}A_i - \sum_{i=1}^{m(t)}A_i x^2(t-h_i). \end{aligned} \tag{3.13}$$

From (3.10), (3.11), and (3.13) for $V(t) = V_1(t) + V_2(t)$ we have

$$\mathbf{E}_t\Delta V(t) \leq (a^2 + b^2 + 2a\mu_1(\epsilon) + \gamma\mu_2(\epsilon) - 1)x^2(t), \quad t > t_0 - \tau. \tag{3.14}$$

From (3.2), it follows that $\mathbf{E}_t\Delta V(t) \leq 0$ for enough small ϵ. It is easy to see that the functional $V(t)$ satisfies the conditions of Theorem 2.1. Therefore, using Remark 2.1, we get that the trivial solution of Equations (2.1)–(2.2) is stable in probability. □

Let us obtain now another condition of the stability in probability of the trivial solution of Equations (2.1)–(2.2).

Denote

$$\alpha = \sum_{i=1}^{\infty}\left|\sum_{j=i}^{\infty}a_j\right|, \quad \beta = \sum_{j=0}^{\infty}a_j. \tag{3.15}$$

Theorem 3.2. *Let $\gamma < \infty$ and*

$$\beta^2 + 2\alpha|1-\beta| + b^2 < 1. \tag{3.16}$$

Then the trivial solution of Equations (2.1)–(2.2) *is stable in probability.*

Proof. It is easy to see that Equation (2.1) can be represented in the form

$$\begin{aligned}x(t+\tau) &= \beta x(t) + \Delta F(t) + \sum_{i=0}^{m(t)} b_i x(t-h_i)\xi(t+\tau)\\ &\quad + g(t, x(t), x(t-h_1), \ldots, x(t-h_{m(t)})),\end{aligned} \tag{3.17}$$

where

$$F(t) = -\sum_{l=1}^{m(t)} x(t-h_l)\sum_{i=l}^{\infty}a_i, \quad \Delta F(t) = F(t+\tau) - F(t). \tag{3.18}$$

Following the general method of Lyapunov functionals construction we will construct Lyapunov functional $V(t)$ in the form $V(t) = V_1(t) + V_2(t)$ again but now with

$$V_1(t) = (x(t) - F(t))^2.$$

Calculating $\mathbf{E}_t \Delta V_1(t)$ we get

$$\begin{aligned}
\mathbf{E}_t \Delta V_1(t) &= \mathbf{E}_t (x(t+\tau) - F(t+\tau))^2 - (x(t) - F(t))^2 \\
&= \mathbf{E}_t \Bigg(\beta x(t) - F(t) + \sum_{i=0}^{m(t)} b_i x(t-h_i) \xi(t+\tau) \\
&\qquad + g(t, x(t), x(t-h_1), \ldots, x(t-h_{m(t)})) \Bigg)^2 - (x(t) - F(t))^2 \\
&= (\beta^2 - 1) x^2(t) + \left(\sum_{i=0}^{m(t)} b_i x(t-h_i) \right)^2 + 2(1-\beta) x(t) F(t) \\
&\quad + g^2(t, x(t), x(t-h_1), \ldots, x(t-h_{m(t)})) \\
&\quad + 2\beta x(t) g(t, x(t), x(t-h_1), \ldots, x(t-h_{m(t)})) \\
&\quad - 2F(t) g(t, x(t), x(t-h_1), \ldots, x(t-h_{m(t)})). \qquad (3.19)
\end{aligned}$$

Using the assumption $x(s) \in \mathrm{U}_\epsilon = \{x : |x| \le \epsilon\}$ for $s \le t$, inequalities (3.5), (3.6), and

$$\begin{aligned}
&2|x(t)F(t)| \\
&\quad \le \sum_{l=1}^{m(t)} \left| \sum_{i=l}^{\infty} a_i \right| (x^2(t) + x^2(t-h_l)) \le \alpha x^2(t) + \sum_{l=1}^{m(t)} \left| \sum_{i=l}^{\infty} a_i \right| x^2(t-h_l), \\
&2|x(t) g(t, x(t), x(t-h_1), \ldots, x(t-h_{m(t)}))| \\
&\quad \le \sum_{i=0}^{m(t)} \gamma_i |x(t-h_i)|^{\nu_i - 1} (x^2(t) + x^2(t-h_i)) \\
&\quad \le (\mu_1(\epsilon) + \gamma_0 \epsilon^{\nu_0 - 1}) x^2(t) + \sum_{l=1}^{m(t)} \gamma_l \epsilon^{\nu_l - 1} x^2(t-h_l), \\
&2|F(t) g(t, x(t), x(t-h_1), \ldots, x(t-h_{m(t)}))| \\
&\quad \le \sum_{i=0}^{m(t)} \gamma_i \sum_{l=1}^{m(t)} \left| \sum_{j=l}^{\infty} a_j \right| \epsilon^{\nu_i - 1} (x^2(t-h_l) + x^2(t-h_i)) \\
&\quad \le \alpha \gamma_0 \epsilon^{\nu_0 - 1} x^2(t) + \sum_{l=1}^{m(t)} \left(\mu_1(\epsilon) \left| \sum_{j=l}^{\infty} a_j \right| + \alpha \gamma_l \epsilon^{\nu_l - 1} \right) x^2(t-h_l),
\end{aligned}$$

we have

$$\mathbf{E}_t \Delta V_1(t) \le \big(\beta^2 - 1 + b|b_0| + \alpha|1-\beta| + \gamma\gamma_0 \epsilon^{2(\nu_0-1)} + |\beta|\mu_1(\epsilon) + (\alpha + |\beta|)\gamma_0 \epsilon^{\nu_0-1}\big)x^2(t) + \sum_{i=1}^{m(t)} A_i x^2(t-h_i), \tag{3.20}$$

with

$$A_i = b|b_i| + \gamma\gamma_i \epsilon^{2(\nu_i-1)} + (\mu_1(\epsilon) + |1-\beta|)\left|\sum_{j=i}^{\infty} a_j\right| + (\alpha + |\beta|)\gamma_i \epsilon^{\nu_i-1}. \tag{3.21}$$

Define now the functional V_2 by (3.12), (3.21). Then using (3.20), (3.13), similar to (3.14) we get

$$\mathbf{E}_t \Delta V(t) \le \big(\beta^2 - 1 + 2\alpha|1-\beta| + b^2 + 2(\alpha + |\beta|)\mu_1(\epsilon) + \gamma\mu_2(\epsilon)\big)x^2(t). \tag{3.22}$$

Here $t > t_0 - \tau$, $\mu_1(\epsilon)$ and $\mu_2(\epsilon)$ are defined in (3.8).

From here and (3.16) it follows that for enough small ϵ there exists $c_3 > 0$ such that

$$\mathbf{E}_t \Delta V(t) \le -c_3 x^2(t), \quad t > t_0 - \tau. \tag{3.23}$$

From (3.23) it follows that the functional $V(t)$ satisfies condition (2.10). It is easy to check that the functional $V(t)$ also satisfies condition (2.9); but it does not satisfy condition (2.8). So we cannot here use Theorem 2.1 and have to find another way to prove.

Consider the random variable T which is described by (2.12). From (2.16) we have

$$\mathbf{P}_{t_0}\left\{\sup_{t \ge t_0} |x(t)| > \epsilon\right\} \le \mathbf{P}_{t_0}\{|x(T)| \ge \epsilon\} \le \frac{\mathbf{E}_{t_0} x^2(T)}{\epsilon^2}. \tag{3.24}$$

To estimate $\mathbf{E}_{t_0} x^2(T)$ note that

$$V(T) \ge (x(T) - F(T))^2 \ge x^2(T) - 2x(T)F(T). \tag{3.25}$$

Besides via (3.18), (3.15), and (2.14) we have

$$\begin{aligned} 2x(T)F(T) &\le \sum_{l=1}^{m(T)} \left|\sum_{j=l}^{\infty} a_j\right| (x^2(T) + x^2(T-h_l)) \\ &\le \alpha x^2(T) + \sum_{l=1}^{q-1} \left|\sum_{j=l}^{\infty} a_j\right| x^2(T-h_l) + \sum_{l=q}^{m(T)} \left|\sum_{j=l}^{\infty} a_j\right| x^2(T-h_l) \end{aligned}$$

Since $T - h_l \leq t_0$ for $l \geq q$ then via (2.7) we obtain

$$2x(T)F(T) \leq \alpha x^2(T) + \sum_{l=1}^{q-1} \left| \sum_{j=l}^{\infty} a_j \right| x^2(T - h_l) + \alpha\delta^2. \tag{3.26}$$

From (3.16) it follows that $\alpha < 1$. So, substituting (3.26) into (3.25) we obtain

$$\mathbf{E}_{t_0} x^2(T) \leq (1-\alpha)^{-1} \left\{ \mathbf{E}_{t_0} V(T) + \sum_{l=1}^{q-1} \left| \sum_{j=l}^{\infty} a_j \right| \mathbf{E}_{t_0} x^2(T - h_l) + \alpha\delta^2 \right\}. \tag{3.27}$$

Note that $t_0 < T - h_l < T$ for $1 \leq l < q$. So, from (3.23) it follows $c_3 \mathbf{E}_{t_0} x^2(T - h_l) \leq \mathbf{E}_{t_0} V(T - h_l)$. Besides, similar to (2.15) one can show that $\mathbf{E}_{t_0} V(T - h_l) \leq V(s)$, where $0 \leq l < q$, $s = T - q\tau \leq t_0$. Hence, via (3.27), (2.9), (3.15), (2.7) we have

$$\begin{aligned} \mathbf{E}_{t_0} x^2(T) &\leq (1-\alpha)^{-1} \left\{ V(s) + \sum_{l=1}^{q} \left| \sum_{j=l}^{\infty} a_j \right| \frac{V(s)}{c_3} + \alpha\delta^2 \right\} \\ &\leq (1-\alpha)^{-1} \left\{ c_2 \|\phi\|_0^2 + \sum_{l=1}^{q} \left| \sum_{j=l}^{\infty} a_j \right| \frac{c_2 \|\phi\|_0^2}{c_3} + \alpha\delta^2 \right\} \\ &\leq (1-\alpha)^{-1} \left\{ c_2 + \frac{\alpha c_2}{c_3} + \alpha \right\} \delta^2 := C\delta^2. \end{aligned} \tag{3.28}$$

From here and (3.24) by $\delta = \epsilon\sqrt{\epsilon_1/C}$, (2.6) follows. Therefore, the trivial solution of Equations (2.1)–(2.2) is stable in probability. □

Remark 3.1. Note that condition 3.16 can be transformed to the form

$$b^2 < (1-\beta)(1+\beta-2\alpha), \quad |\beta| < 1.$$

Remark 3.2. As was shown in [8] for stochastic differential equations with delays and in [6] for stochastic difference equations with discrete time one can show that inequalities (3.2) and (3.16) are sufficient conditions for the asymptotic mean square stability of the trivial solution of Equations (2.1)–(2.2) in the case when $g \equiv 0$. It means that investigation of the stability in probability of nonlinear stochastic difference equations with continuous variable with order of nonlinearity more than one can be reduced to the investigation of the asymptotic mean square stability of the linear part of this equation.

4. EXAMPLES

Example 4.1. Consider the nonlinear difference equation

$$x(t+\tau) = \sum_{i=0}^{m(t)} 2^{-(i+1)}\big[(-1)^i a + b\xi(t+\tau)\big]x(t-h_i) + \gamma \sum_{i=0}^{m(t)} 2^{-(i+1)} x^{\nu_i}(t-h_i), \tag{4.1}$$

where $m(t) = m + [\frac{t-t_0}{\tau}]$, $m \geq 0$, $h_i = i\tau$, $\nu_i > 1$, $i = 0, 1, \ldots$.

From Theorem 3.1 it follows that if

$$a^2 + b^2 < 1 \tag{4.2}$$

then the trivial solution of Equation (4.1) is stable in probability.

To use Theorem 3.2 note that $\alpha = 3^{-1}|a|$, $\beta = 3^{-1}a$. Via Remark 3.1 we get condition of the stability in probability in the form

$$|b| < \begin{cases} 1 - 3^{-1}a, & \text{if } a \in [0, 3), \\ \sqrt{(1-3^{-1}a)(1+a)}, & \text{if } a \in (-1, 0). \end{cases} \tag{4.3}$$

On Figure 1 the stability regions are shown obtained by condition (4.2) (Curve No. 1) and condition (4.3) (Curve No. 2). One can see that both these regions supplement each other.

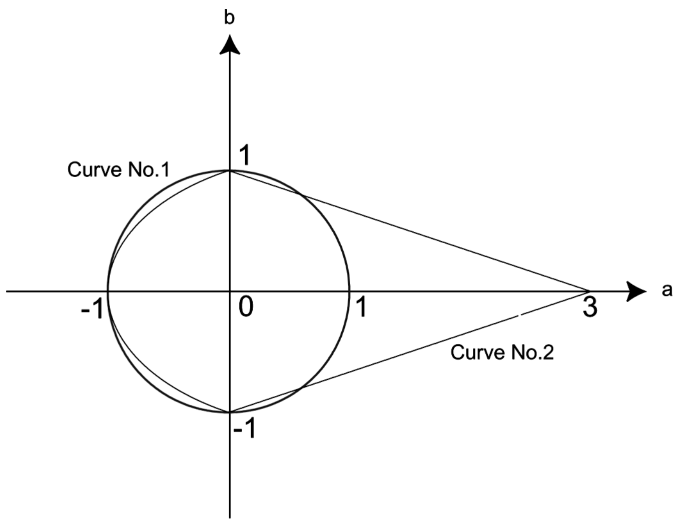

Figure 1. Stability regions for Equation (4.1).

Example 4.2. Consider the nonlinear difference equation

$$x(t+\tau) = \frac{ax(t)}{1+\gamma\sin(x(t-m(t)\tau))} + bx(t-k\tau)\xi(t+\tau), \tag{4.4}$$

where $m(t) = m + [\frac{t-t_0}{\tau}]$, $m \geq 0$, $k \geq 0$ and

$$|\gamma| < 1. \tag{4.5}$$

For the functional

$$V(t) = x^2(t) + b^2 \sum_{j=1}^{k} x^2(t-j\tau)$$

via inequalities (4.5) and $|\sin(x)| \leq 1$ we have

$$\begin{aligned}
\mathbf{E}_t\Delta V(t) &= \mathbf{E}_t x^2(t+\tau) + b^2\sum_{j=1}^{k} x^2(t+\tau-j\tau) - x^2(t) - b^2\sum_{j=1}^{k} x^2(t-j\tau)\\
&= \mathbf{E}_t\left(\frac{ax(t)}{1+\gamma\sin(x(t-m(t)\tau))} + bx(t-k\tau)\xi(t+\tau)\right)^2\\
&\quad - x^2(t) + b^2(x^2(t) - x^2(t-k\tau))\\
&= \frac{a^2x^2(t)}{(1+\gamma\sin(x(t-m(t)\tau)))^2} - x^2(t) + b^2x^2(t)\\
&\leq \left(\frac{a^2}{(1-|\gamma|)^2} + b^2 - 1\right)x^2(t).
\end{aligned}$$

From [10] it follows that the inequality

$$\frac{a^2}{(1-|\gamma|)^2} + b^2 < 1 \tag{4.6}$$

is a sufficient condition for the asymptotic mean square stability of the trivial solution of Equation (4.4).

To get a sufficient condition for the stability in probability of the trivial solution of Equation (4.4) rewrite it in the form

$$x(t+\tau) = ax(t) + g(x(t), x(t-m(t)\tau)) + bx(t-k\tau)\xi_i, \tag{4.7}$$

where

$$g(x, y) = -\frac{a\gamma x\sin(y)}{1+\gamma\sin(y)}.$$

Using the inequality $|\sin(y)| \leq |y|$ in numerator of the fraction and conditions (4.5), $|\sin(y)| \leq 1$ in the denominator we have

$$|g(x, y)| \leq \frac{|a\gamma xy|}{1 - |\gamma|} \leq \frac{|a\gamma|}{2(1 - |\gamma|)}(x^2 + y^2).$$

It means that the function $g(x, y)$ satisfies condition (2.4). Therefore, the condition

$$a^2 + b^2 < 1,$$

which can be obtained from (4.6) by $\gamma = 0$, is a sufficient condition for the asymptotic mean square stability of the linear part of Equation (4.7) and as it follows from Theorem 2.1 is a sufficient condition for the stability in probability of the trivial solution of Equation (4.4) for all γ satisfying condition (4.5).

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