Mathematics and Computers in Simulation 45 (1998) 269–277

Stability of epidemic model with time delays influenced by stochastic perturbations[1]

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Abstract

Many processes in automatic regulation, physics, mechanics, biology, economy, ecology etc. can be modelled by hereditary equations (see, e.g. [1–6]). One of the main problems for the theory of stochastic hereditary equations and their applications is connected with stability. Many stability results were obtained by the construction of appropriate Lyapunov functionals. In [7–11], the procedure is proposed, allowing, in some sense, to formalize the algorithm of the corresponding Lyapunov functionals construction for stochastic functional differential equations, for stochastic difference equations. In this paper, stability conditions are obtained by using this procedure for the mathematical model of the spread of infections diseases with delays influenced by stochastic perturbations. © 1998 IMACS/Elsevier Science B.V.

Keywords: SIR-model; Stochastic perturbations; Stability conditions; Lyapunov functionals; Construction method

1. Problem statement

Consider the mathematical model of the spread of infections diseases. Let $S(t)$ be the number of members of a population susceptible to the disease, $I(t)$ be the number of infective members and $R(t)$ be the number of members who have been removed from the possibility of infection through full immunity. Then the epidemic model can be described by the system [4,13]

$$\dot{S}(t) = -\beta S(t) \int_0^h f(s) I(t-s)\,\mathrm{d}s - \mu_1 S(t) + b$$

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[1]This paper was written during a visit of V. Kolmanovskii and L. Shaikhet in Italy (Napoli, Urbino).

PII S0378-4754(97)00106-7

$$\dot{I}(t) = \beta S(t) \int_0^h f(s) I(t-s)\,ds - (\mu_2 + \lambda) I(t) \tag{1}$$

$$\dot{R}(t) = \lambda I(t) - \mu_3 R(t)$$

It is assumed that β, b, λ, μ_1, μ_2, μ_3 are positive constants, $\mu_1=\min(\mu_1, \mu_2, \mu_3)$, $f(s)$ is nonnegative function, such that

$$\int_0^h f(s)\,ds = 1, \quad \int_0^h s f(s)\,ds < \infty$$

It is easy to see that positive point of equilibrium for system (1) is given by $E_+=(S^*, I^*, R^*)$, where

$$S^* = \frac{\mu_2 + \lambda}{\beta}, \quad I^* = \frac{b - \mu_1 S^*}{\beta S^*}, \quad R^* = \frac{\lambda I^*}{\mu_3} \tag{2}$$

provided that

$$S^* = \frac{\mu_2 + \lambda}{\beta} < \frac{b}{\mu_1} \tag{3}$$

Remark that models of type (1) were considered in numerous papers [4,12,13]. A particular case of this model with fixed delay was proposed first in [12]. The stability properties of the system (1) by $\mu_1=\mu_2=\mu_3=b=\mu$ were considered in [13].

Here we assume that stochastic perturbations are of white noise type, which are directly proportional to distances $S(t)$, $I(t)$, $R(t)$ from values of S^*, I^*, R^*, influence on the $\dot{S}(t), \dot{I}(t), \dot{R}(t)$, respectively. By this way, the system (1) will be reduced to the form

$$\dot{S}(t) = -\beta S(t) \int_0^h f(s) I(t-s)\,ds - \mu_1 S(t) + b + \sigma_1 (S(t) - S^*) \dot{w}_1(t)$$

$$\dot{I}(t) = \beta S(t) \int_0^h f(s) I(t-s)\,ds - (\mu_2 + \lambda) I(t) + \sigma_2 (I(t) - I^*) \dot{w}_2(t) \tag{4}$$

$$\dot{R}(t) = \lambda I(t) - \mu_3 R(t) + \sigma_3 (R(t) - R^*) \dot{w}_3(t)$$

Here σ_1, σ_2, σ_3 are constants, $w_1(t)$, $w_2(t)$, $w_3(t)$ are independent from each other standard Wiener processes [14].

Let us centre the system (4) on the positive equilibrium E_+ by the change of variables $u_1=S-S^*$, $u_2=I-I^*$, $u_3=R-R^*$. By this way, we obtain

$$\dot{u}_1 = -(\beta I^* + \mu_1) u_1 - \beta S^* \int_0^h f(s) u_2(t-s)\,ds - \beta u_1 \int_0^h f(s) u_2(t-s)\,ds + \sigma_1 u_1 \dot{w}_1$$

$$\dot{u}_2 = \beta I^* u_1 - \beta S^* u_2 + \beta S^* \int_0^h f(s) u_2(t-s)\,\mathrm{d}s + \beta u_1 \int_0^h f(s) u_2(t-s)\,\mathrm{d}s + \sigma_2 u_2 \dot{w}_2, \tag{5}$$

$$\dot{u}_3 = \lambda u_2 - \mu_3 u_3 + \sigma_3 u_3 \dot{w}_3$$

It is easy to see that the stability of the system (4) equilibrium is equivalent to the stability of zero solution of Eq. (5).

Below we will obtain the sufficient conditions for stability in a probability sense of zero solution of system (5). Along with the system (5) we will consider the linear part of the system (5)

$$\dot{z}_1 = -(\beta I^* + \mu_1) z_1 - \beta S^* \int_0^h f(s) z_2(t-s)\,\mathrm{d}s + \sigma_1 z_1 \dot{w}_1$$

$$\dot{z}_2 = \beta I^* z_1 - \beta S^* z_2 + \beta S^* \int_0^h f(s) z_2(t-s)\,\mathrm{d}s + \sigma_2 z_2 \dot{w}_2 \tag{6}$$

$$\dot{z}_3 = \lambda z_2 - \mu_3 z_3 + \sigma_3 z_3 \dot{w}_3$$

and the auxiliary system without delays

$$\begin{aligned} \dot{y}_1 &= -(\beta I^* + \mu_1) y_1 + \sigma_1 y_1 \dot{w}_1 \\ \dot{y}_2 &= \beta I^* y_1 - \beta S^* y_2 + \sigma_2 y_2 \dot{w}_2 \\ \dot{y}_3 &= \lambda y_2 - \mu_3 y_3 + \sigma_3 y_3 \dot{w}_3 \end{aligned} \tag{7}$$

2. Definitions, auxiliary statements

Consider the stochastic differential equation [14]

$$\mathrm{d}x(t) = a(t, x_t)\mathrm{d}t + b(t, x_t)\mathrm{d}w(t), \quad x_0 = \varphi \in H. \tag{8}$$

Let $\{\Omega, \sigma, \mathbf{P}\}$ be the probability space, $\{f_t, t \geq 0\}$ be the family of σ-algebras, $f_t \in \sigma$, H be the space of f_0-adapted functions $\varphi(s) \in R^n$, $s \leq 0$, $\|\varphi\|_0 = \sup_{s\leq 0}|\varphi(s)|$, $\|\varphi\|_1^2 = \sup_{s\leq 0}\mathbf{M}|\varphi(s)|^2$, $\mathbf{M}$ is the mathematical expectation, $x_t = x(t+s)$, $s \leq 0$, $w(t)$ is m-dimensional f_t-adapted Wiener process, n-dimensional vector $a(t, \varphi)$ and $n\times m$-dimensional matrix $b(t, \varphi)$ are defined by $t \geq 0$, $\varphi \in H$, $a(t, 0)=0$, $b(t, 0)=0$.

Generating operator L of Eq. (8) is defined [14] by formula

$$LV(t,\varphi) = \lim_{\Delta\to 0} \frac{\mathbf{M}_{t,\varphi} V(t+\Delta,\ y_{t+\Delta}) - V(t,\varphi)}{\Delta}$$

Here a scalar functional $V(t, \varphi)$ is defined by $t \geq 0$, $\varphi \in H$ and $x(s)$ is the solution of Eq. (8) by $s \geq t$ with initial function $x_t = \varphi \in H$.

Let us describe one class of functionals $V(t, \varphi)$ for which the operator L can be calculated in final form. We reduce the arbitrary functional $V(t, \varphi)$, $t \geq 0$, $\varphi \in H$, to the form $V(t, \varphi) = V(t, \varphi(0), \varphi(s))$, $s<0$

and define the function

$$V_\varphi(t,x)=V(t,\varphi)=V(t,x_t)=V(t,x,x(t+s)),\quad s<0,\ \varphi=x_t,\ x=\varphi(0)=x(t)$$

Let D be the class of functionals $V(t, \varphi)$ for which function $V_\varphi(t, x)$ has two continuous derivations with respect to x and one bounded derivative with respect to t for almost all $t\geq 0$. For functionals from D the generating operator L of Eq. (8) is defined and is equal to

$$LV(t,x_t)=\frac{\partial V_\varphi(t,x)}{\partial t}+a'(t,x_t)\frac{\partial V_\varphi(t,x)}{\partial x}+\frac{1}{2}Tr\left[b'(t,x_t)\frac{\partial^2 V_\varphi(t,x)}{\partial x^2}b(t,x_t)\right]$$

Definition 1 The zero solution of Eq. (8) is called mean square stable if for any $\epsilon>0$ there exists a $\delta>0$ such that $\mathbf{M}|x(t)|^2<\epsilon$ for any $t\geq 0$ provided that $\|\varphi\|_1^2<\delta$.

Definition 2 The zero solution of Eq. (8) is called asymptotically mean square stable if it is mean square stable and $\lim_{t\to\infty}\mathbf{M}|x(t)|^2=0$.

Definition 3 The zero solution of Eq. (8) is called stable in probability if for any $\epsilon_1>0$ and $\epsilon_2>0$ there exists $\delta>0$ such that solution $x(t)=x(t, \varphi)$ of Eq. (8) satisfies

$$\mathbf{P}\{|x(t,\varphi)|>\epsilon_1\}<\epsilon_2$$

for any initial function $\varphi\in H$ such that $\mathbf{P}\{|\varphi|\leq\delta\}=1$. Here $\mathbf{P}\{\cdot\}$ is the probability of the event enclosed in braces.

Theorem 4 *Let there exists the functional* $V(t, \varphi)\in D$ *such that*

$$c_1\mathbf{M}|x(t)|^2\leq\mathbf{M}V(t,x_t)\leq c_2\|x_t\|_1^2,\quad \mathbf{M}LV(t,x_t)\leq -c_3\mathbf{M}|x(t)|^2$$

$c_i>0$. *Then the zero solution of* Eq. (8) *is asymptotically mean square stable.*

Theorem 5 *Let there exists the functional* $V(t, \varphi)\in D$ *such that*

$$c_1|x(t)|^2\leq V(t,x_t)\leq c_2\|x_t\|_0^2,\quad LV(t,x_t)\leq 0$$

$c_i>0$, *for any function* $\varphi\in H$ *such that* $\mathbf{P}\{|\varphi|\leq\delta\}=1$, *where* $\delta>0$ *is sufficiently small. Then the zero solution of* Eq. (8) *is stable in probability.*

The proofs of these theorems can be found in [1,2].

By this way, the construction of stability conditions is reduced to construction of Lyapunov functionals. Below we will use the general method of Lyapunov functionals construction [7–11] allowing to obtain stability conditions immediately in terms of parameters of system under consideration.

3. Formal description of the Lyapunov functional: construction procedure

Let us consider the stochastic differential equation of neutral type

$$\mathrm{d}(x(t)-G(t,x_t))=a_1(t,x_t)\mathrm{d}t+a_2(t,x_t)\mathrm{d}\xi(t),\quad t\geq 0,\ x(t)\in R^n \tag{9}$$

Here $x_t = x(t+s)$, $s \leq 0$, $\xi(t) \in R^m$ is a standard Wiener process, $G(t,0)=a_i(t, 0)=0$, $i=1, 2$, functionals a_i are assumed satisfying usual conditions sufficient for the existence of the solution of Eq. (9) with initial conditions $x_0 = \varphi(s)$, where $\varphi \in R^n$ is a given function.

The problem is to construct stability conditions of the trivial solution with respect to the disturbances of the initial condition. From Theorems 4 and 5, it follows that construction of the stability conditions can be reduced to the construction of special Lyapunov functionals $V(t, x_t)$ satisfying the assumptions of these theorems.

The proposed procedure of Lyapunov functionals construction consists of four steps.

1. Let us transform the Eq. (9) to the form

$$\mathrm{d}z(t, x_t) = (b_1(t, x(t)) + c_1(t, x_t))\mathrm{d}t + (b_2(t, x(t)) + c_2(t, x_t))\mathrm{d}\xi(t) \tag{10}$$

   where $z(t, x_t)$ is some functional on x_t, $z(t, 0)=0$, functionals b_i, $i=1, 2$, depend on t and $x(t)$ only and do not depend on the previous values $x(t+s)$, $s<0$, of the solution, $b_i(t, 0)=0$.
2. Consider equation without memory

$$\mathrm{d}y(t) = b_1(t, y(t))\mathrm{d}t + b_2(t, y(t))\mathrm{d}\xi(t) \tag{11}$$

   Let us assume that the zero solution of Eq. (11) is uniformly asymptotically mean square stable and therefore there exists Lyapunov function $v(t, y)$, for which the condition $L_0 v(t, y) \leq -|y|^2$ hold. Here L_0 is generating operator of Eq. (11).
3. We'll construct the Lyapunov functional $V(t, x_t)$ in the form $V=V_1+V_2$. Let us replace argument y of the function $v(t, y)$ on the functional $z(t, x_t)$ from left-hand part of Eq. (10). As a result we obtain the main component $V_1(t, x_t)=v(t, z(t, x_t))$ of the functional $V(t, x_t)$.
4. Usually the functional V_1 almost satisfies the requirements of stability theorem for Eq. (1). In order that these conditions would be completely satisfied the auxiliary component V_2 can be easily chosen by standard way.

Remark that the representation (10) is not unique. This fact allows using different representations (10) to construct different Lyapunov functionals and as a result to get different sufficient stability conditions.

4. Main results

In this section, as a result we will obtain conditions of stability in probability for the zero solution of nonlinear system (5). Preliminary to this, let us prove some auxiliary statements.

Lemma 6 *Let be*

$$\sigma_1^2 < 2\mu_1, \quad \sigma_2^2 < 2(\mu_2 + \lambda), \quad \sigma_3^2 < 2\mu_3 \tag{12}$$

Then the zero solution of the system (7) *is asymptotic mean square stable.*

Proof Let us show that the function

$$v = py_1^2 + y_2^2 + p^2 y_3^2 + q(y_1 + y_2)^2 \tag{13}$$

is Lyapunov function for the system (7) for some $q > 0$ and $p > 0$. Let L_0 be the generating operator [14] of the system (7). Then

$$\begin{aligned} L_0 v &= -2(py_1 + q(y_1+y_2))(\beta I^* + \mu_1)y_1 + 2\beta(y_2 + q(y_1+y_2))(I^* y_1 - S^* y_2) \\ &\quad +2p^2 y_3(\lambda y_2 - \mu_3 y_3) + (p+q)\sigma_1^2 y_1^2 + (1+q)\sigma_2^2 y_2^2 + p^2\sigma_3^2 y_3^2 \\ &\leq -2p(\beta I^* + \mu_1)y_1^2 - 2q(y_1+y_2)(\mu_1 y_1 + \beta S^* y_2) + 2\beta I^* y_1 y_2 - 2\beta S^* y_2^2 \\ &\quad +p^2\lambda(y_2^2/p + p y_3^2) - 2p^2\mu_3 y_3^2 + (p+q)\sigma_1^2 y_1^2 + (1+q)\sigma_2^2 y_2^2 + p^2\sigma_3^2 y_3^2 \\ &\leq y_1^2(p+q)(\sigma_1^2 - 2\mu_1) + y_2^2((1+q)(\sigma_2^2 - 2\beta S^*) + p\lambda) \\ &\quad +p^2 y_3^2(p\lambda - 2\mu_3 + \sigma_3^2) + 2y_1 y_2(\beta I^* - q(\mu_1 + \beta S^*)) \end{aligned}$$

Let be

$$q = \frac{\beta I^*}{\mu_1 + \beta S^*} \tag{14}$$

By this way we obtain

$$L_0 v \leq -y_1^2(p+q)(2\mu_1 - \sigma_1^2) - y_2^2((1+q)(2\beta S^* - \sigma_2^2) - p\lambda) - p^2 y_3^2(2\mu_3 - \sigma_3^2 - p\lambda)$$

Let us choose p such that

$$p < \frac{1}{\lambda}\min\left[(1+q)(2\beta S^* - \sigma_2^2), 2\mu_3 - \sigma_3^2\right]$$

From Eq.(12) it follows that there exists $c > 0$ such that $L_0 v \leq -c|y|^2$, where $y = (y_1, y_2, y_3)$. It means (Theorem 4) that the zero solution of the system (7) is asymptotic mean square stable. Lemma is proved.

Theorem 7 *Let be*

$$\sigma_1^2 < 2\mu_1, \qquad \sigma_2^2 < \frac{2q(\mu_2 + \lambda)}{1+q}, \qquad \sigma_3^2 < 2\mu_3 \tag{15}$$

where q is described by Eq.(14). *Then the zero solution of the system (6) is asymptotic mean square stable.*

Proof Following of the formal procedure of Lyapunov functionals construction we will construct Lyapunov functional V for the system (6) in the form $V = V_1 + V_2$, where V_1 is Lyapunov function (13) for the auxiliary system (7) without delays:

$$V_1 = pz_1^2 + z_2^2 + p^2 z_3^2 + q(z_1 + z_2)^2 \tag{16}$$

Let L_1 be the generating operator [14] of the system (6). Then

$$\begin{aligned}
L_1V_1 = {} & -2(pz_1 + q(z_1+z_2))((\beta I^* + \mu_1)z_1 + \beta S^* \int_0^h f(s) z_2(t-s)\,\mathrm{d}s) \\
& + 2\beta(z_2 + q(z_1+z_2))(I^* z_1 - S^* z_2 + S^* \int_0^h f(s) z_2(t-s)\,\mathrm{d}s) + 2p^2 z_3(\lambda z_2 - \mu_3 z_3) \\
& + (p+q)\sigma_1^2 z_1^2 + (1+q)\sigma_2^2 z_2^2 + p^2\sigma_3^2 z_3^2 + 2q\sigma_1\sigma_2 z_1 z_2 \le z_1^2[(p+q)(\sigma_1^2 - 2\mu_1) - 2p\beta I^*] \\
& + z_2^2[(1+q)(\sigma_2^2 - 2\beta S^*) + p\lambda] + p^2 z_3^2(\sigma_3^2 - 2\mu_3 + p\lambda) \\
& + 2z_1 z_2(\beta I^* + q(\sigma_1\sigma_2 - \mu_1 - \beta S^*)) + 2\beta S^*(z_2 - p z_1)\int_0^h f(s) z_2(t-s)\,\mathrm{d}s
\end{aligned}$$

Using Eq. (14) we obtain

$$\begin{aligned}
L_1V_1 & \le z_1^2 q(\sigma_1^2 - 2\mu_1) + z_2^2[(1+q)(\sigma_2^2 - 2\beta S^*) + p\lambda] + p^2 z_3^2(\sigma_3^2 - 2\mu_3 + p\lambda) \\
& \quad + p\beta S^*[z_1^2 + \int_0^h f(s) z_2^2(t-s)\,\mathrm{d}s)] + \beta S^*[z_2^2 + \int_0^h f(s) z_2^2(t-s)\,\mathrm{d}s)] \\
& = z_1^2[q(\sigma_1^2 - 2\mu_1) + p\beta S^*] + z_2^2[(1+q)(\sigma_2^2 - 2\beta S^*) + p\lambda + \beta S^*] \\
& \quad + p^2 z_3^2(\sigma_3^2 - 2\mu_3 + p\lambda) + \beta S^*(1+p)\int_0^h f(s) z_2^2(t-s)\,\mathrm{d}s
\end{aligned}$$

Following [7–11] let us choose V_2 in the form

$$V_2 = \beta S^*(1+p)\int_0^h f(s)\int_{t-s}^t z_2^2(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \tag{17}$$

Using Eq. (15), for functional $V{=}V_1{+}V_2$ we obtain

$$\begin{aligned}
L_1V \le {} & -z_1^2[q(2\mu_1 - \sigma_1^2) - p\beta S^*)] - z_2^2[2q\beta S^* - (1+q)\sigma_2^2 - p(\lambda + \beta S^*)] \\
& - p^2 z_3^2(2\mu_3 - \sigma_3^2 - p\lambda)
\end{aligned} \tag{18}$$

From Eq. (15) it follows that there exists $p{>}0$ such that

$$p < \min\left[\frac{q(2\mu_1 - \sigma_1^2)}{\beta S^*}, \frac{2q\beta S^* - (1+q)\sigma_2^2}{\lambda + \beta S^*}, \frac{2\mu_3 - \sigma_3^2}{\lambda}\right]$$

Therefore, there exists $c{>}0$ such that $L_1V \le -c|z|^2$, where $z{=}(z_1, z_2, z_3)$. It means (Theorem 4) that the zero solution of the system (6) is asymptotic mean square stable. Theorem is proved.

Remark 8 It is known [15] that if the initial nonlinear system has a nonlinearity order more than one, then the conditions providing the asymptotic mean square stability of linear part of the initial system, in the same time provide the stability in probability of the initial system. Let us show it for the system (5).

Theorem 9 *Let the conditions of Theorem* 7 *hold. Then the zero solution of the system* (5) *is stable in probability.*

Proof Let L be the generating operator of the system (5). Consider the functional $V=V_1+V_2$, where V_1 and V_2 are defined by Eq. (16) and Eq. (17), i.e.

$$V = pu_1^2 + u_2^2 + p^2u_3^2 + q(u_1+u_2)^2 + \beta S^*(1+p)\int_0^h f(s)\int_{t-s}^t u_2^2(\tau)\,d\tau\,ds$$

Then analogously to Eq. (18) we obtain

$$\begin{aligned} LV = {} & -2(pu_1 + q(u_1+u_2))\left[(\beta I^* + \mu_1)u_1 + \beta S^*\int_0^h f(s)u_2(t-s)\,ds + \beta u_1\int_0^h f(s)u_2(t-s)\,ds\right] \\ & + 2\beta(u_2 + q(u_1+u_2))\left[I^*u_1 - S^*u_2 + S^*\int_0^h f(s)u_2(t-s)\,ds + u_1\int_0^h f(s)u_2(t-s)\,ds\right] \\ & + 2p^2u_3(\lambda u_2 - \mu_3u_3) + (1+p)\beta S^*u_2^2 - (1+p)\beta S^*\int_0^h f(s)u_2^2(t-s)\,ds + (p+q)\sigma_1^2u_1^2 \\ & + (1+q)\sigma_2^2u_2^2 + p^2\sigma_3^2u_3^2 + 2q\sigma_1\sigma_2u_1u_2 \le -u_1^2[q(2\mu_1-\sigma_1^2) - p\beta S^*] \\ & - u_2^2[2q\beta S^* - (1+q)\sigma_2^2 - p(\lambda + \beta S^*)] - p^2u_3^2(2\mu_3 - \sigma_3^2 - p\lambda) \\ & + 2\beta u_1(u_2 - pu_1)\int_0^h f(s)u_2(t-s)\,ds \end{aligned}$$

Let us suppose that $\mathbf{P}\{|u_2(s)| < \delta\} = 1$. Then

$$2\beta\left|u_1(u_2 - pu_1)\int_0^h f(s)u_2(t-s)\,ds\right| \le \beta\delta(u_1^2(1+2p) + u_2^2)$$

Therefore,

$$\begin{aligned} LV \le {} & -u_1^2[q(2\mu_1 - \sigma_1^2) - p\beta S^* - \beta\delta(1+2p)] - u_2^2[2q\beta S^* - (1+q)\sigma_2^2 - p(\lambda+\beta S^*) - \beta\delta] \\ & - p^2u_3^2(2\mu_3 - \sigma_3^2 - p\lambda) \end{aligned}$$

Hence, for sufficiently small $\delta>0$ we obtain $LV\le 0$. It means (Theorem 5) that the zero solution of the system (5) is stable in probability. Theorem is proved.

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