Stochastic Volterra integro-differential equations: stability and numerical methods

Leonid E. Shaikhet*& Jason A. Roberts†

March 9, 2004

Abstract

We consider the reliability of some numerical methods in preserving the stability properties of the linear stochastic functional differential equation

$$\dot{x}(t) = \alpha x(t) + \beta \int_0^t x(s)ds + \sigma x(t-\tau)\dot{W}(t),$$

where $\alpha, \beta, \sigma, \tau \geq 0$ are real constants, and $W(t)$ is a standard Wiener process. We adopt the shorthand notation of $\dot{x}(t)$ to represent the differential $dx(t)$ etc.

Our choice of test equation is a stochastic perturbation of the classical deterministic Brunner & Lambert test equation for $\sigma = 0$ and so our investigation may be thought of as an extension of their work.

Sufficient conditions for the asymptotic mean square stability of solutions to both the differential equation and discrete analogues are derived using the general method of Lyapunov functionals construction proposed by Kolmanovskii & Shaikhet which has previously been successfully employed for deterministic and stochastic differential and difference equations with delay.

The areas of the regions of asymptotic stability for each θ method, indicated by the sufficient conditions for the discrete system, are shown to be equal and we show that an upper bound can be put on the time-step parameter for the numerical method fo which the system is asymptotically mean-square stable.

We illustrate our results by means of numerical experiments and various stability diagrams. We examine the extent to which the continuous system can tolerate stochastic perturbations before losing its stability properties and we illustrate how one may accurately choose a numerical method to preserve the stability properties of the original problem in the numerical solution. Our numerical experiments also indicate that the quality of the sufficient conditions is very high.

*Donetsk State Academy of Management, Ukraine, leonid.shaikhet@usa.net, collaboration supported by NATO, grant ref. PST.EV.979727

†Mathematics Department, University College Chester, England, j.roberts@chester.ac.uk

1 Introduction

Volterra integro-differential equations arise in the modelling of hereditary systems (i.e. systems where the past influences the present) such as population growth, pollution, financial markets and mechanical systems (see [4], [1] for example). The long-term behaviour and stability of such systems is an important area for investigation. For example - will a population decline to dangerously low levels? Could a small change in the environmental conditions have drastic consequences on the long-term survival of the population? There is a growing body of work devoted to such investigations (see [19], [6] for example). Analytical solutions to such problems are generally unavailable and numerical methods are adopted for obtaining approximate solutions. A natural question to ask is "do the numerical solutions preserve the stability properties of the exact solution?". We refer the reader to a number of works where the answers to such questions are investigated: [2], [3], [7], [8], [5], [21].

Many real-world phenomena are subject to random noise or perturbations (for example, freak weather conditions may adversely affect the supports of a bridge, possibly changing the long-term integrity of the structure). It is a natural extension of the deterministic work carried out by ourselves and others to consider the stablility of stochastic systems and of numerical solutions to such systems. We refer the readers to a number of texts which discuss the role of stochastic systems in mathematical modelling: [9], [1], [20].

In this paper we consider the scalar linear test equation

$$\dot{x}(t) = \alpha x(t) + \beta \int_0^t x(s)ds + \sigma x(t-\tau)\dot{W}(t), \tag{1.1}$$

$$x(s) = \varphi(s), \;\; s \in [-\tau, 0],$$

where $\alpha, \beta, \sigma, \tau \geq 0$ are real constants, and $W(t)$ is a standard Wiener process.

In particular if $\sigma = 0$ then this equation reduces to the deterministic linear test equation of Brunner and Lambert [2].

When considering stability of a system we must decide on a suitable definition for stability. There are a number of definitions for the stability of stochastic systems. A common choice of definition amongst numerical analysts investigating stochastic differential equations is that of mean square stability and asymptotic mean square stability. We derive asymptotic mean square stability conditions for the linear test equation (1.1). An analagous approach is used to derive conditions for asymptotic mean square stability of a linear stochastic difference equation. It is shown that our choice of numerical methods are special cases of this particular difference equation, thereby allowing us to produce stability conditions for the numerical solutions to the original problem. Finally, we present some stability diagrams and numerical experiments to illustrate our results.

The main conclusion of our investigation here can be formulated in the following way: if the trivial solution of the initial functional differential equation is asymptotically mean square stable then there exist a way and a step of discretization of this equation that the trivial solution of the corresponding difference equation is asymptotically mean square stable too. Moreover, it is possible to find an upper bound for the step of discretization for which the corresponding discrete analogue preserves the properties of stability.

The conditions for asymptotic mean square stability are obtained here by virtue of Kolmanovskii and Shaikhet's general method of Lyapunov functionals construction ([11] to [17]) which is applicable for both differential and difference equations, both for deterministic and stochastic systems with delay.

Let us remind ourselves here of some definitions and statements which will be used.

Let $\{\Omega, \mathcal{F}, \mathbf{P}\}$ be a basic probability space with a family of σ-algebras $f_t \subset \mathcal{F}$, $t \geq 0$, H be a space of f_0-adapted functions $\varphi(s)$, $s \leq 0$, $\mathbf{E}$ is the sign of expectation.

Consider a stochastic differential equation with aftereffect

$$\dot{x}(t) = a(t, x(t)) + b(t, x(t))\dot{W}(t), \ \ x_0 = \varphi \in \mathcal{H}. \tag{1.2}$$

Hence $W(t) \in \mathbb{R}^m$ is an m-dimensional Wiener process, the functionals $a(t, \varphi) \in \mathbb{R}^n$ and $b(t, \varphi) \in \mathbb{R}^{n \times m}$ are defined for $t \geq 0$, $\varphi \in H$, $a(t, 0) = 0$, $b(t, 0) = 0$, $x_t(s) = x(t + s)$, $s \leq 0$, is a trajectory of the process $x(s)$ for $s \leq t$.

Definition 1.1 *The trivial solution of equation (1.2) is called*

(i) mean square stable if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $\mathbf{E}|x(t)|^2 < \epsilon$ for all $t \geq 0$ if $\sup_{s \leq 0} \mathbf{E}|\varphi(s)|^2 < \delta$;

(ii) asymptotically mean square stable if it is mean square stable and $\lim_{t\to\infty} \mathbf{E}|x(t)|^2 = 0$ for every initial function $\varphi \in H$.

Let D be a space of functionals $V(t, \varphi)$, for which $t \geq 0$, $\varphi \in H$, for which the function

$$V_\varphi(t, x) = V(t, x_t) = V(t, x(t), x(s), s \leq t), \ \ x = x(t),$$

has one continuous derivative with respect to t and two continuous derivatives with respect to x. For each functional V from D the differential operator L is defined by the formula

$$LV(t, \varphi) = \frac{\partial}{\partial t} V_\varphi(t, x) + a'(t, \varphi) \frac{\partial}{\partial x} V_\varphi(t, x) + \frac{1}{2} \mathrm{tr} \left[b'(t, \varphi) \frac{\partial^2}{\partial x^2} V_\varphi(t, x) b(t, \varphi) \right], \tag{1.3}$$

where the prime symbol ' denotes transpose.

Theorem 1.1 *Let there exist a functional $V = V(t, \varphi) \in D$ such that*

$$\begin{array}{rcl} \mathbf{E}V(t, x_t) & \geq & c_1 \mathbf{E}|x(t)|^2, \\ \mathbf{E}V(0, \varphi) & \leq & c_2 \sup_{s \leq 0} \mathbf{E}|\varphi(s)|^2, \\ \mathbf{E}LV(t, x_t) & \leq & -c_3 \mathbf{E}|x(t)|^2, \end{array}$$

where $c_i > 0$, $i = 1, 2, 3$. Then the trivial solution of equation (1.2) is asymptotically mean square stable.

Let $\{\Omega, \mathcal{F}, \mathbf{P}\}$ be a basic probability space, $f_i \in \mathcal{F}$, $i \in Z = \{0, 1, \ldots\}$ be a sequence of σ-algebras, $\xi_i \in \mathbb{R}^m$, $i \in Z$ be f_{i+1}-adapted and mutually independent random variables, $\mathbf{E}\xi_i = 0$, $\mathbf{E}\xi_i\xi_i' = I$, where I is an identity matrix.

Consider a stochastic difference equation

$$x_{i+1} = a(i, x_{-m}, \ldots, x_i) + b(i, x_{-m}, \ldots, x_i)\xi_i, \ i \in Z. \tag{1.4}$$

Here $a \in \mathbb{R}^n$, $b \in \mathbb{R}^{n \times m}$, $a(i, 0, \ldots, 0) = 0$, $b(i, 0, \ldots, 0) = 0$, $x_i = \varphi_i$, $i \in [-m, 0]$.

Definition 1.2 *The trivial solution of equation (1.4) is called:*

(i) mean square stable if for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $\mathbf{E}|x_i|^2 < \epsilon$, $i \in Z$, if $\sup_{i \in [-m, 0]} \mathbf{E}|\varphi_i|^2 < \delta$;

(ii) asymptotically mean square stable if $\lim_{i \to \infty} \mathbf{E}|x_i|^2 = 0$ for every initial function φ_i.

Theorem 1.2 *Let there exist a nonnegative functional $V_i = V(i, x_{-m}, \ldots, x_i)$, which satisfies the conditions*

$$\mathbf{E}V(0, x_{-m}, \ldots, x_0) \leq c_1 \sup_{i \leq 0} \mathbf{E}|\varphi_i|^2,$$

$$\mathbf{E}\Delta V_i \leq -c_2 \mathbf{E}|x_i|^2, \ i \in Z,$$

where $c_1 > 0$, $c_2 > 0$, $\Delta V_i = V_{i+1} - V_i$. Then the trivial solution of equation (3.1) is asymptotically mean square stable.

2 A linear stochastic Volterra integro-differential equation

Consider equation (1.1). It is well known [10] that for $\beta = 0$ the inequality

$$2\alpha + \sigma^2 < 0 \tag{2.1}$$

is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of equation (1.1).

If $\sigma = 0$ then equation (1.1) reduces to the Brunner and Lambert test equation [2] and also takes the differential form

$$\ddot{x}(t) = \alpha\dot{x}(t) + \beta x(t).$$

In this case the inequalities

$$\alpha < 0, \ \beta < 0, \tag{2.2}$$

are the necessary and sufficient condition for asymptotic stability of the trivial solution of equation (1.1).

We proceed in the following way to obtain asymptotic mean square stability conditions for the trivial solution of (1.1) via Lyapunov's second method. Following conditions (2.1), (2.2) we will suppose that the conditions

$$2\alpha + \sigma^2 < 0, \ \beta < 0, \tag{2.3}$$

hold.

We transform equation (1.1) in the following way. Let

$$y_1(t) = \int_0^t x(s)ds, \ y_2(t) = x(t).$$

Then equation (1.1) is transformed into the system of equations

$$\begin{aligned} \dot{y}_1(t) &= y_2(t) \\ \dot{y}_2(t) &= \beta y_1(t) + \alpha y_2(t) + \sigma y_2(t-\tau)\dot{W}(t) \end{aligned}$$

or in the matrix form

$$\dot{y}(t) = Ay(t) + By(t-\tau)\dot{W}(t), \tag{2.4}$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \ A = \begin{pmatrix} 0 & 1 \\ \beta & \alpha \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}.$$

Following the general method of Lyapunov functionals construction [11], [12] we will construct a Lyapunov functional for equation (2.4) in the form $V = V_1 + V_2$, where the main part V_1 of the functional V must be chosen as a Lyapunov function for some auxiliary differential equation without delay (in this case it is equation (2.4) with $B = 0$.) Let us choose V_1 in the form $V_1 = y'(t)Py(t)$ where $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$ is a positive definite matrix. Calculating for equation (2.4) the operator L defined by (1.3) we obtain

$$LV_1 = y'(t)\left(PA + A'P\right)y(t) + y'(t-\tau)B'PBy(t-\tau). \tag{2.5}$$

Let us choose the additional functional V_2 in the form

$$V_2 = \int_{t-\tau}^t y'(s)B'PBy(s)ds.$$

Then

$$LV_2 = y'(t)B'PBy(t) - y'(t-\tau)B'PBy(t-\tau) \tag{2.6}$$

and from (2.5), (2.6) for the functional $V = V_1 + V_2$ it follows

$$LV = y'(t)\left(PA + A'P + B'PB\right)y(t). \tag{2.7}$$

Suppose that the matrix P is a positive definite solution of the matrix equation

$$PA + A'P + B'PB = -I, \tag{2.8}$$

where I is the identity matrix. Matrix equation (2.8) is equivalent to the system of the equations

$$\begin{aligned} 2\beta p_{12} &= -1, \\ p_{11} + \alpha p_{12} + \beta p_{22} &= 0, \\ 2p_{12} + \left(2\alpha + \sigma^2\right) p_{22} &= -1, \end{aligned}$$

with the solution

$$p_{11} = \frac{\alpha}{2\beta} - \frac{1-\beta}{2\alpha+\sigma^2},\ p_{12} = -\frac{1}{2\beta},\ p_{22} = \frac{1-\beta}{\beta\left(2\alpha+\sigma^2\right)}. \tag{2.9}$$

Note that by conditions (2.3) $p_{11} > 0$ and $p_{22} > 0$. Also, using (2.3) we have

$$\begin{aligned} p_{11}p_{22} &= \left(\frac{\alpha}{2\beta} - \frac{1-\beta}{2\alpha+\sigma^2}\right)\left(\frac{1-\beta}{\beta\left(2\alpha+\sigma^2\right)}\right) \\ &> \frac{\alpha(1-\beta)}{2\beta^2\left(2\alpha+\sigma^2\right)} \\ &> \frac{1}{4\beta^2}\frac{2\alpha}{\left(2\alpha+\sigma^2\right)} \\ &\geq \frac{1}{4\beta^2} \\ &= p_{12}^2. \end{aligned}$$

Therefore the matrix P with elements (2.9) is positive definite, as required. From here and (2.7), (2.8) it follows that there exists a positive definite functional V, for which $LV = -\left|y(t)\right|^2$. Recalling our originally supposed conditions, (2.1) and (2.3) we can now state the following result.

Theorem 2.1 *The system of inequalities*

$$2\alpha + \sigma^2 < 0,\ \ \beta \leq 0, \tag{2.10}$$

is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of equation (1.1).

3 A linear stochastic Volterra difference equation

Let $\{\Omega, \mathcal{F}, P\}$ be a basic probability space, $f_i \in \mathcal{F}$, $i \in Z = \{0, 1, \ldots\}$, be a sequence of σ-algebras, ξ_i, $i \in Z$, be scalar f_{i+1}-adapted and mutually independent random variables, $\mathbf{E}\xi_i = 0$, $\mathbf{E}\xi_i^2 = 1$, where $\mathbf{E}$ is the sign for expectation. Consider the scalar stochastic difference equation

$$x_1 = (a+b)x_0 + \sigma x_{-m}\xi_0,$$

$$x_2 = ax_1 + b\left(\theta x_0 + (1-\theta)x_1\right) + \sigma x_{1-m}\xi_1,$$

$$x_{i+1} = ax_i + b\left(\theta x_0 + \sum_{j=1}^{i-1} x_j + (1-\theta)x_i\right) + \sigma x_{i-m}\xi_i, \;\; i \geq 2. \tag{3.1}$$

Here a, b, σ are constants, $\theta \in [0,1]$, $m \geq 0$ is integer. Note that if $b = 0$ then the inequality

$$a^2 + \sigma^2 < 1 \tag{3.2}$$

is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of equation (3.1) [22].

Suppose that $b \neq 0$. We transform equation (3.1) for $i \geq 2$ in the following way:

$$\begin{aligned} x_{i+1} &= (a + b(1-\theta))x_i + bx_{i-1} + b\left(\theta x_0 + \sum_{j=0}^{i-2} x_j\right) + \sigma x_{i-m}\xi_i \\ &= (a + b(1-\theta))x_i + bx_{i-1} + \sigma x_{i-m}\xi_i + x_i \\ &\quad - (a + b(1-\theta))x_{i-1} - \sigma x_{i-1-m}\xi_{i-1} \\ &= (a + b(1-\theta) + 1)x_i + (b\theta - a)x_{i-1} + \sigma x_{i-m}\xi_i - \sigma x_{i-1-m}\xi_{i-1} \end{aligned}$$

As a result we obtain equation (3.1) in the form

$$x_{i+1} = Ax_i + Bx_{i-1} + \sigma_1 x_{i-m}\xi_i + \sigma_2 x_{i-1-m}\xi_{i-1}, i \geq 2, \tag{3.3}$$

where

$$A = a + b(1-\theta) + 1, \;\; B = b\theta - a, \;\; \sigma_1 = \sigma, \;\; \sigma_2 = -\sigma. \tag{3.4}$$

It is known [22] that for $\sigma_2 = 0$ the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of equation (3.3) is

$$|A| < 1 - B, \;\; |B| < 1, \tag{3.5}$$

$$\sigma_1^2 < \frac{1+B}{1-B}\left((1-B)^2 - A^2\right). \tag{3.6}$$

We now obtain a necessary and sufficient condition for asymptotic mean square stability of the trivial solution of equation (3.3) for arbitrary σ_1 and σ_2. Let

$$x(i) = \begin{pmatrix} x_{i-1} \\ x_i \end{pmatrix}, \;\; A_1 = \begin{pmatrix} 0 & 1 \\ B & A \end{pmatrix}, \;\; B_k = \begin{pmatrix} 0 \\ \sigma_k \end{pmatrix}, \;\; k = 1, 2. \tag{3.7}$$

Then equation (3.3) takes the following matrix form:

$$x(i+1) = A_1 x(i) + B_1 x_{i-m}\xi_i + B_2 x_{i-1-m}\xi_{i-1}. \tag{3.8}$$

Following the general method of Lyapunov functional construction let us construct a Lyapunov functional V_i for equation (3.8) in the form $V_i = V_{1i} + V_{2i}$, where the main part V_{1i} of the functional V_i must be chosen as a Lyapunov function for some auxiliary difference equation without delay (in this case it is equation (3.8) with $B_1 = B_2 = 0$.) Let us choose V_{1i} in the form

$$V_{1i} = x'(i)Dx(i),\ D = \begin{pmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{pmatrix}, \tag{3.9}$$

and D is a positive semi-definite solution of the matrix equation

$$A_1' D A_1 - D = -U,\ U = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.10}$$

with $d_{22} > 0$. Calculating $\mathbf{E}\Delta V_{1i} = \mathbf{E}\left(V_{1,i+1} - V_{1i}\right)$, by virtue of (3.9), (3.8) we obtain

$$\begin{aligned}
\mathbf{E}\Delta V_{1i} &= \mathbf{E}\left(x'(i+1)Dx(i+1) - x'(i)Dx(i)\right) \\
&= \mathbf{E}((A_1 x(i) + B_1 x_{i-m}\xi_i + B_2 x_{i-1-m}\xi_{i-1})' D(A_1 x(i) \\
&\quad + B_1 x_{i-m}\xi_i + B_2 x_{i-1-m}\xi_{i-1}) - x'(i)Dx(i)) \\
&= \mathbf{E}(x'(i)\left(A_1' D A_1 - D\right)x(i) + B_1' D B_1 x_{i-m}^2 \xi_i^2 \\
&\quad + B_2' D B_2 x_{i-1-m}^2 \xi_{i-1}^2 + 2B_1' D A_1 x(i) x_{i-m}\xi_i \\
&\quad + 2B_2' D A_1 x(i) x_{i-1-m}\xi_{i-1} + 2B_1' D B_2 x_{i-m} x_{i-1-m}\xi_i\xi_{i-1}).
\end{aligned} \tag{3.11}$$

From (3.10) it follows that

$$\mathbf{E}x'(i)\left(A_1' D A_1 - D\right)x(i) = -\mathbf{E}x_i^2. \tag{3.12}$$

Using (3.7), (3.9) and the properties of ξ_i, we obtain

$$\begin{aligned}
\mathbf{E}x_{i-m}^2\xi_i^2 &= \mathbf{E}x_{i-m}^2, \\
\mathbf{E}x(i)x_{i-m}\xi_i &= 0, \\
\mathbf{E}x_{i-m}x_{i-1-m}\xi_i\xi_{i-1} &= 0, \\
B_2' D A_1 &= \left(\sigma_2 B d_{22}, \sigma_2\left(d_{12} + A d_{22}\right)\right), \\
B_k' D B_k &= \sigma_k^2 d_{22},\ \ k = 1, 2, \\
\mathbf{E}x(i)x_{i-1-m}\xi_{i-1} &= \left(0, \mathbf{E}x_i x_{i-1-m}\xi_{i-1}\right)'.
\end{aligned} \tag{3.13}$$

Using (3.3), we have

$$\begin{aligned}\mathbf{E}x_i x_{i-1-m}\xi_{i-1} &= \mathbf{E}(Ax_{i-1}+Bx_{i-2}+\sigma_1 x_{i-1-m}\xi_{i-1}\\ &\quad +\sigma_2 x_{i-2-m}\xi_{i-2})x_{i-1-m}\xi_{i-1}\\ &= \sigma_1\mathbf{E}x_{i-1-m}^2.\end{aligned} \tag{3.14}$$

Using (3.11) to (3.14) we obtain

$$\begin{aligned}\mathbf{E}\Delta V_{1i} &= -\mathbf{E}x_i^2+\sigma_1^2 d_{22}\mathbf{E}x_{i-m}^2\\ &\quad+\left(\sigma_2^2 d_{22}+2\sigma_1\sigma_2\left(d_{12}+Ad_{22}\right)\right)\mathbf{E}x_{i-1-m}^2.\end{aligned} \tag{3.15}$$

Using (3.7), (3.9) we have

$$\begin{aligned}A_1' D A_1 &= \begin{pmatrix}0 & B\\ 1 & A\end{pmatrix}\begin{pmatrix}d_{11} & d_{12}\\ d_{12} & d_{22}\end{pmatrix}\begin{pmatrix}0 & 1\\ B & A\end{pmatrix}\\ &= \begin{pmatrix}0 & B\\ 1 & A\end{pmatrix}\begin{pmatrix}Bd_{12} & d_{11}+Ad_{12}\\ Bd_{22} & d_{12}+Ad_{22}\end{pmatrix}\\ &= \begin{pmatrix}B^2 d_{22} & B\left(d_{12}+Ad_{22}\right)\\ B\left(d_{12}+Ad_{22}\right) & d_{11}+2Ad_{12}+A^2 d_{22}\end{pmatrix}.\end{aligned} \tag{3.16}$$

From (3.9), (3.10), (3.16) it follows that equation (3.10) can be transformed into the system of equations

$$\begin{aligned}B^2 d_{22}-d_{11} &= 0,\\ B\left(d_{12}+Ad_{22}\right)-d_{12} &= 0,\\ d_{11}+2Ad_{12}+A^2 d_{22}-d_{22} &= -1.\end{aligned} \tag{3.17}$$

The solution of system (3.17) has the form

$$\begin{aligned}d_{11} &= B^2 d_{22},\\ d_{12} &= \frac{AB}{1-B}d_{22},\\ d_{22} &= \left(\frac{1+B}{1-B}\left((1-B)^2-A^2\right)\right)^{-1}.\end{aligned} \tag{3.18}$$

Note that $d_{22}>0$ if and only if condition (3.5) holds. Substituting (3.18) into (3.15), we have

$$\mathbf{E}\Delta V_{i1} = -\mathbf{E}x_i^2+\sigma_1^2 d_{22}\mathbf{E}x_{i-m}^2+\gamma d_{22}\mathbf{E}x_{i-1-m}^2, \tag{3.19}$$

where

$$\gamma = \sigma_2^2 + 2\sigma_1\sigma_2\frac{A}{1-B}. \tag{3.20}$$

Put $\gamma_0 = \max(\gamma, 0)$ and choose the additional functional V_{2i} in the form

$$V_{2i} = d_{22}\left(\left(\sigma_1^2 + \gamma_0\right)\sum_{j=1}^{m} x_{i-j}^2 + \gamma_0 x_{i-1-m}^2\right). \tag{3.21}$$

Then

$$\begin{aligned}
\Delta V_{2i} &= d_{22}\left(\left(\sigma_1^2 + \gamma_0\right)\left(\sum_{j=1}^{m} x_{i+1-j}^2 - \sum_{j=1}^{m} x_{i-j}^2\right) + \gamma_0\left(x_{i-m}^2 - x_{i-1-m}^2\right)\right)\\
&= d_{22}\left(\left(\sigma_1^2 + \gamma_0\right)\left(x_i^2 - x_{i-m}^2\right) + \gamma_0\left(x_{i-m}^2 - x_{i-1-m}^2\right)\right)\\
&= d_{22}\left(\left(\sigma_1^2 + \gamma_0\right)x_i^2 - \sigma_1^2 x_{i-m}^2 - \gamma_0 x_{i-1-m}^2\right).
\end{aligned} \tag{3.22}$$

So, using (3.15), (3.22) for the functional $V_i = V_{1i} + V_{2i}$ we have

$$\mathbf{E}\Delta V_i = -\left(1 - d_{22}\left(\sigma_1^2 + \gamma_0\right)\right)\mathbf{E}x_i^2 + d_{22}\left(\gamma - \gamma_0\right)\mathbf{E}x_{i-1-m}^2. \tag{3.23}$$

If $\gamma \geq 0$ then $\gamma_0 = \gamma$ and, using (3.20), we obtain

$$\mathbf{E}\Delta V_i = -\left(1 - d_{22}\left(\sigma_1^2 + 2\sigma_1\sigma_2\frac{A}{1-B} + \sigma_2^2\right)\right)\mathbf{E}x_i^2. \tag{3.24}$$

From here and representation (3.17) for d_{22} it follows [22] that the inequality

$$\sigma_1^2 + 2\sigma_1\sigma_2\frac{A}{1-B} + \sigma_2^2 < \frac{1+B}{1-B}\left((1-B)^2 - A^2\right) \tag{3.25}$$

is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of equation (3.3).

Consider the situation if $\gamma < 0$. In this case $\gamma_0 = 0$ and (3.23) takes the form

$$\mathbf{E}\Delta V_i = -\left(1 - \sigma_1^2 d_{22}\right)\mathbf{E}x_i^2 + \gamma d_{22}\mathbf{E}x_{i-1-m}^2. \tag{3.26}$$

So, if $\gamma < 0$ then the inequality $\sigma_1^2 d_{22} < 1$ is a sufficient condition of asymptotic mean square stability of the trivial solution of equation (3.3). Let us suppose that $\gamma < 0$ and $\sigma_1^2 d_{22} \geq 1$. Summing (3.26) from $i = 0$ to $i = n$, we have

$$\begin{aligned}
\mathbf{E}V_{n+1} - \mathbf{E}V_0 &= -\left(1 - \sigma_1^2 d_{''2}\right)\sum_{i=0}^{n}\mathbf{E}x_i^2\\
&\quad + \gamma d_{22}\left(\sum_{i=0}^{n-1-m}\mathbf{E}x_i^2 + \sum_{i=-1-m}^{-1}\mathbf{E}x_i^2\right).
\end{aligned}$$

From here, using $V_{n+1} \geq 0$ and $\gamma < 0$, we obtain

$$\left(1 - \sigma_1^2 d_{22}\right)\sum_{i=0}^{n}\mathbf{E}x_i^2 - \gamma d_{22}\sum_{i=0}^{n-1-m}\mathbf{E}x_i^2 \leq \mathbf{E}V_0.$$

or

$$\left(1-d_{22}\left(\sigma_1^2+\gamma\right)\right)\sum_{i=0}^{n}\mathbf{E}x_i^2 \le \mathbf{E}V_0+|\gamma|d_{22}\sum_{i=n-m}^{n}\mathbf{E}x_i^2.$$

Note that by virtue of (3.5) we have

$$\begin{aligned}\sigma_1^2+\gamma &= \sigma_1^2+2\sigma_1\sigma_2\frac{A}{1-B}+\sigma_2^2\\ &> \sigma_1^2-2\left|\sigma_1\sigma_2\right|+\sigma_2^2=\left(|\sigma_1|-|\sigma_2|\right)^2\ge 0.\end{aligned}$$

Therefore, by condition (3.25), that is equivalent to $d_{22}\left(\sigma_1^2+\gamma\right)<1$, each mean square bounded solution of equation (3.3), i.e. $\mathbf{E}x_i^2\le C$, satisfies the condition $\lim_{i\to\infty}\mathbf{E}x_i^2=0$.

So, in any case by condition (3.25) the mean square bounded solution of equation (3.3) is asymptotically mean square trivial, i.e. $\lim_{i\to\infty}\mathbf{E}x_i^2=0.$. Note also that for $\sigma_2=0$ condition (3.25) coincides with (3.6).

Using (3.4), (3.5), we rewrite condition (3.25) in terms of the parameters of equation (3.1):

$$\begin{gathered}\sigma^2<(1-a+b\theta)\left(1+a-b\left(\theta-\frac{1}{2}\right)\right),\\ b\left(\theta-\frac{1}{2}\right)-1<a<b\theta+1,\\ -4<b<0.\end{gathered}\tag{3.27}$$

If $b\to 0$ then condition (3.27) takes the form (3.2). Conditions (3.27), (3.2) can be written also in the form

$$\left(a-\left(\theta-\frac{1}{4}\right)b\right)^2+\sigma^2<\left(1+\frac{b}{4}\right)^2$$

or

$$\begin{gathered}\left(\theta-\frac{1}{4}\right)b-\sqrt{\left(1+\frac{b}{4}\right)^2-\sigma^2}<a\\ <\left(\theta-\frac{1}{4}\right)b+\sqrt{\left(1+\frac{b}{4}\right)^2-\sigma^2},\\ -4\left(1-|\sigma|\right)<b\le 0.\end{gathered}\tag{3.28}$$

Stability regions, obtained by virtue of condition (3.28) for $\sigma=0$ and different values of θ are shown in figure 3.1 with the following key:

1. $\theta=0$,
2. $\theta=0.25$,
3. $\theta=0.5$,
4. $\theta=0.75$,

5. $\theta = 1$.

Stability regions, obtained by virtue of condition (3.28) for $\theta = 1$ and different values of σ^2 are shown in figure 3.2 with the following key:

1. $\sigma^2 = 0$,
2. $\sigma^2 = 0.1$,
3. $\sigma^2 = 0.2$,
4. $\sigma^2 = 0.3$,
5. $\sigma^2 = 0.4$,
6. $\sigma^2 = 0.5$,
7. $\sigma^2 = 0.6$,
8. $\sigma^2 = 0.7$,
9. $\sigma^2 = 0.8$,
10. $\sigma^2 = 0.9$.

Figure 3.3 uses the same key as figure 3.2 and is for $\theta = 0.375$.

Remark 3.1 Note that the stability region, given by condition (3.28) depends on θ and σ, but the area S of this stability region depends on σ only and does not depend on θ, i.e. $S = S(\sigma)$. It is easy to see that

$$\begin{aligned} S(\sigma) &= 2\int_{-4(1-|\sigma|)}^{0} \sqrt{\left(1+\frac{b}{4}\right)^2 - \sigma^2 db} \\ &= 8\int_{|\sigma|}^{1} \sqrt{x^2-\sigma^2}dx. \end{aligned}$$

Putting $t = x + \sqrt{x^2-\sigma^2}$, we obtain

$$\begin{aligned} S(\sigma) &= 2\int_{|\sigma|}^{1+\sqrt{1-\sigma^2}} \frac{\left(t^2-\sigma^2\right)^2}{t^3} dt \\ &= 2\int_{|\sigma|}^{1+\sqrt{1-\sigma^2}} \left(t + \frac{\sigma^4}{t^3} - \frac{2\sigma^2}{t}\right) dt \\ &= \left[\frac{t^4-\sigma^4}{t^2} - 4\sigma^2 \ln(t)\right]_{|\sigma|}^{1+\sqrt{1-\sigma^2}} \\ &= 4\left(\sqrt{1-\sigma^2} - \sigma^2 \ln\left(\frac{1+\sqrt{1-\sigma^2}}{|\sigma|}\right)\right). \end{aligned}$$

In particular, $S(0) = 4$, $S(0.5) = 2\sqrt{3} - \ln\left(2+\sqrt{3}\right) > 2$, $S(1) = 0$.

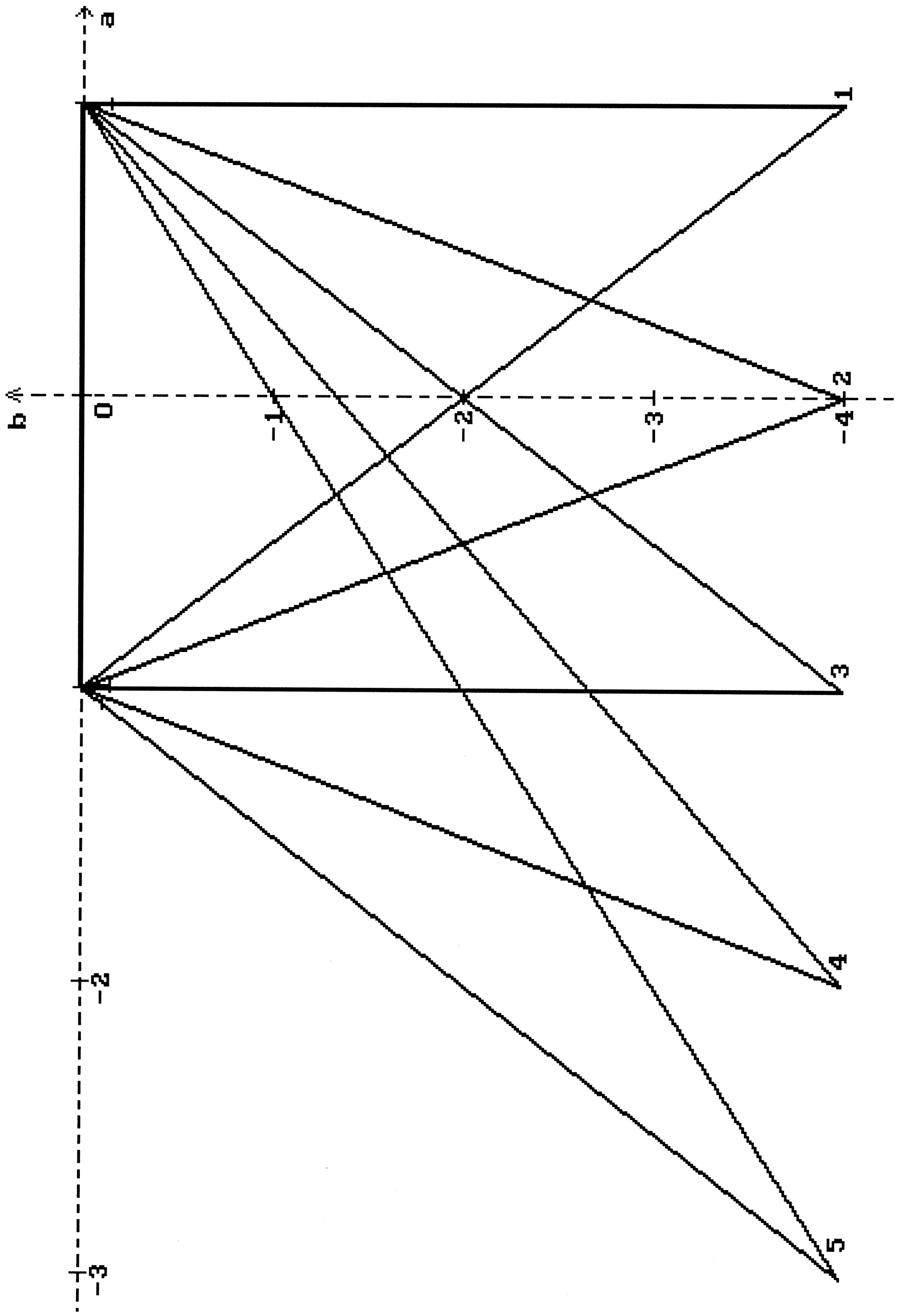

Figure 3.1: Stability diagram, $\sigma = 0$, differing θ values

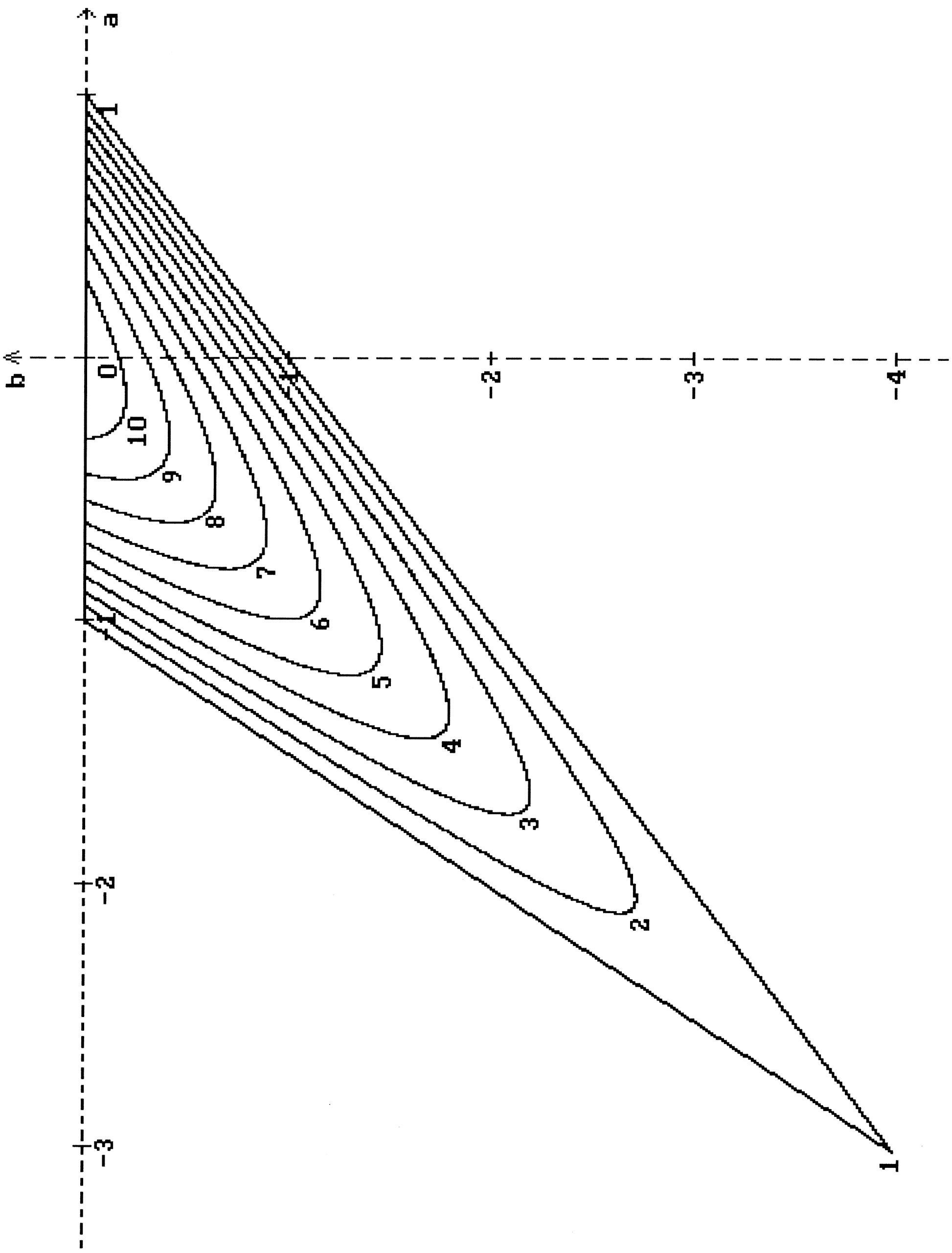

Figure 3.2: Stability diagram, $\theta = 1$, differing σ^2 values

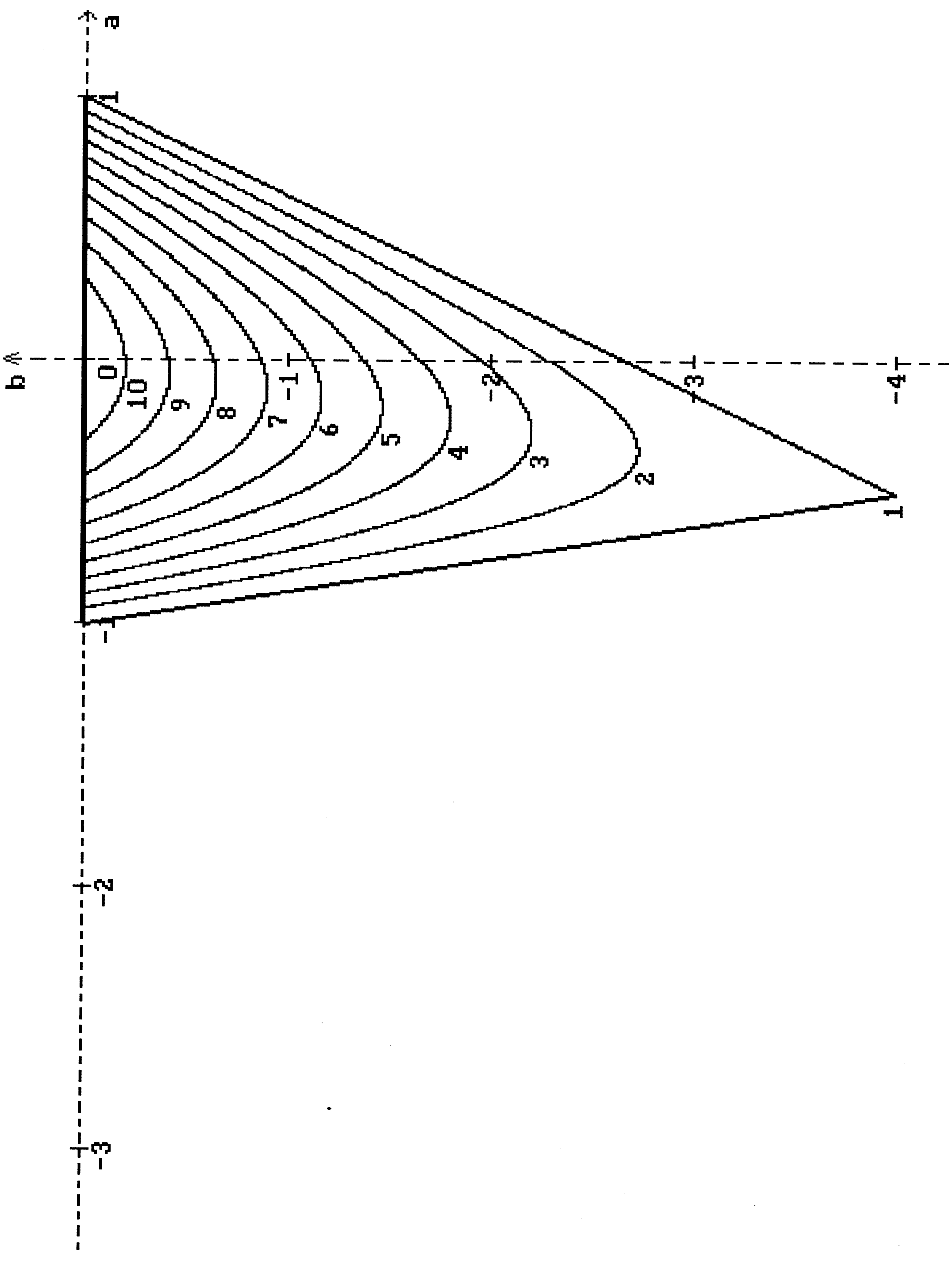

Figure 3.3: Stability diagram, $\theta = 0.375$, differing σ^2 values

4 Stability of difference analogues to the integro-differential equation

If we discretise equation (1.1) using a numerical method based on the Euler-Maruyama scheme for the stochastic differential equation part and a θ method to approximate the integral with a quadrature, then we obtain a family of numerical methods of the form

$$x_1 = \left(1 + \alpha h + \beta h^2\right) x_0 + \sigma h^{\frac{1}{2}} x_{-m} \xi_0$$

$$x_2 = (1 + \alpha h)x_1 + \beta h^2 \left(\theta x_0 + (1 - \theta)x_1\right) + \sigma h^{\frac{1}{2}} x_{1-m} \xi_1$$

$$x_{i+1} = (1 + \alpha h)\, x_i + \beta h^2 \left(\theta x_0 + \sum_{j=1}^{i-1} x_j + (1 - \theta)\, x_i \right) + \sigma h^{\frac{1}{2}} x_{i-m} \xi_i, \qquad (4.1)$$

$$i \geq 2,$$

where $\theta \in [0, 1]$, $\tau = mh$, $\xi_i = h^{-\frac{1}{2}} \left(\xi\left(t_{i+1}\right) - \xi\left(t_i\right)\right)$. Equation (4.1) has the form of equation (3.1) with

$$a = 1 + \alpha h,\ b = \beta h^2,\ \sigma \to \sigma h^{\frac{1}{2}}$$

Stability condition (3.28) for equation (4.1) therefore takes the form

$$\frac{1}{h}\left(-1 + \left(\theta - \frac{1}{4}\right)\beta h^2 - \sqrt{\left(1 + \frac{1}{4}\beta h^2\right)^2 - \sigma h} \right) < \alpha$$

$$< \frac{1}{h}\left(-1 + \left(\theta - \frac{1}{4}\right)\beta h^2 + \sqrt{\left(1 + \frac{1}{4}\beta h^2\right)^2 - \sigma^2 h} \right),$$

$$-4\left(1 - |\sigma| h^{\frac{1}{2}}\right) < \beta h^2 \leq 0. \qquad (4.2)$$

The stability regions in the (α, β) space, obtained by condition (4.2) for $\theta = 1$, $\sigma^2 = 0$ are shown in figure 4.1 for different values of the stepsize h of the numerical method, using the following key:

1. $h = 0$,
2. $h = 0.01$,
3. $h = 0.02$,
4. $h = 0.03$,
5. $h = 0.04$,
6. $h = 0.05$,
7. $h = 0.06$,

8. $h = 0.07$,
9. $h = 0.08$,
10. $h = 0.1$,
11. $h = 0.15$.

Figure 4.2 and figure 4.3 show similar pictures with $\theta = 1$ and h as indicated above but with $\sigma^2 = 1$ and $\sigma^2 = 3$ respectively.

Figure 4.4 illustrates the stability region in the (α, β) space for $\sigma^2 = 0$, $h = 0.05$ and different values θ (i.e. different numerical schemes) according to the following key:

1. $\theta = 0$,
2. $\theta = 0.25$,
3. $\theta = 0.5$,
4. $\theta = 0.75$,
5. $\theta = 1$.

Figure 4.5 is a similar picture for the same values of h and θ but with $\sigma^2 = 1$.

If we calculate the infimum with respect to θ in the left-hand part and the supremum in the right-hand part of inequalities (4.2) we obtain

$$\frac{1}{h}\left(-1 + \frac{3}{4}\beta h^2 - \sqrt{\left(1 + \frac{1}{4}\beta h^2\right)^2 - \sigma h}\right) < \alpha$$

$$< \frac{1}{h}\left(-1 - \frac{1}{4}\beta h^2 + \sqrt{\left(1 + \frac{1}{4}\beta h^2\right)^2 - \sigma^2 h}\right),$$

$$-4\left(1 - |\sigma| h^{\frac{1}{2}}\right) < \beta h^2 \leq 0. \tag{4.3}$$

It is easy to check that if $h \to 0$ then condition (4.3) coincides with condition (2.10). It leads to the useful statement.

Theorem 4.1 *If α, β and σ satisfy condition (2.10) then there exists a small enough h that condition (4.3) holds too. And if α, β, σ and h satisfy condition (4.3) then there exists a $\theta \in [0, 1]$ such that condition (4.2) holds too and therefore the trivial solution of equation (4.1) is asymptotically mean square stable.*

The stability regions obtained by condition (4.3) for $h = 0.1$ and different values of σ are shown in figure 4.6, according to the following key:

1. $\sigma^2 = 0.5$,

2. $\sigma^2 = 1$,

3. $\sigma^2 = 2$,

4. $\sigma^2 = 3$.

Figure 4.7 shows a similar picture for $\sigma^2 = 1$ and different values of h:

1. $h = 0.1$,

2. $h = 0.065$,

3. $h = 0.045$,

4. $h = 0.035$.

5 Upper bound for the step of discretisation

From condition (4.2) it follows

$$f(h) = \theta\left(\theta - \frac{1}{2}\right)\beta^2 h^3 - \left(2\theta - \frac{1}{2}\right)\alpha\beta h^2 + \left(\alpha^2 - 2\beta\theta\right)h + 2\alpha + \sigma^2 > 0. \quad (5.1)$$

Using the representation (5.1) consider different possible cases for determining an upper bound for the step of discretisation.

5.1

Let $\beta = 0$. From (5.1), (2.10) we obtain

$$f(h) = \alpha^2 h + 2\alpha + \sigma^2 < 0$$

for $h \in [0, h_1)$, where

$$h_1 = -\frac{2\alpha + \sigma^2}{\alpha^2} > 0.$$

For example, if $\alpha = -30$, $\beta = 0$, $\sigma^2 = 1$ then $h_1 \approx 0.0656$. Changing α to $\alpha = -40$, we obtain $h_1 \approx 0.0494$. On figure 5.1 which coincides with figure 4.5 ($\sigma^2 = 1$, $h = 0.05$) the points $A_1(-30, 0)$ and $A_2(-40, 0)$ are shown. One can see that the point A_1 belongs to the stability region but the point A_2 does not belong since $h = 0.05 > h_1 = 0.0494$.

Suppose now that $\beta < 0$ and consider the following possibilities for θ.

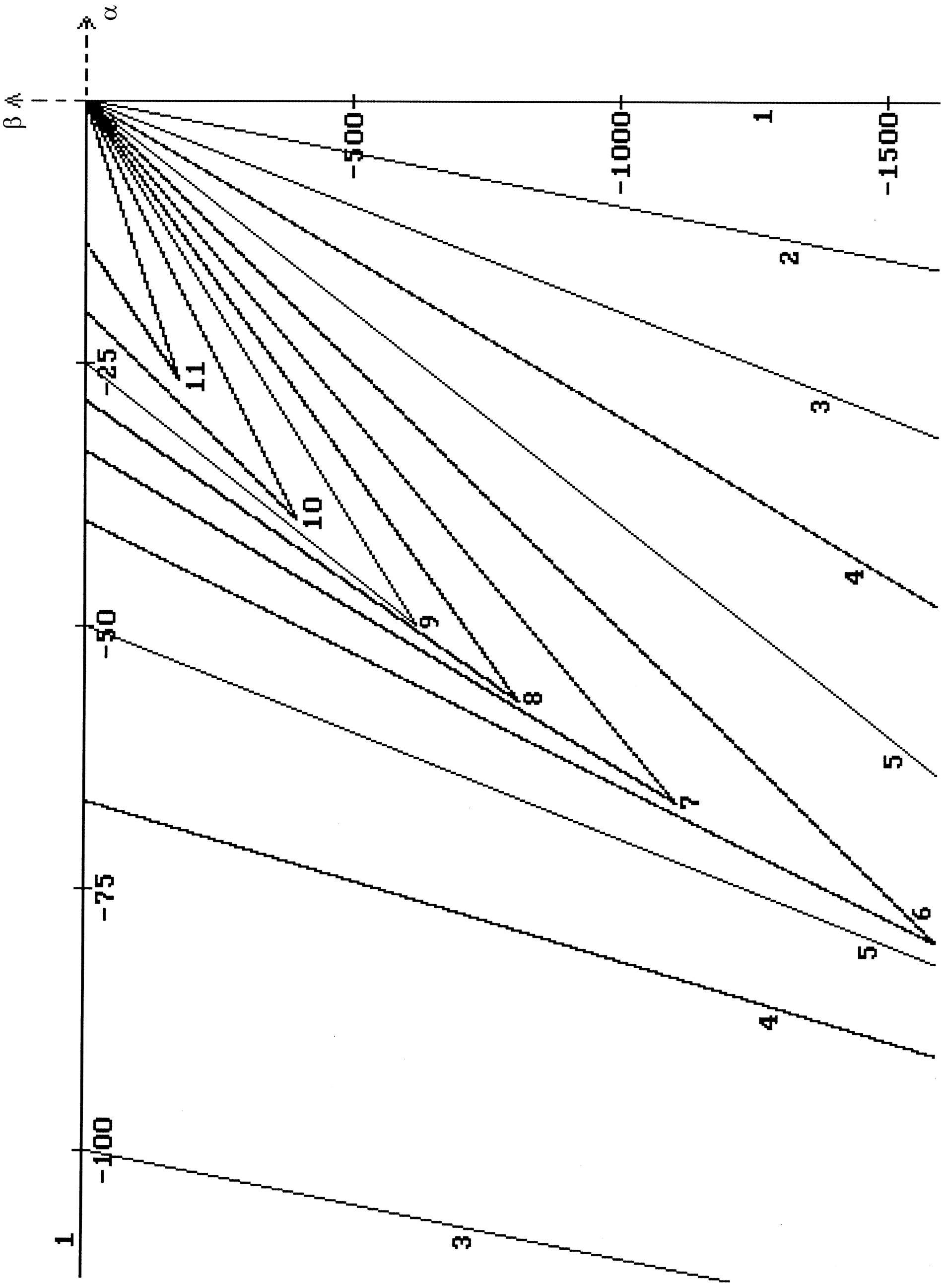

Figure 4.1: Stability diagram, $\theta = 1$, $\sigma^2 = 0$, differing h values

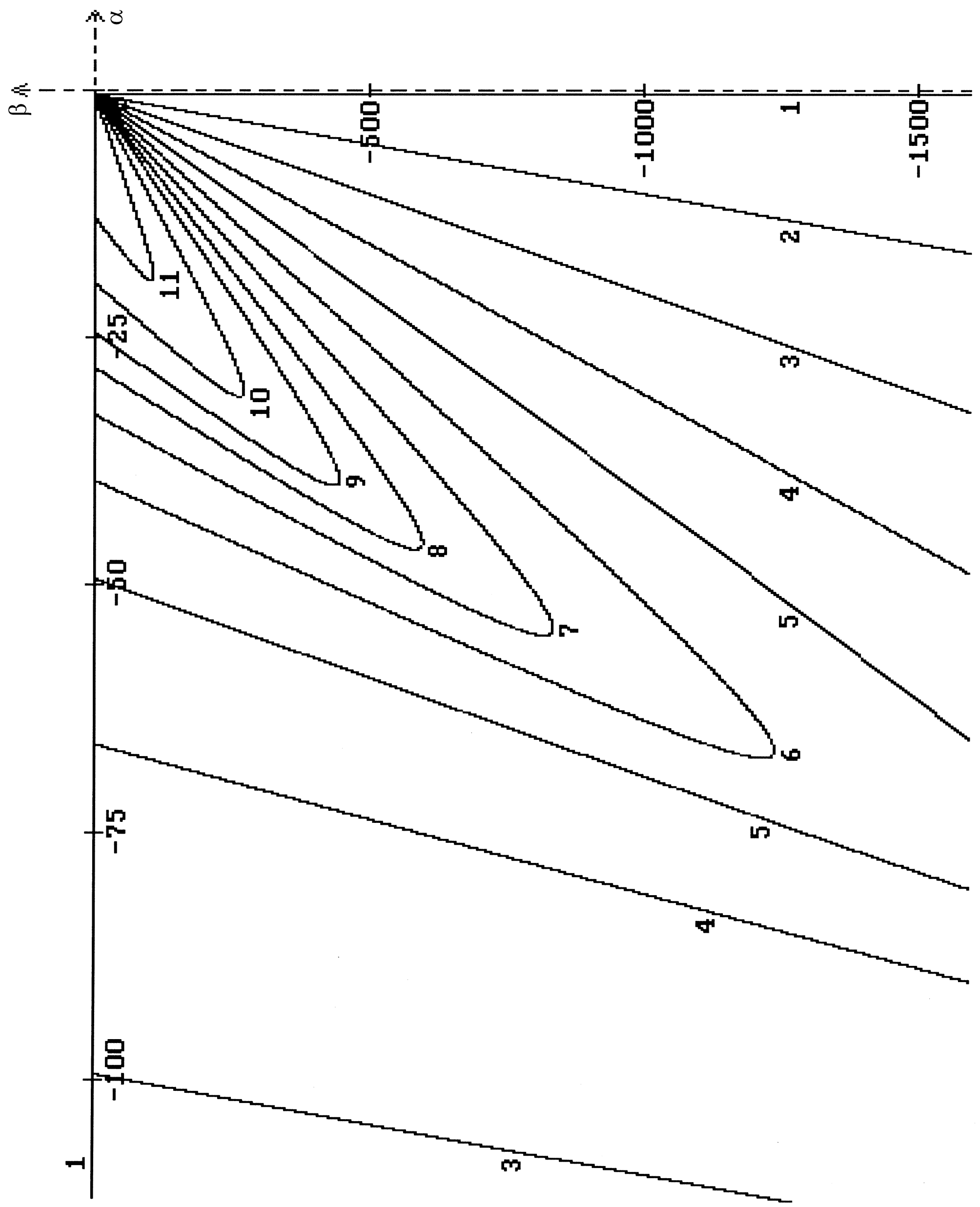

Figure 4.2: Stability diagram, $\theta = 1$, $\sigma^2 = 1$, differing h values

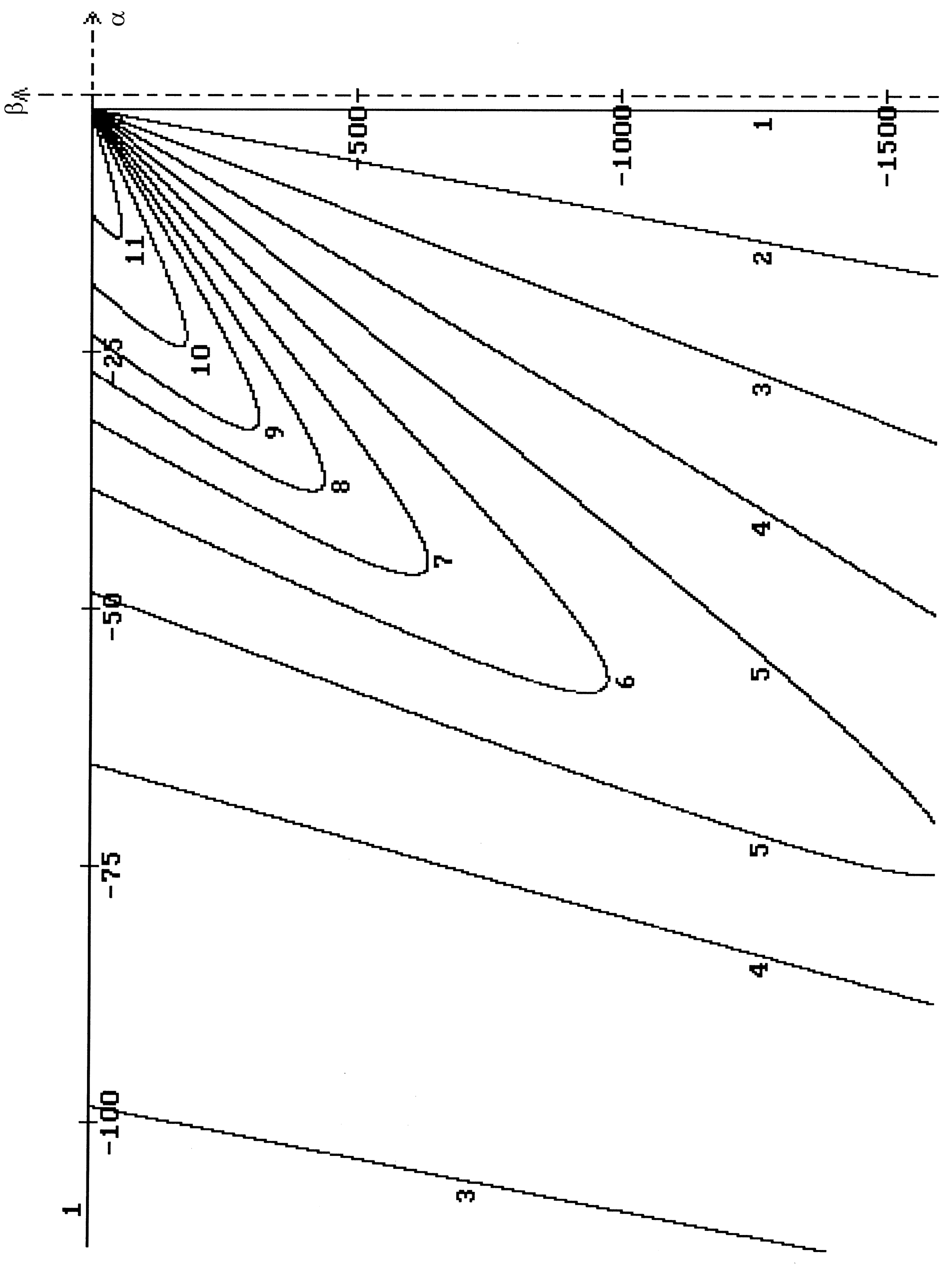

Figure 4.3: Stability diagram, $\theta = 1$, $\sigma^2 = 3$, differing h values

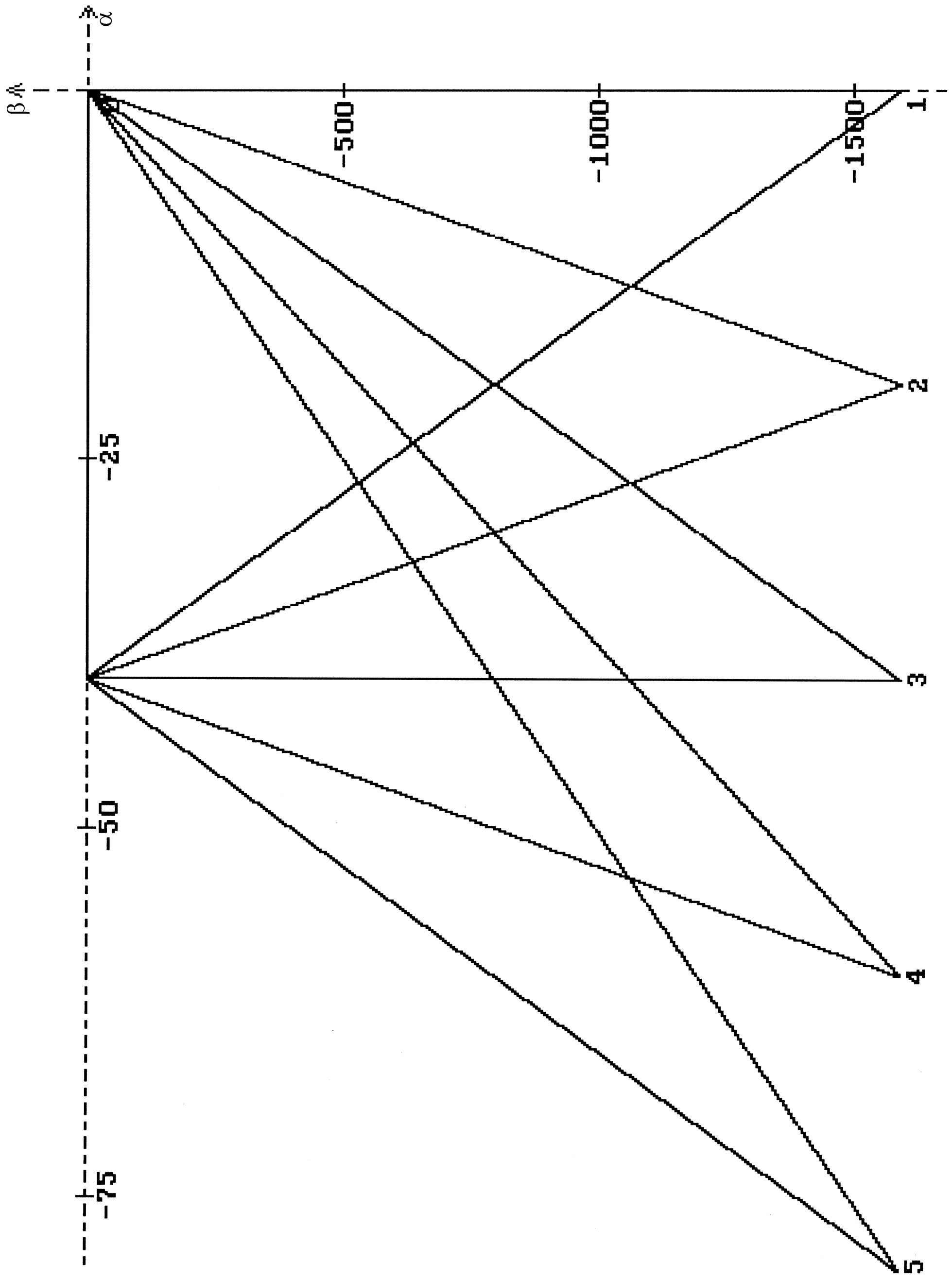

Figure 4.4: Stability diagram, $\sigma^2 = 0$, $h = 0.05$, differing θ values

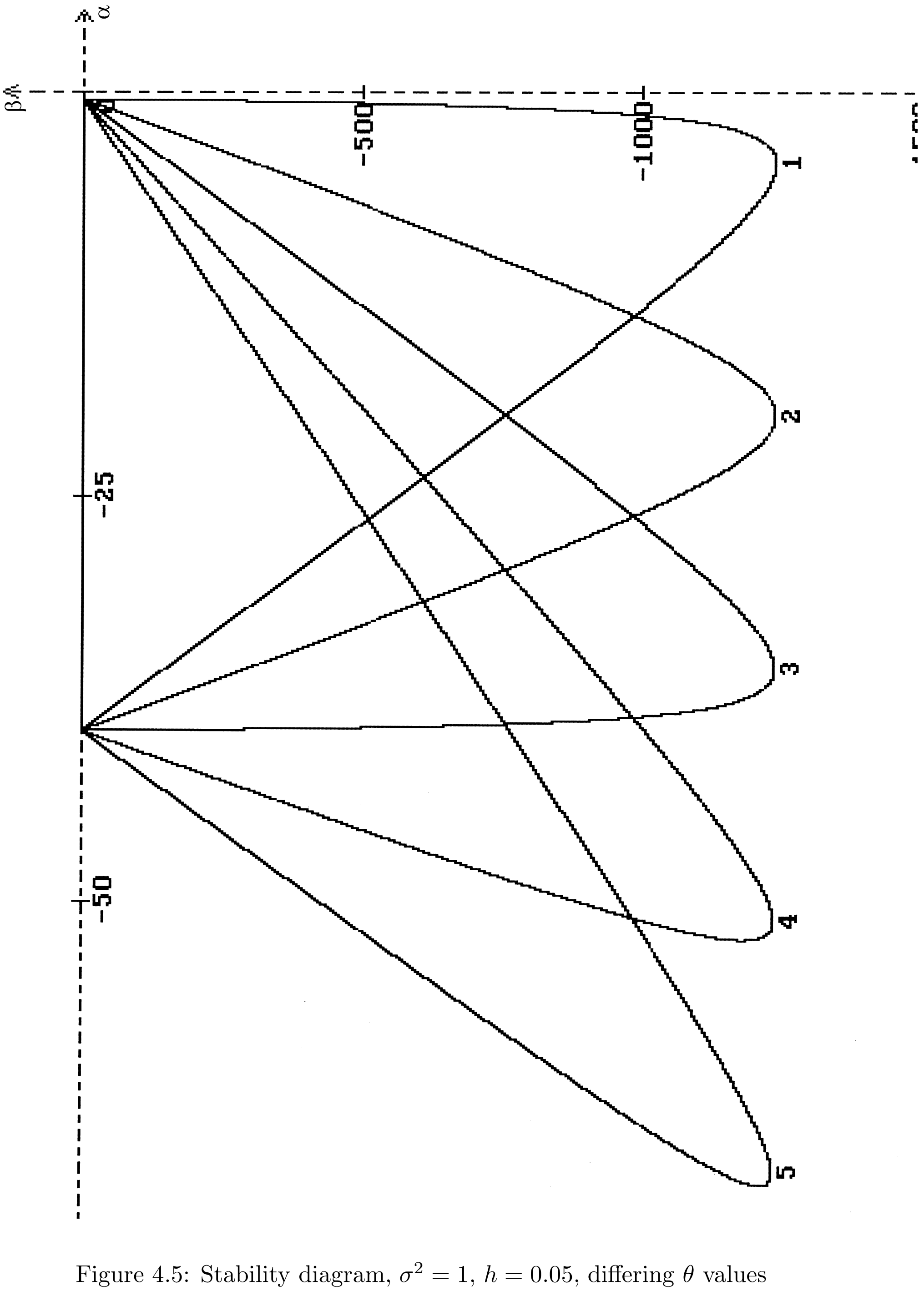

Figure 4.5: Stability diagram, $\sigma^2 = 1$, $h = 0.05$, differing θ values

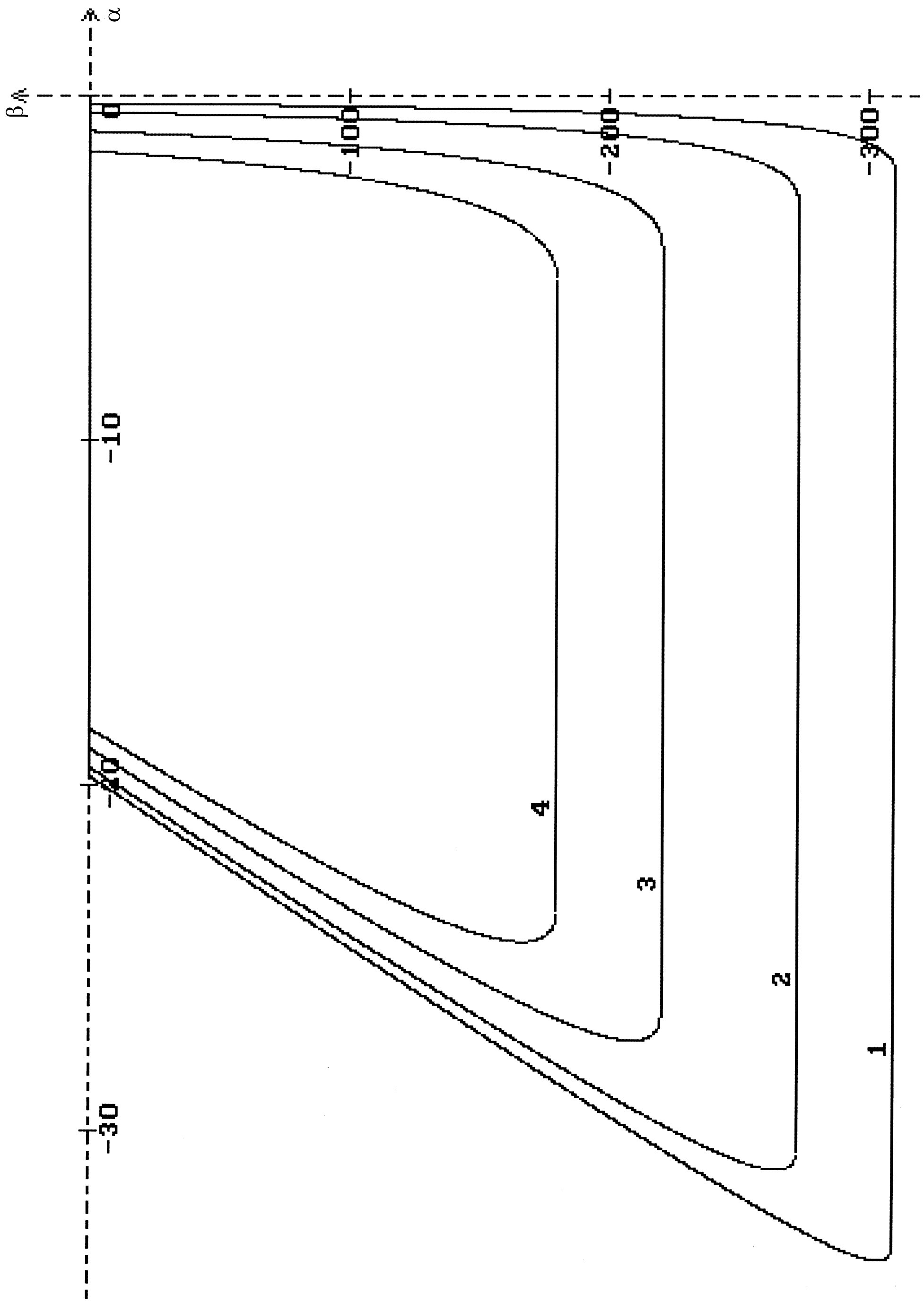

Figure 4.6: Stability diagram, $h = 0.1$, differing σ^2 values

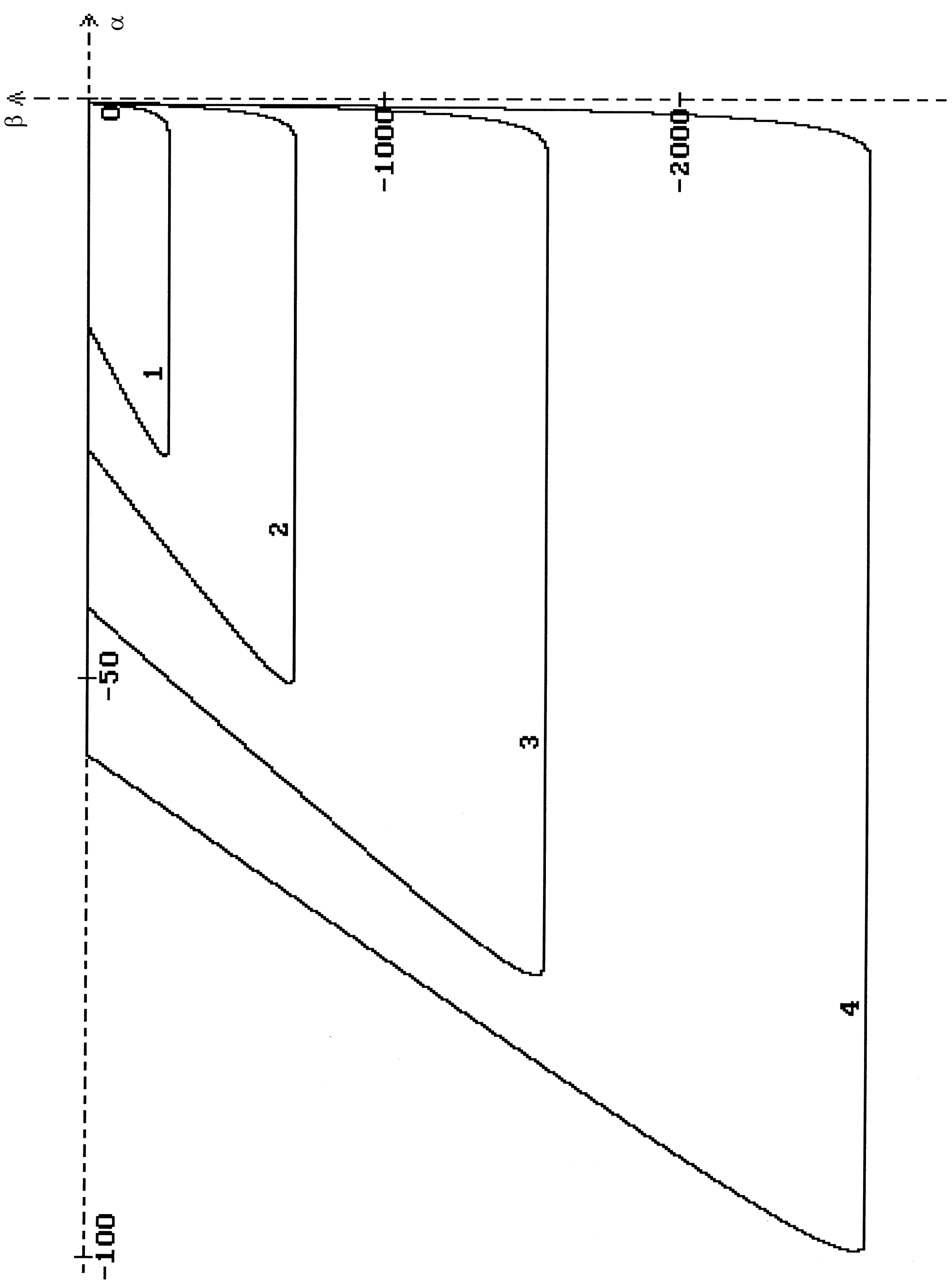

Figure 4.7: Stability diagram, $\sigma^2 = 1$, differing h values

5.2

Let $\theta = 0$. Then
$$f(h) = \frac{1}{2}\alpha\beta h^2 + \alpha^2 h + 2\alpha + \sigma^2.$$
Since $2\alpha + \sigma^2 < 0$ and $\alpha\beta > 0$ then $f(h) < 0$ for $h \in [0, h_1)$, where
$$h_1 = \frac{\sqrt{\alpha^4 - 2\alpha\beta\left(2\alpha + \sigma^2\right)} - \sigma^2}{\alpha\beta} > 0.$$
For example, if $\alpha = -10$, $\beta = -1000$, $\sigma^2 = 1$ then $h_1 \approx 0.0524$. Changing β to $\beta = -1200$ we obtain $h_1 \approx 0.0486 < 0.05$. On figure 5.1 the point $B_1(-10, -1000)$ belongs to the stability region with $\theta = 0$ and the point $B_2(-10, -1200)$ does not belong.

5.3

Let $\theta = \frac{1}{2}$. Then
$$f(h) = -\frac{1}{2}\alpha\beta h^2 + \left(\alpha^2 - \beta\right) h + 2\alpha + \sigma^2.$$
Since
$$D = \left(\alpha^2 - \beta\right)^2 + 2\alpha\beta\left(2\alpha + \sigma^2\right) = \left(\alpha^2 + \beta\right)^2 + 2\alpha\beta\sigma^2 > 0$$
then $f(h) < 0$ for $h \in [0, h_1)$, where
$$h_1 = \frac{\alpha^2 - \beta - \sqrt{D}}{\alpha\beta} > 0.$$
For example, if $\alpha = -30$, $\beta = -1000$, $\sigma^2 = 1$ then $h_1 \approx 0.0545$. Changing β on $\beta = -1200$ we obtain $h_1 \approx 0.0472$. On figure 5.1 the point $C_1(-30, 1000)$ belongs to the stability region with $\theta = \frac{1}{2}$ and the point $C_2(-30, -1200)$ does not belong to this region.

5.4

Let $\theta \in \left(\frac{1}{2}, 1\right]$. From (5.1) and (2.10) it follows that $f(h) < 0$ for $h \leq 0$. So $f(h) < 0$ for $h \in [0, h_1)$, where h_1 is the least root of the equation $f(h) = 0$. For example, if $\alpha = -40$, $\beta = -1000$, $\sigma^2 = 1$, $\theta = 0.75$ we obtain
$$f(h) = 187500h^3 - 40000h^2 + 3100h - 79 = 0$$
and $h_1 \approx 0.0511$. Changing β to $\beta = -1200$ we obtain
$$f(h) = 270000h^3 - 48000h^2 + 3400h - 79 = 0$$
with $h_1 \approx 0.0431$. On figure 5.1 the point $D_1(-40, -1000)$ belongs to the stability region with $\theta = \frac{3}{4}$ but the point $D_2(-40, -1200)$ does not belong to this region.

5.5

Let $\theta \in \left(0, \frac{1}{2}\right)$. From (5.1) and (2.10) it follows that $f(0) < 0$ and $\frac{df}{dh}(0) > 0$. It means that $f(h) < 0$ for $h \in [0, h_1)$ where h_1 is the least positive root of the equation $f(h) = 0$. For example, if $\alpha = -20$, $\beta = -1200$, $\sigma^2 = 1$, $\theta = \frac{1}{4}$ then

$$f(h) = -90000h^3 + 1000h - 39$$

and $h_1 \approx 0.0508$. Changing β to $\beta = -1300$ we obtain

$$f(h) = -105625h^3 + 1050h - 39 = 0$$

with $h_1 \approx 0.0489$. On figure 5.1 the point $E_1(-20, -1000)$ belongs to the stability region with $\theta = \frac{1}{4}$ but the point $E_2(-20, -1300)$ does not belong to this region.

6 Numerical experiments

We illustrate some of our results with trajectories of equation (4.1).

Figure 6.1 shows 50 trajectories of equation (4.1) with $m = 0$ (i.e. without delay), $x_0 = 1$, $\alpha = -55$, $\beta = -1000$, $\sigma^2 = 1$, $h = 0.05$, $\theta = 1$. The dark line represents the arithmetic mean of the trajectories, as it does for all the figures in this section. It is clear that we have a stable system. If we change the parameter h to $h = 0.06$ we suddenly don't have a stable system (as shown in figure 6.3), as expected from examining figure 4.2. Figures 6.2 and 6.4 illustrate the trajectories of the Wiener process $W(t_k) = h^{\frac{1}{2}} \sum_{i=0}^{k-1} \xi_i$, $k = 0, 1, \ldots$, where ξ_i are mutually independent, normally distributed random variables with mean zero and unit variance.

Figure 4.5 shows the regions of stability for different θ methods. We illustrate this point with figures 6.5, 6.6 and 6.7. Each figure shows 50 trajectories with identical parameter values except for θ. For figure 6.5 $\theta = 0$, for figure 6.6 $\theta = 1$, and for figure 6.7 $\theta = 0.5$. The interesting point here is that for particular parameter values where the integro-differential equation is mean square stable we can choose a θ method which replicates this stability property. In figure 6.5 the sufficient conditions for asymptotic mean square stability of the discrete system (i.e. $-38.8603 < \alpha < -0.5147$, given the other parameters) are not satisfied and the trajectories are indeed unstable, whereas in figures 6.6 and 6.7 the conditions (i.e. $-40.1103 < \alpha < -1.7647$ for 6.6 and $-39.4853 < \alpha < -1.1397$ for 6.7, given the other parameters) are satisfied and we have asymptotic mean-square stability. Figure 6.8 uses the same parameters as figure 6.5 except that $\alpha = -39$. In this case the sufficient conditions are not satisfied for the discrete analogue (we are very close to satisfying them though) but we still have asymptotic mean square stability, thus verifying that our conditions are only sufficient and not necessary and sufficient. However we believe our experiments indicate that the sufficient conditions are very good ones.

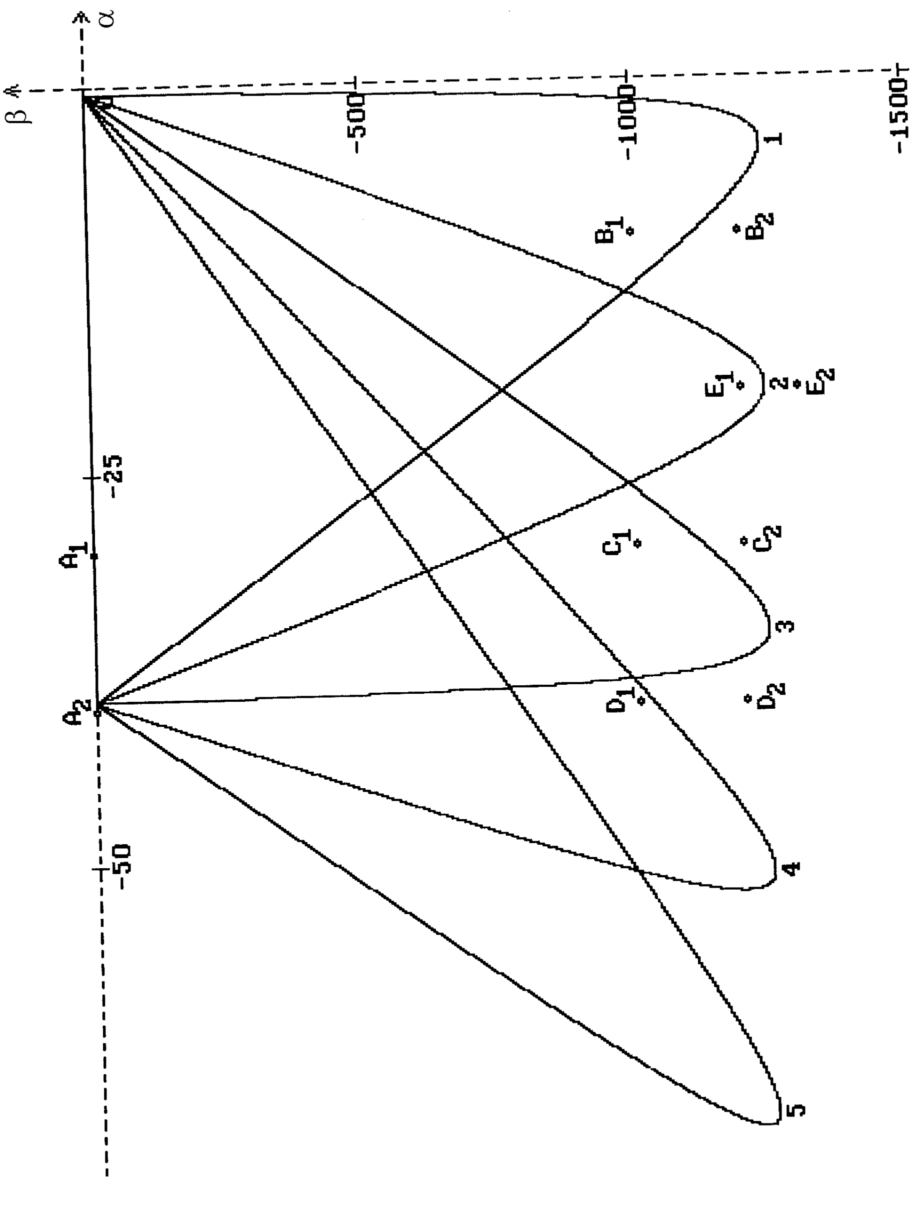

Figure 5.1: Stability diagram, $\sigma^2 = 1$, $h = 0.05$, differing θ values

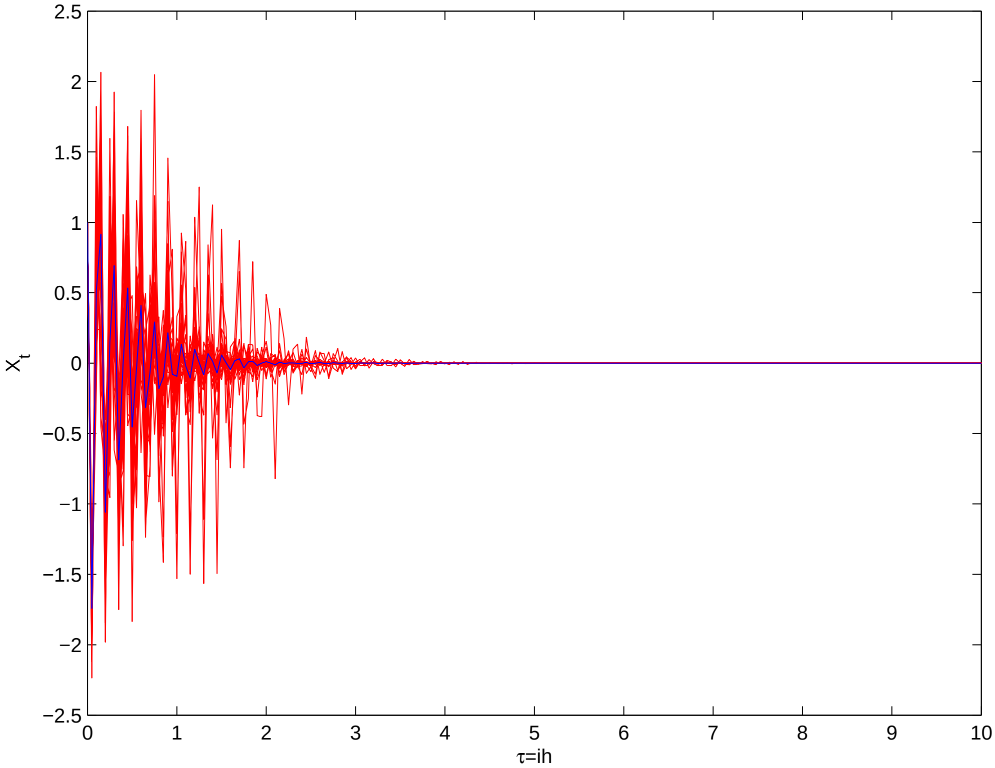

Figure 6.1: Trajectories of equation (4.1) with $m = 0$, $\alpha = -55$, $\beta = -1000$, $\sigma^2 = 1$, $h = 0.05$, $\theta = 1$, $x_0 = 1$

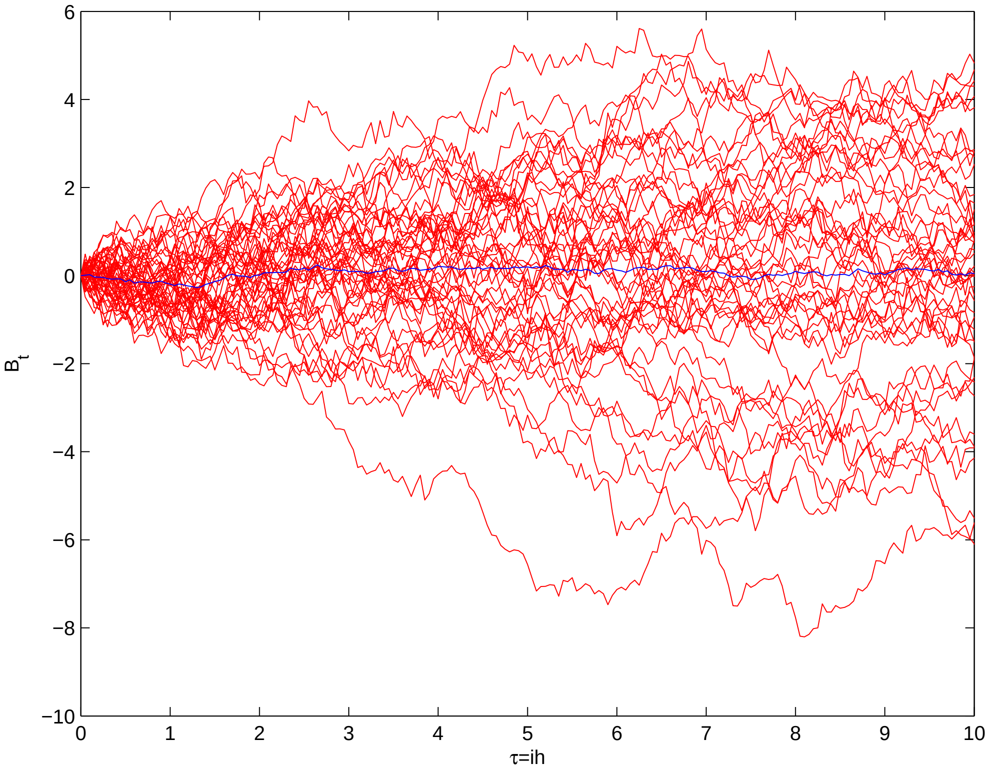

Figure 6.2: Trajectories of Wiener process for figure 6.1

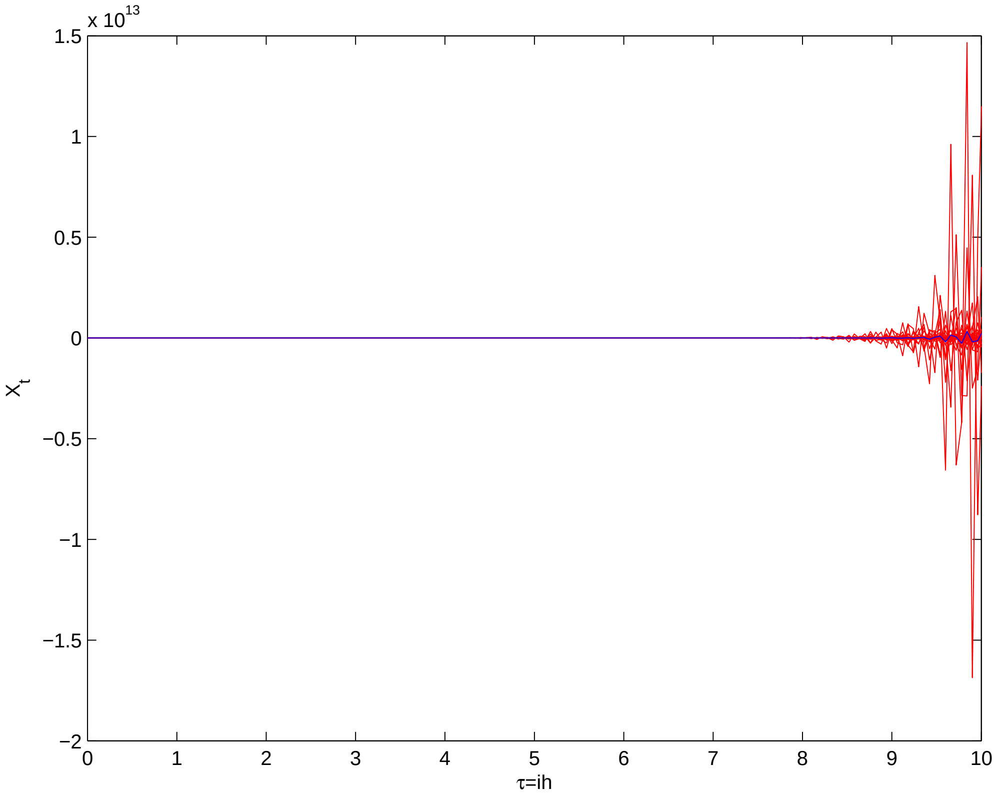

Figure 6.3: Trajectories of equation (4.1) with $m = 0$, $\alpha = -55$, $\beta = -1000$, $\sigma^2 = 1$, $h = 0.06$, $\theta = 1$, $x_0 = 1$

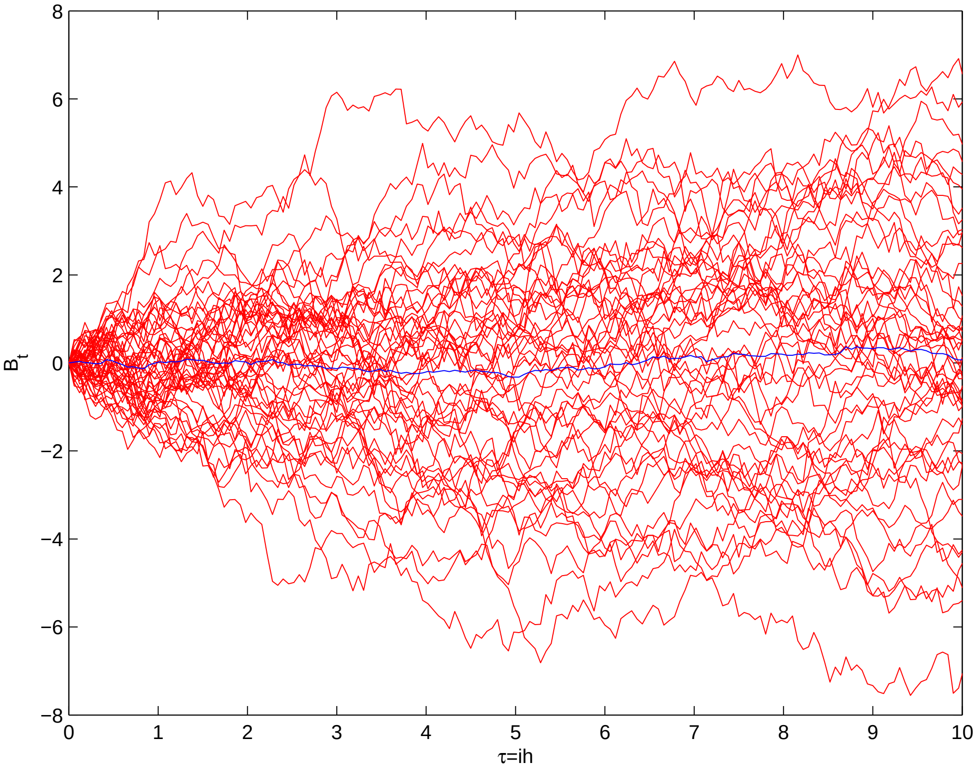

Figure 6.4: Trajectories of Wiener process for figure 6.3

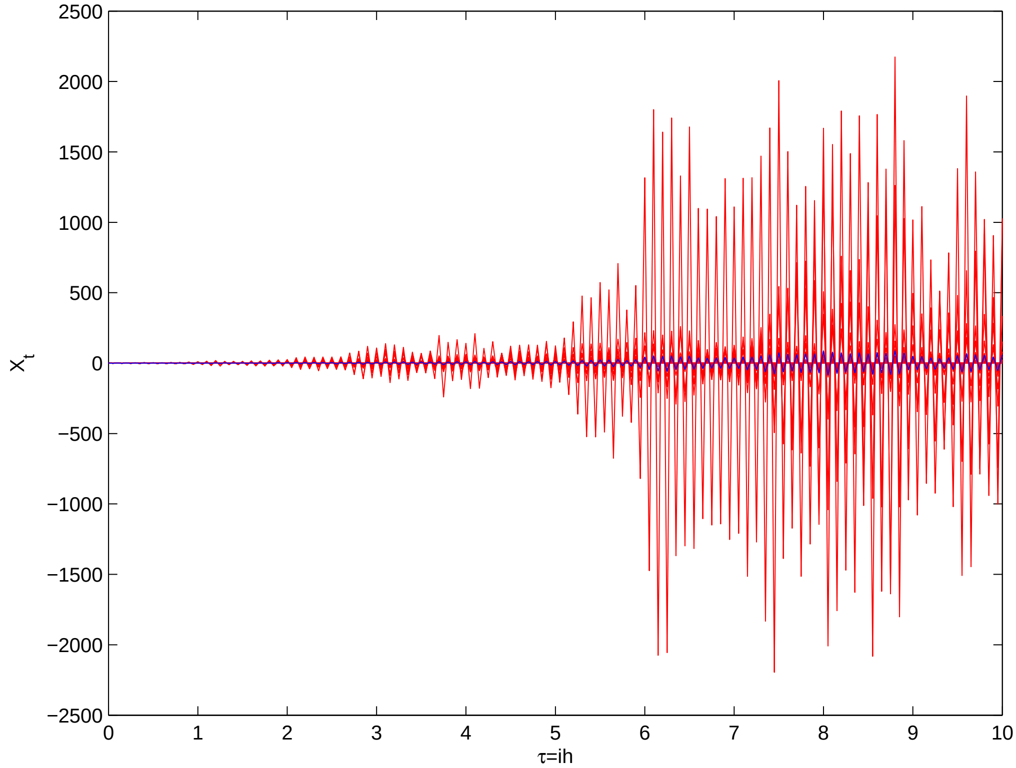

Figure 6.5: Trajectories of equation (4.1) with $m = 0$, $\alpha = -40$, $\beta = -25$, $\sigma^2 = 1$, $h = 0.05$, $\theta = 0$, $x_0 = 1$

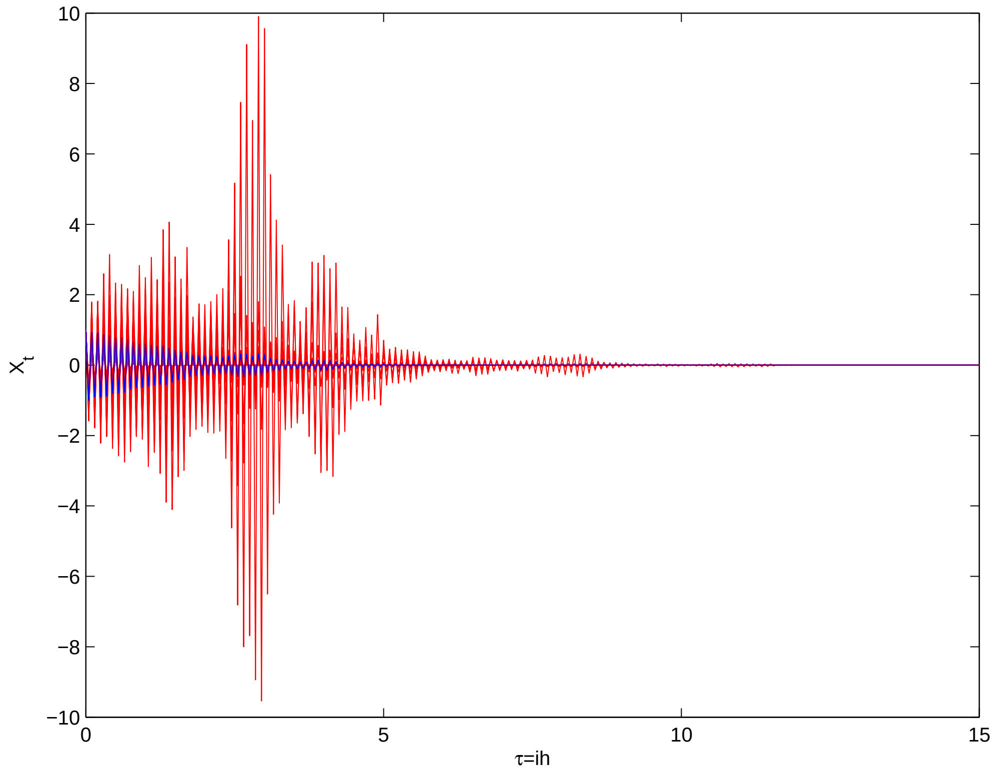

Figure 6.6: Trajectories of equation (4.1) with $m = 0$, $\alpha = -40$, $\beta = -25$, $\sigma^2 = 1$, $h = 0.05$, $\theta = 1$, $x_0 = 1$

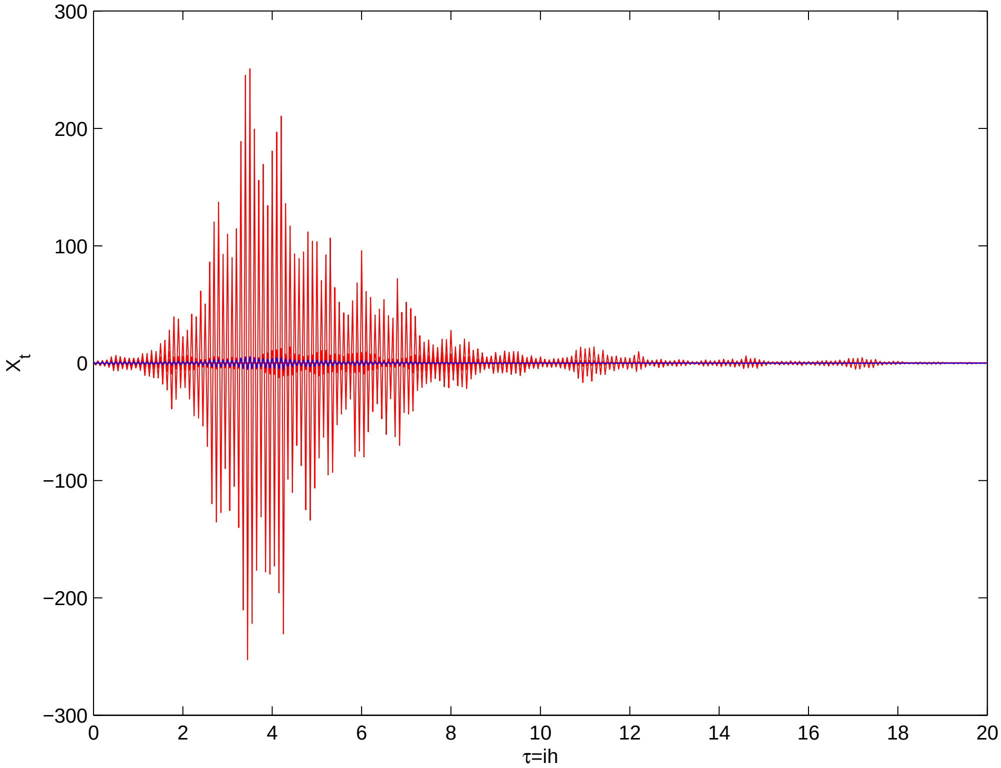

Figure 6.7: Trajectories of equation (4.1) with $m = 0$, $\alpha = -40$, $\beta = -25$, $\sigma^2 = 1$, $h = 0.05$, $\theta = 0.5$, $x_0 = 1$

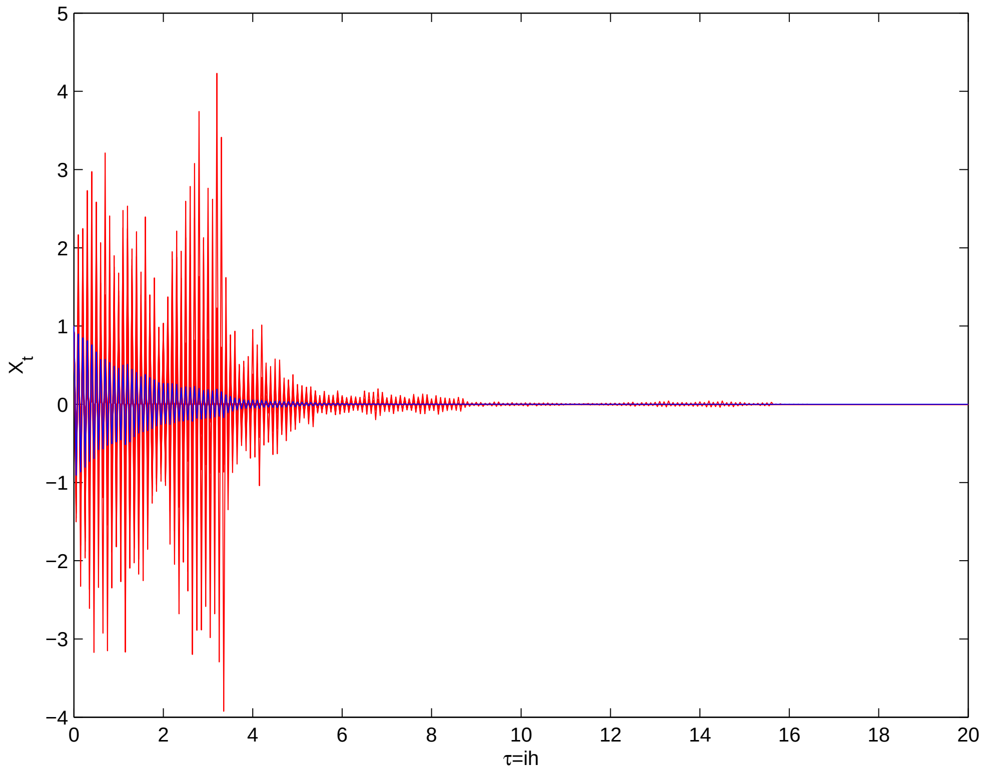

Figure 6.8: Trajectories of equation (4.1) with $m = 0$, $\alpha = -39$, $\beta = -25$, $\sigma^2 = 1$, $h = 0.05$, $\theta = 0$, $x_0 = 1$

7 Acknowledgements

This work has been carried out with the financial assistance of NATO, reference PST.EV.979727, to whom the authors wish to express their thanks. We would also like to thank Dr John Edwards and Prof Neville Ford of University College Chester and Prof Christopher Baker of the University of Manchester for helpful comments relating to early drafts of the work.

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