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ALMOST SURE STABILITY OF SOME STOCHASTIC DYNAMICAL SYSTEMS WITH MEMORY

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Abstract. Almost sure asymptotic stability of stochastic difference and differential equations with non-anticipating memory terms is studied in $\mathbb{R}^1$. Sufficient criteria are obtained with help of Lyapunov-Krasovskiĭ-type functionals, martingale decomposition and semi-martingale convergence theorems. The results allow numerical methods for stochastic differential equations with memory to be studied in terms of their ability to reproduce almost sure stability.

1. **Introduction.** More and more, stochastic difference equations and stochastic differential equations with non-anticipating memory terms are used to model dynamic phenomena with noisy data or random perturbations. In particular, one is interested in their qualitative behavior. We are going to study almost sure stability of trivial solutions of them in this paper. For this purpose, we consider discrete one-dimensional dynamical system

$$x_{n+1} = x_n + \kappa_n \Phi(x_n) - a_n \Phi(x_{n+1}) + f_n(x_n) + g_n((x_l)_{l\le n}) + \sigma_n((x_l)_{l\le n})\xi_{n+1} \quad (1)$$

driven by martingale-differences $\{\xi_n\}_{n\in\mathbb{N}}$, where Φ and f_n are real-valued functions, g_n and σ_n are real-valued functionals, and κ_n and a_n are non-random parameters, $\mathbb{N} = \{0, 1, ...\}$, and their continuous time analogue in form of stochastic differential equations with memory. Sets of non-trivial sufficient conditions are worked out and verified to guarantee almost sure stability for these systems. As the major stochastic dynamic techniques, we combine Lyapunov-Krasovskiĭ-type functionals approach

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with the martingale decomposition and convergence theory in both discrete and continuous time.

The concept of almost sure asymptotic stability under investigation is defined as follows. Suppose that a complete, filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ as underlying stochastic basis with non-decreasing filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is given. In passing, we note that "a.s." is used as the abbreviation for such wordings as "$\mathbb{P}$-almost sure" or "$\mathbb{P}$-almost surely", respectively, wherever convenient.

Definition 1.1. The difference equation (1) is said to have a **trivial (equilibrium) solution** $x^* = 0$ if, for all $n \in \mathbb{N}$, we have $\Phi(0) = f_n(0) = 0$ and $g_n((x_l^*)_{l\le n}) = \sigma_n((x_l^*)_{l\le n}) = 0$, where all $x_l^* = 0$. A trivial (equilibrium) solution $x^* = 0$ of (1) is said to be **globally a.s. asymptotically stable** if, for all $\{x_l\}_{l\le 0} \neq 0$ (a.s.) which are independent of the σ-algebra $\sigma(\xi_n : n \in \mathbb{N})$, we have $\mathbb{P}(\lim_{n\to\infty} x_n = 0) = 1$.

Many authors dealt with stability of both discrete and continuous stochastic dynamical systems. There is a textbook of "biblical" character on this topic due to Khasminskiǐ [5]. Most of the authors treat the concept of moment stability and stability in probability. For example, see Arnold [2], Gard [4], Mao [9], Mohammed [11], Schurz [16], [17], [19], [20] or Shaikhet [24], [25], [26], [27], [28], [29]. An innovative Lyapunov-Krasovskiǐ-type approach in probability and mean square sense has been developed in a series of papers of Kolmanovskiǐ and Shaikhet [6], [7]. In contrast to those works, less is known on stability of stochastic dynamical systems with memory in almost sure sense. There are some initial-step papers of Rodkina and Schurz [13], [14], [15], Schurz [21], [22] to investigate those systems with applications to numerical methods in that direction. Moreover, a martingale-approach has been presented by Melnikov and Rodkina [10] in view of stochastic approximations and later has been applied to difference equations in papers of Rodkina, Mao, Kolmanovskii [12], Appleby, Mao, Rodkina [1], Hamaya and Rodkina [3]. Certainly, we cannot cite here all contributions to stochastic stability since the entire literature is already overwhelming, however definitely not with respect to nonlinear stochastic systems with memory in almost sure sense.

The paper is organized as follows. In Section 2 we compile some auxiliary statements from stochastic analysis which we shall need in the proofs later. Section 3 investigates the almost sure stability problem for stochastic difference equations (1) with exclusively discrete time. A series of examples is presented there to illustrate the fulfillment of our major hypotheses. Section 4 reports on results for their analogue of stochastic differential equations (SDEs) in continuous time. Eventually, we discuss first applications to discretization methods of continuous time SDEs in Section 5. Stability of partial nonlinear drift-implicit stochastic θ-methods for general differential systems with memory is investigated there. The presented analysis allows SDEs to be compared with their numerical discretization with sufficiently small step sizes in terms of their ability to reproduce almost sure stability.

2. Auxiliary statements and technical notation.

Suppose that $\{\xi_n\}_{n\in\mathbb{N}}$ is a one-dimensional real-valued $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$-martingale-difference (for details see [8] and [32]). Let $\mathcal{B}(S)$ denote the set of all Borel-sets of the set S.

The following Lemma 2.1 is a generalization of the well-known Doob decomposition of sub-martingales (for details, see [8] and [32]).

Lemma 2.1. *Let the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ be a $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$-martingale-difference. Then there is a $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$-martingale-difference $\mu = \{\mu_n\}_{n\in\mathbb{N}}$ and a positive $(\mathcal{F}_{n-1}, \mathcal{B}(\mathbb{R}^1))$-measurable (i.e. predictable) stochastic sequence $\eta = \{\eta_n\}_{n\in\mathbb{N}}$ such that, for every*

$n = 1, 2, \ldots$ *almost surely (a.s.),*

$$\xi_n^2 = \mu_n + \eta_n. \tag{2}$$

The process $\eta = \{\eta_n\}_{n\in\mathbb{N}}$ in Lemma 2.1 can be represented by $\eta_n = \mathbb{E}\left[\xi_n^2|\mathcal{F}_{n-1}\right]$. Moreover, $\eta = \{\eta_n\}_{n\in\mathbb{N}}$ is a non-random sequence when ξ_n are independent random variables. In this case, we have

$$\eta_n = \mathbb{E}\,\xi_n^2 \quad \text{and} \quad \mu_n = \xi_n^2 - \mathbb{E}\,\xi_n^2. \tag{3}$$

To establish asymptotic stability we shall also make use of an application of well-known martingale convergence theorems [32] in two different forms: discrete Lemma 2.2 (which is proved in [1]) and continuous Lemma 2.3.

Lemma 2.2. *Let $\{Z_n\}_{n\in\mathbb{N}}$ be a non-negative $(\mathcal{F}_n, \mathcal{B}(\mathbb{R}^1))$-measurable process with the properties $\mathbb{E}\,|Z_n| < \infty$ ($\forall n \in \mathbb{N}$) and*

$$Z_{n+1} \le Z_n + U_n - V_n + \nu_{n+1}, \quad n = 0, 1, 2, \ldots, \tag{4}$$

where $\{\nu_n\}_{n\in\mathbb{N}}$ is an $\mathcal{F}_n$-martingale-difference, $\{U_n\}_{n\in\mathbb{N}}$ and $\{V_n\}_{n\in\mathbb{N}}$ are non-negative $(\mathcal{F}_n, \mathcal{B}(\mathbb{R}^1))$-measurable processes with $\mathbb{E}\,|U_n|, \mathbb{E}\,|V_n| < \infty$ ($\forall n \in \mathbb{N}$). Then

$$\left\{\omega \in \Omega : \sum_{n=1}^{\infty} U_n < \infty\right\} \subseteq \left\{\omega \in \Omega : \sum_{n=1}^{\infty} V_n < \infty\right\} \bigcap \{\omega \in \Omega : Z_n(\omega) \to\}.$$

Here $\{\omega \in \Omega : Z_n(\omega) \to\}$ denotes the set of all $\omega \in \Omega$ for which $\lim\limits_{n\to\infty} Z_n(\omega)$ exists and is finite.

Remark 1. Lemma 2.2 remains correct if $V_n \ge 0$ only for $n \ge N_0$, where $N_0 \in \mathbb{N}$ is non-random.

Lemma 2.3. *Suppose that Z_t, $t \ge 0$, is a non-negative continuous $\{\mathcal{F}_t\}_{t\ge 0}$-semi-martingale and its stochastic differential dZ_t is estimated by the inequality*

$$dZ_t \le dA_t^{(1)} - dA_t^{(2)} + dM_t, \quad t \ge 0, \tag{5}$$

where $A_t^{(i)}$, $i = 1, 2$, are continuous non-decreasing $(\mathcal{F}_t, \mathcal{B}(\mathbb{R}^1))$-measurable processes with $A_0^{(i)} = 0$ and M_t is a continuous $\{\mathcal{F}_t\}_{t\ge 0}$-martingale started at $M_0 = 0$. Then

$$\{\omega \in \Omega : A_\infty^{(1)}(\omega) < \infty\} \subseteq \{\omega \in \Omega : Z_t(\omega) \to\} \cap \{\omega \in \Omega : A_\infty^{(2)}(\omega) < \infty\} \quad a.s.$$

Here $\{\omega \in \Omega : Z_t(\omega) \to\}$ denotes the set of all $\omega \in \Omega$ for which $\lim\limits_{t\to\infty} Z_t(\omega)$ exists and is finite.

Proof. Define $Y_t := Z_0 + A_t^{(1)} - A_t^{(2)} + M_t - Z_t$ for all $t \ge 0$. Then, due to differential inequality (5), we find that $dY_t = dA_t^{(1)} - dA_t^{(2)} + dM_t - dZ_t \ge 0$ for all $t \ge 0$, and the process Y_t, $t \ge 0$ started at $Y_0 = 0$ is continuous, $(\mathcal{F}_t, \mathcal{B}(\mathbb{R}^1))$-measurable, non-negative and non-decreasing. The following decomposition for the $\{\mathcal{F}_t\}_{t\ge 0}$-semi-martingale process Z_t holds

$$Z_t = Z_0 + A_t^{(1)} - \left(A_t^{(2)} + Y_t\right) + M_t, \quad t \ge 0,$$

where $A_t^{(1)}$, $\left(A_t^{(2)} + Y_t\right)$ are $(\mathcal{F}_t, \mathcal{B}(\mathbb{R}^1))$-measurable and non-decreasing. Since they are also continuous, they are predictable. Now we can apply Theorem 7 (Chapter 2, p. 139, [8]) and obtain that a.s.

$$\{\omega \in \Omega : A_\infty^{(1)}(\omega) < \infty\} \subseteq \{\omega \in \Omega : Z_t(\omega) \to\} \cap \{\omega \in \Omega : A_\infty^{(2)}(\omega) + Y_\infty(\omega) < \infty\},$$

$$\text{but} \quad \{\omega \in \Omega : A_\infty^{(2)}(\omega) + Y_\infty(\omega) < \infty\} \subseteq \{\omega \in \Omega : A_\infty^{(2)}(\omega) < \infty\} \quad a.s.,$$

since $Y_t \ge 0$ (a.s.) for all $t \ge 0$, hence the conclusion of Lemma 2.3 easily follows. □

In the proofs in what follows, the following elementary inequalities shall be used.

Lemma 2.4. *For arbitrary positive numbers a, b, α and $\beta \in \mathbb{R}^1$, we have*

$$a^\alpha b^\beta \leq \frac{\alpha}{\alpha+\beta} a^{\alpha+\beta} + \frac{\beta}{\alpha+\beta} b^{\alpha+\beta}, \tag{6}$$

$$\left(\sqrt{a}+\sqrt{b}\right)^2 \leq (1+\alpha)a + (1+\alpha^{-1})b. \tag{7}$$

The equality in latter inequality is reached for $\alpha = \sqrt{ba^{-1}}$.

Furthermore, for an arbitrary functional put $\Delta V_n = V_{n+1} - V_n$. In particular, we shall investigate functionals of the form

$$V_n = \sum_{i=1}^{\infty} \alpha_{i,n-1} f(x_{n-i}) \quad \text{with} \quad \alpha_{i,n} = \sum_{j=i}^{\infty} \beta_{j,n}. \tag{8}$$

Lemma 2.5. *The increments of functional $(V_n)_{n\in\mathbb{N}}$, defined by (8) possess the representation*

$$\Delta V_n = \alpha_{1,n} f(x_n) + \sum_{i=1}^{\infty} \left(\sum_{j=i+1}^{\infty} (\beta_{j,n} - \beta_{j,n-1}) - \beta_{i,n-1} \right) f(x_{n-i}). \tag{9}$$

If in addition $\beta_{j,n}$ is non-increasing in n for all $j \in \mathbb{N}$ and $f(x)$ is non-negative function then

$$\Delta V_n \leq \alpha_{1,n} f(x_n) - \sum_{i=1}^{\infty} \beta_{i,n-1} f(x_{n-i}), \quad n \in \mathbb{N}. \tag{10}$$

Proof. It is enough to note that

$$\begin{aligned}
\Delta V_n &= \sum_{i=1}^{\infty} \alpha_{i,n} f(x_{n+1-i}) - \sum_{i=1}^{\infty} \alpha_{i,n-1} f(x_{n-i}) \\
&= \sum_{i=0}^{\infty} \alpha_{i+1,n} f(x_{n-i}) - \sum_{i=1}^{\infty} \alpha_{i,n-1} f(x_{n-i}) \\
&= \alpha_{1,n} f(x_n) + \sum_{i=1}^{\infty} (\alpha_{i+1,n} - \alpha_{i,n-1}) f(x_{n-i}) \\
&= \alpha_{1,n} f(x_n) + \sum_{i=1}^{\infty} \left(\sum_{j=i+1}^{\infty} (\beta_{j,n} - \beta_{j,n-1}) - \beta_{i,n-1} \right) f(x_{n-i}).
\end{aligned}$$

□

3. **Stability of discrete time stochastic equation.** Consider discrete time stochastic difference equations (1) with memory under the following set of major hypotheses.

(H0) a_n is a non-increasing sequence of non-random, non-negative real numbers and κ_n is a sequence of non-random real numbers satisfying the condition

$$a_n > |\kappa_n|, \quad n \in \mathbb{N}. \tag{11}$$

(H1) The function $\Phi : \mathbb{R}^1 \to \mathbb{R}^1$ is continuous and

$$\forall u \in \mathbb{R}^1 \;:\; u\Phi(u) \geq cu^{2m} \quad \text{and} \quad \Phi(u) = 0 \quad \text{then} \quad u = 0, \tag{12}$$

where m is a positive integer and $c > 0$ is a real constant.

(H2) There are non-negative non-random numbers δ_n, ν_n, $\gamma_{i,n}^{(0)}$, $\gamma_{i,n}^{(1)}$, $\gamma_{i,n}^{(2)} \in \mathbb{R}_+$ such that

$$f_n^2(x_n) \leq \delta_n \Phi^2(x_n), \quad |g_n((x_l)_{l\leq n})| \leq \sum_{i=0}^{\infty} \gamma_{i,n}^{(0)} |x_{n-i}|^{2m-1}, \tag{13}$$

$$\sigma_n^2((x_l)_{l\leq n}) \leq \nu_n + \sum_{i=0}^{\infty} \gamma_{i,n}^{(1)} x_{n-i}\Phi(x_{n-i}) + \sum_{i=0}^{\infty} \gamma_{i,n}^{(2)} \Phi^2(x_{n-i}), \tag{14}$$

$$\sum_{n=1}^{\infty} \nu_n \eta_{n+1} < \infty, \quad \text{a.s.} \tag{15}$$

(H3)

$$\epsilon_n^{(1)} = 2\left(a_n - \kappa_n - \sqrt{\delta_n} - \frac{\rho_n^{(0)}}{c}\right) - \eta_{n+1}\rho_n^{(1)} \geq 0, \tag{16}$$

$$\begin{aligned} \epsilon_n^{(2)} &= a_n^2 - \kappa_n^2 - \eta_{n+1}\rho_n^{(2)} \\ &- |\kappa_n|\left[2\sqrt{\delta_n} + \left(1 + \frac{1}{c^2}\right)\rho_n^{(0)}\right] - \left(\sqrt{\delta_n} + \frac{\rho_n^{(0)}}{c}\right)^2 \geq 0, \end{aligned} \tag{17}$$

where

$$\rho_n^{(l)} = \sum_{j=0}^{\infty} \gamma_{j,n}^{(l)}, \qquad l = 0, 1, 2, \qquad n \in \mathbb{N}. \tag{18}$$

(H4) $\exists \varepsilon > 0$ and $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0 \;:\; \epsilon_n^{(1)} + \epsilon_n^{(2)} > \varepsilon > 0$.

(H5) All $\beta_{i,n}^{(1)}$ and $\beta_{i,n}^{(2)}$ are non-increasing in n, where

$$\begin{aligned} \beta_{i,n}^{(1)} &= \gamma_{i,n}^{(1)}\eta_{n+1} + \gamma_{i,n}^{(0)}\frac{2m-1}{mc}, \\ \beta_{i,n}^{(2)} &= \gamma_{i,n}^{(2)}\eta_{n+1} + \gamma_{i,n}^{(0)}\left(|\kappa_n| + \rho_n^{(0)} + c\sqrt{\delta_n}\right)\frac{1}{c^2}. \end{aligned} \tag{19}$$

Despite of its apparent (visible) complexity, hypotheses (H0)-(H4) are fulfilled in many cases. In what follows, we present 3 examples satisfying those major hypotheses.

Example 1.

- $a_n \equiv a > 0$,
- $\kappa_n \equiv \kappa > 0$,
- $a > \kappa$,
- $f_n \equiv 0$ and therefore $\delta_n \equiv 0$,
- $\Phi(u) = u^{2m-1}$, and therefore $c = 1$,
- $g_n \equiv 0$,
- $\gamma_{i,n}^{(l)} \equiv 0$, $l = 1, 2, 3$, $i, n \in \mathbf{N}$,
- $\eta_n \equiv 1$,
- $\nu_n = \frac{1}{n^2}$.

In this case equation (1) takes the form

$$x_{n+1} = x_n + \kappa_n x_n^{2m-1} - a_n x_{n+1}^{2m-1} + \frac{1}{n}\xi_{n+1}, \tag{20}$$

and

$$\varepsilon_n^{(1)} = 2(a-\kappa) > 0, \quad \varepsilon_n^{(2)} = a^2 - \kappa^2 > 0.$$

Example 2.

- $a_n \equiv a > 0$,
- $\kappa_n \equiv \kappa > 0$,
- $a > \kappa$,
- $\Phi(u) = u^{2m-1}$, and therefore $c = 1$,
- $\delta_n \equiv \delta < \left(\frac{a-\kappa}{2}\right)^2$,
- $f_n(u) = f(u) = \varphi(u)u^{2m-1}$, where $|\varphi(u)|^2 \le \delta$ for all $u \in \mathbb{R}$,
- $\eta_n \equiv 1$,
- $\nu_n = \frac{1}{n^2}$,
- $\gamma_{i,n}^{(l)} = \gamma_i^{(l)} = \frac{\gamma(l)}{(i+1)^2}, \quad l = 1,2,3, \quad i,n \in \mathbf{N}$.

We set

$$S_2 = \sum_{i=0}^{\infty} \frac{1}{(i+1)^2} \tag{21}$$

and note that

$$1 < S_2 < 2.$$

Let

$$\gamma(0) < \frac{a-\kappa}{8}, \quad \gamma(1) < \frac{a-\kappa}{4}, \quad \gamma(2) < \frac{(a-\kappa)(39a-7\kappa)}{128}. \tag{22}$$

Therefore

$$\rho_n^{(l)} = \sum_{j=0}^{\infty} \gamma_{j,n}^{(l)} = \gamma(l)S_2, \quad l = 0,1,2. \tag{23}$$

We can estimate $\varepsilon_n^{(1)}$ by

$$\varepsilon_n^{(1)} \equiv \varepsilon^{(1)} = 2(a-\kappa-\sqrt{\delta}-\gamma(0)S_2) - \gamma(1)S_0 \ge 2\left(\frac{a-\kappa}{2} - (2\gamma(0)+\gamma(1))\right) > 0,$$

if

$$2\gamma(0) + \gamma(1) < \frac{a-\kappa}{2},$$

which is indeed satisfied due to (22). An estimation of $\varepsilon_n^{(2)}$ yields that

$$\begin{aligned}
\varepsilon_n^{(2)} &\equiv \varepsilon^{(2)} = a^2 - \kappa^2 - \gamma(2)S_2 - |\kappa|\left(2\sqrt{\delta} + 2\gamma(0)S_2\right) - \left(\sqrt{\delta} + \gamma(0)S_2\right)^2 \\
&> a^2 - \kappa^2 - \kappa(a-\kappa) - 2\gamma(2) - 4\kappa\gamma(0) - \left(\frac{a-\kappa}{2} + 2\gamma(0)\right)^2 \\
&> (a-\kappa)a - 2\frac{(a-\kappa)(39a-7\kappa)}{128} - 4\kappa\frac{a-\kappa}{8} - \Big((a-\kappa)(1/2+1/8)\Big)^2 \\
&= (a-\kappa)\left(a - \frac{39a-7\kappa}{64} - \frac{\kappa}{2} - \frac{25(a-\kappa)}{64}\right) = 0.
\end{aligned}$$

Thus

$$\varepsilon = \min\left\{\varepsilon^{(1)}, \varepsilon^{(2)}\right\}.$$

Example 3.

- $a_n = a + \kappa_n$, $a > 0$, $\kappa_n \geq 0$ $\forall n \in \mathbf{N}$,
- $\kappa_n \downarrow$ (i.e. κ_n is decreasing),
- $\Phi(u) = \psi(u)u^{2m-1}$,
- $\psi(u) > 1$ is arbitrary continuous function, and therefore $c = 1$,
- $\delta_n < \frac{a^2}{4}$,
- $f_n = \varphi_n(u)u^{2m-1}$, where $|\varphi_n(u)|^2 \leq \delta_n$ for all $u \in \mathbb{R}$, $n \in \mathbf{N}$,
- $\eta_n = \frac{1}{n}$,
- $\nu_n = \frac{1}{n}$,
- $\gamma_{i,n}^{(l)} = \frac{1}{n}\frac{\gamma(l)}{(i+1)^2}$, $\gamma(l) > 0$, $l = 1, 2, 3$, $i, n \in \mathbf{N}$.

We have

$$\rho_n^{(l)} = \frac{1}{n}\gamma(l)S_2 > \frac{2}{n}\gamma(l), \quad l = 0, 1, 2.$$

Then we can estimate $\varepsilon_n^{(1)}$ by

$$\varepsilon_n^{(1)} > 2(a - \frac{a}{2} - \frac{2}{n}\gamma(0)) - \frac{2\gamma(1)}{n^2} \to a,$$

when $n \to \infty$. For the estimation of $\varepsilon_n^{(2)}$, we arrive at

$$\begin{aligned}
\varepsilon_n^{(2)} &> (a + \kappa_n)^2 - \kappa_n^2 - \frac{2}{n^2}\gamma(2) - |\kappa_n|\left(a + \frac{4}{n}\gamma(0)\right) - \left(\frac{a}{2} + \frac{2}{n}\gamma(0)\right)^2 \\
&= a^2 + 2a\kappa_n - a\kappa_n - \frac{a^2}{4} + o(n^{-1}) = \frac{3}{4}a^2 + a\kappa_n + o(n^{-1}),
\end{aligned}$$

which also tends to a positive limit as $n \to \infty$.

Consequently, we can recognize that the hypotheses (H0) - (H5) are satisfied in several meaningful examples. For more examples, see below or [13], [14] with $\Phi(u) = u^3$ in recent literature.

Theorem 3.1. *Let ξ_n, $n \in \mathbb{N}$, be square-integrable, independent random variables with $\mathbb{E}\,\xi_n = 0$, $\mathbb{E}\,\xi_n^2 = \eta_n$ and hypotheses (H0) - (H5) be fulfilled. Then, for every $(\mathcal{F}_0, \mathcal{B}(\mathbb{R}^1))$-measurable initial condition $\{x_l\}_{l\leq 0}$, the solution $\{x_n\}_{n\in\mathbb{N}}$ of equation (1) has the property that $\lim_{n\to\infty} x_n = 0$ almost surely. If additionally Φ, f_n, g_n and σ_n have 0 as its trivial equilibrium then 0 is a globally a.s. asymptotically stable equilibrium for equation (1).*

Proof. Rewrite equation (1) in the equivalent form

$$\begin{aligned}
y_{n+1} + (a_n - a_{n+1})\Phi(x_{n+1}) \quad &= \quad y_n - (a_n - \kappa_n)\Phi(x_n) \\
&\quad + f_n(x_n) + g_n((x_l)_{l\leq n}) + \sigma_n((x_l)_{l\leq n})\xi_{n+1}
\end{aligned} \tag{24}$$

where

$$y_n \quad = \quad x_n + a_n\Phi(x_n). \tag{25}$$

Following the general method of Lyapunov functionals construction as suggested by [6], [7], we shall construct Lyapunov functional V_n for equation (1) in the form $V_n = V_{1,n} + V_{2,n}$ with

$$V_{1,n} = y_n^2 = (x_n + a_n\Phi(x_n))^2.$$

For estimation of the increments $\Delta V_{1,n} = y_{n+1}^2 - y_n^2$, note that from the inequalities $a_n \geq a_{n+1}$ and $y_n\Phi(x_n) = x_n\Phi(x_n) + a_n\Phi^2(x_n) \geq 0$ it follows that

$$y_{n+1}^2 \leq (y_{n+1} + (a_n - a_{n+1})\Phi(x_{n+1}))^2.$$

So, we arrive at

$$\Delta V_{1,n} \le (y_{n+1} + (a_n - a_{n+1})\Phi(x_{n+1}))^2 - y_n^2.$$

Applying (24) and (3) we estimate

$$\begin{aligned}
\Delta V_{1,n} \quad \le \quad & [y_n - (a_n - \kappa_n)\Phi(x_n) \\
& + f_n(x_n) + g_n((x_l)_{l\le n}) + \sigma_n((x_l)_{l\le n})\xi_{n+1}]^2 - y_n^2 \\
= \quad & -2(a_n - \kappa_n) y_n \Phi(x_n) + (a_n - \kappa_n)^2 \Phi^2(x_n) \\
& +2(y_n - (a_n - \kappa_n)\Phi(x_n))[f_n(x_n) + g_n((x_l)_{l\le n})] \\
& +[f_n(x_n) + g_n((x_l)_{l\le n})]^2 + \sigma_n^2((x_l)_{l\le n})\eta_{n+1} + \zeta_{n+1} \qquad (26)
\end{aligned}$$

where

$$\begin{aligned}
\zeta_{n+1} \quad = \quad & 2[x_n + \kappa_n\Phi(x_n) + f_n(x_n) + g_n((x_l)_{l\le n})]\sigma_n((x_l)_{l\le n})\xi_{n+1} \\
& +\sigma_n^2((x_l)_{l\le n})\mu_{n+1} \qquad (27)
\end{aligned}$$

and μ_n is a martingale-difference defined by (3).

Via (25), the identity $y_n - (a_n - \kappa_n)\Phi(x_n) = x_n + \kappa_n\Phi(x_n)$, and inequality (6) from Lemma 2.4, we continue estimation (26) with some $\alpha_n > 0$

$$\begin{aligned}
\Delta V_{1,n} \quad \le \quad & -2(a_n - \kappa_n)(x_n + a_n\Phi(x_n))\Phi(x_n) + (a_n - \kappa_n)^2\Phi^2(x_n) \\
& +2(x_n + \kappa_n\Phi(x_n))[f_n(x_n) + g_n((x_l)_{l\le n})] + \sigma_n^2((x_l)_{l\le n})\eta_{n+1} \\
& +(1+\alpha_n)f_n^2(x_n) + (1+\alpha_n^{-1})g_n^2((x_l)_{l\le n}) + \zeta_{n+1}. \qquad (28)
\end{aligned}$$

Using (18) and inequalities (13), we get

$$\begin{aligned}
g_n^2((x_l)_{l\le n}) \quad \le \quad & \rho_n^{(0)} \sum_{i=0}^{\infty} \gamma_{i,n}^{(0)} |x_{n-i}|^{2(2m-1)} \le \frac{\rho_n^{(0)}}{c^2} \sum_{i=0}^{\infty} \gamma_{i,n}^{(0)} \Phi^2(x_{n-i}) \\
= \quad & \frac{\rho_n^{(0)}}{c^2} \left(\gamma_{0,n}^{(0)} \Phi^2(x_n) + \sum_{i=1}^{\infty} \gamma_{i,n}^{(0)} \Phi^2(x_{n-i}) \right), \qquad (29) \\
|f_n(x_n)(x_n + \kappa_n\Phi(x_n))| \quad \le \quad & \sqrt{\delta_n} |\Phi(x_n)||x_n + \kappa_n\Phi(x_n)| \\
\le \quad & \sqrt{\delta_n} \left[x_n\Phi(x_n) + |\kappa_n|\Phi^2(x_n) \right]. \qquad (30)
\end{aligned}$$

Via inequality (6) of Lemma 2.4 we obtain that

$$\begin{aligned}
& |g_n((x_l)_{l\le n})x_n| \le \sum_{i=0}^{\infty} \gamma_{i,n}^{(0)} |x_{n-i}|^{2m-1}|x_n| \\
\le \quad & \frac{1}{2m} \sum_{i=0}^{\infty} \gamma_{i,n}^{(0)} \left(|x_n|^{2m} + (2m-1)|x_{n-i}|^{2m} \right) \\
= \quad & \frac{1}{2m} \left[\left(\rho_n^{(0)} + (2m-1)\gamma_{0,n}^{(0)} \right) |x_n|^{2m} + (2m-1) \sum_{i=1}^{\infty} \gamma_{i,n}^{(0)} |x_{n-i}|^{2m} \right] \\
\le \quad & \frac{1}{2mc} \left[\left(\rho_n^{(0)} + (2m-1)\gamma_{0,n}^{(0)} \right) x_n\Phi(x_n) + (2m-1) \sum_{i=1}^{\infty} \gamma_{i,n}^{(0)} x_{n-i}\Phi(x_{n-i}) \right]. \quad (31)
\end{aligned}$$

Besides

$$\begin{aligned}|g_n((x_l)_{l\le n})\Phi(x_n)| &\le \sum_{i=0}^{\infty}\gamma_{i,n}^{(0)}|x_{n-i}|^{2m-1}|\Phi(x_n)|\\ &\le \frac{1}{2}\sum_{i=0}^{\infty}\gamma_{i,n}^{(0)}\left(\Phi^2(x_n)+|x_{n-i}|^{4m-2}\right)\\ &\le \frac{1}{2}\sum_{i=0}^{\infty}\gamma_{i,n}^{(0)}\left(\Phi^2(x_n)+\frac{\Phi^2(x_{n-i})}{c^2}\right)\\ &= \frac{1}{2}\left[\left(\rho_n^{(0)}+\frac{\gamma_{0,n}^{(0)}}{c^2}\right)\Phi^2(x_n)+\frac{1}{c^2}\sum_{i=1}^{\infty}\gamma_{i,n}^{(0)}\Phi^2(x_{n-i})\right].\end{aligned} \tag{32}$$

Substituting (30)-(32) into (28) yields that

$$\begin{aligned}\Delta V_{1,n} \le\ & -2(a_n-\kappa_n)(x_n+a_n\Phi(x_n))\Phi(x_n)\\ & +(a_n-\kappa_n)^2\Phi^2(x_n)+2\sqrt{\delta_n}\left[x_n\Phi(x_n)+|\kappa_n|\Phi^2(x_n)\right]\\ & +\frac{1}{mc}\left[\left(\rho_n^{(0)}+(2m-1)\gamma_{0,n}^{(0)}\right)x_n\Phi(x_n)+(2m-1)\sum_{i=1}^{\infty}\gamma_{i,n}^{(0)}x_{n-i}\Phi(x_{n-i})\right]\\ & +(1+\alpha_n)\delta_n\Phi^2(x_n)+(1+\alpha_n^{-1})\frac{\rho_n^{(0)}}{c^2}\left(\gamma_{0,n}^{(0)}\Phi^2(x_n)+\sum_{i=1}^{\infty}\gamma_{i,n}^{(0)}\Phi^2(x_{n-i})\right)\\ & +\eta_{n+1}\left(\nu_n+\sum_{i=0}^{\infty}\gamma_{i,n}^{(1)}x_{n-i}\Phi(x_{n-i})+\sum_{i=0}^{\infty}\gamma_{i,n}^{(2)}\Phi^2(x_{n-i})\right)\\ & +|\kappa_n|\left[\left(\rho_n^{(0)}+\frac{\gamma_{0,n}^{(0)}}{c^2}\right)\Phi^2(x_n)+\frac{1}{c^2}\sum_{i=1}^{\infty}\gamma_{i,n}^{(0)}\Phi^2(x_{n-i})\right]+\zeta_{n+1}.\end{aligned}$$

Now, rewrite this inequality by summing over all terms involving $x_n\Phi(x_n)$ and $\Phi^2(x_n)$, respectively. This leads to

$$\begin{aligned}\Delta V_{1,n} \le\ & \left[-2(a_n-\kappa_n)+2\sqrt{\delta_n}+\eta_{n+1}\gamma_{0,n}^{(1)}+\frac{1}{mc}\left(\rho_n^{(0)}+(2m-1)\gamma_{0,n}^{(0)}\right)\right]x_n\Phi(x_n)\\ & +\Bigg[-\left(a_n^2-\kappa_n^2\right)+|\kappa_n|\left(2\sqrt{\delta_n}+\rho_n^{(0)}+\frac{\gamma_{0,n}^{(0)}}{c^2}\right)\\ & +\eta_{n+1}\gamma_{0,n}^{(2)}+(1+\alpha_n)\delta_n+(1+\alpha_n^{-1})\frac{\rho_n^{(0)}\gamma_{0,n}^{(0)}}{c^2}\Bigg]\Phi^2(x_n)\\ & +\sum_{i=1}^{\infty}\beta_{i,n}^{(1)}x_{n-i}\Phi(x_{n-i})+\sum_{i=1}^{\infty}\beta_{i,n}^{(2)}\Phi^2(x_{n-i})\zeta_{n+1}+\nu_n\eta_{n+1},\end{aligned}$$

where

$$\begin{aligned}\beta_{i,n}^{(1)} &= \eta_{n+1}\gamma_{i,n}^{(1)}+\gamma_{i,n}^{(0)}\frac{2m-1}{mc},\\ \beta_{i,n}^{(2)} &= \eta_{n+1}\gamma_{i,n}^{(2)}+\gamma_{i,n}^{(0)}\left(|\kappa_n|+(1+\alpha_n^{-1})\rho_n^{(0)}\right)\frac{1}{c^2}.\end{aligned} \tag{33}$$

Now, set

$$V_{2,n} = \sum_{i=1}^{\infty} \alpha_{i,n-1}^{(1)} x_{n-i} \Phi(x_{n-i}) + \sum_{i=1}^{\infty} \alpha_{i,n-1}^{(2)} \Phi^2(x_{n-i}),$$

with

$$\alpha_{i,n}^{(l)} = \sum_{j=i}^{\infty} \beta_{j,n}^{(l)}, \qquad l = 1, 2. \tag{34}$$

From Lemma 2.5, it follows that

$$\begin{aligned} \Delta V_{2,n} &= \alpha_{1,n}^{(1)} x_n \Phi(x_n) + \alpha_{1,n}^{(2)} \Phi^2(x_n) \\ &\quad + \sum_{i=1}^{\infty} \left(\sum_{j=i+1}^{\infty} (\beta_{j,n}^{(1)} - \beta_{j,n-1}^{(1)}) - \beta_{i,n-1}^{(1)} \right) x_{n-i} \Phi(x_{n-i}) \\ &\quad + \sum_{i=1}^{\infty} \left(\sum_{j=i+1}^{\infty} (\beta_{j,n}^{(2)} - \beta_{j,n-1}^{(2)}) - \beta_{i,n-1}^{(2)} \right) \Phi^2(x_{n-i}). \end{aligned}$$

As a result, for the functional $V_n = V_{1,n} + V_{2,n}$ we have

$$\begin{aligned} \Delta V_n \leq & \left[-2(a_n - \kappa_n) + 2\sqrt{\delta_n} \eta_{n+1} \gamma_{0,n}^{(1)} + \frac{1}{mc} \left(\rho_n^{(0)} + (2m-1) \gamma_{0,n}^{(0)} \right) + \alpha_{1,n}^{(1)} \right] x_n \Phi(x_n) \\ & + \left[-\left(a_n^2 - \kappa_n^2 \right) + |\kappa_n| \left(2\sqrt{\delta_n} + \rho_n^{(0)} + \frac{\gamma_{0,n}^{(0)}}{c^2} \right) \right. \\ & \left. + \eta_{n+1} \gamma_{0,n}^{(2)} + (1+\alpha_n)\delta_n + (1+\alpha_n^{-1}) \frac{\rho_n^{(0)} \gamma_{0,n}^{(0)}}{c^2} + \alpha_{1,n}^{(2)} \right] \Phi^2(x_n) \\ & + \sum_{i=1}^{\infty} \left(\sum_{j=i}^{\infty} (\beta_{j,n}^{(1)} - \beta_{j,n-1}^{(1)}) \right) x_{n-i} \Phi(x_{n-i}) \\ & + \sum_{i=1}^{\infty} \left(\sum_{j=i}^{\infty} (\beta_{j,n}^{(2)} - \beta_{j,n-1}^{(2)}) \right) \Phi^2(x_{n-i}) + \lambda_n \eta_{n+1} + \zeta_{n+1}. \end{aligned}$$

Via representations (16), (33) and (34) we obtain

$$\begin{aligned} \Delta V_n \leq & -\epsilon_n^{(1)} x_n \Phi(x_n) + \left[-(a_n^2 - \kappa_n^2) + |\kappa_n| \left(2\sqrt{\delta_n} + \left(1 + \frac{1}{c^2} \right) \rho_n^{(0)} \right) \right. \\ & \left. + \eta_{n+1} \rho_n^{(2)} + (1+\alpha_n)\delta_n + (1+\alpha_n^{-1}) \left(\frac{\rho_n^{(0)}}{c} \right)^2 \right] \Phi^2(x_n) \\ & + \sum_{i=1}^{\infty} \left(\sum_{j=i}^{\infty} (\beta_{j,n}^{(1)} - \beta_{j,n-1}^{(1)}) \right) x_{n-i} \Phi(x_{n-i}) \\ & + \sum_{i=1}^{\infty} \left(\sum_{j=i}^{\infty} (\beta_{j,n}^{(2)} - \beta_{j,n-1}^{(2)}) \right) \Phi^2(x_{n-i}) + \lambda_n \eta_{n+1} + \zeta_{n+1}. \end{aligned}$$

To minimize right hand part of this inequality, we have to use Lemma 2.4 with inequality (7). For this purpose, put $\alpha_n = \rho_n^{(0)} \left(c\sqrt{\delta_n}\right)^{-1}$. With this α_n representation (33) coincides with (19). Via hypothesis (H5) $\beta_{j,n}^{(m)}$ are non-increasing in $n \in \mathbb{N}$. Thus, using (17), we arrive at the estimate

$$\Delta V_n \le -\epsilon_n^{(1)} x_n \Phi(x_n) - \epsilon_n^{(2)} \Phi^2(x_n) + \lambda_n \eta_{n+1} + \zeta_{n+1}, \quad n = 0, 1, \ldots .$$

By summing this inequality over n, we obtain the decomposition

$$V_n \le V_0 + A_n^{(1)} - A_n^{(2)} + M_n$$

with

$$A_n^{(1)} = \sum_{i=0}^{n-1} \lambda_i \eta_{i+1}, \quad A_n^{(2)} = \sum_{i=0}^{n-1} \left(\epsilon_i^{(1)} x_i \Phi(x_i) + \epsilon_i^{(2)} \Phi^2(x_i)\right), \quad M_n = \sum_{i=1}^{n} \zeta_i,$$

for all $n \in \mathbb{N}$. Note that $\lim_{n\to\infty} A_n^{(1)} < \infty$ due to condition (15) in hypothesis (H2). Eventually, we may apply Lemma 2.2 to the sequence $Z_n := V_0 + A_n^{(1)} - A_n^{(2)} + M_n \ge V_n$. In fact, $Z_n \ge V_n$ is positive and all assumptions of Lemma 2.2 are satisfied. Therefore, $Z_n \ge 0$ converges (a.s.) to a unique, finite limit $Z_\infty = \lim_{n\to\infty} Z_n$ and $A_n^{(2)}$ also converges to a unique, finite limit $A_\infty^{(2)} = \lim_{n\to\infty} A_n^{(2)}$ (a.s.) as n tends to ∞. In particular, this implies that V_n is a positive, bounded sequence (a.s.) for all finite values V_0. By construction of $\{V_n\}_{n\in\mathbb{N}}$ and $y_n^2 = (x_n + a_n\Phi(x_n))^2$, y_n must satisfy

$$0 \le \limsup_{n\to\infty} y_n^2 < \infty \quad (a.s.)$$

Note that $y_n^2 \ge x_n^2 + a_n^2 \Phi^2(x_n)$ under (H1). Hence, we may easily conclude that

$$0 \le \limsup_{n\to\infty} x_n^2 < \infty \quad (a.s.)$$

Let us prove that $\lim_{n\to\infty} x_n^2 = 0$. Suppose, indirectly, that the opposite is true. Then there exists a.s. a finite $c_0^2(\omega) > 0$ on $\Omega_1 = \{\omega : \limsup_{n\to\infty} x_n^2(\omega) = c_0^2(\omega) > 0\}$ with $\mathbb{P}(\Omega_1) = p_1 > 0$. There also exists a subsequence $\{x_{n_k}\}_{n_k\in\mathbb{N}}$, an a.s. finite random variable $c_1 = c_1(\omega) > 0$ and an integer $N(\omega)$ such that

$$|x_{n_k}(\omega)| \ \ge\ c_0(\omega), \quad x_{n_k}\Phi(x_{n_k}) \ge c_1(\omega), \quad \Phi^2(x_{n_k}) \ge c_1(\omega) \tag{35}$$

for all $n_k \ge N(\omega)$ on $\omega \in \Omega_1$. Let $I_N^n = \{n_k \in \mathbb{N} : n \ge n_k \ge N(\omega), (35) \text{ holds}\}$. Note that the cardinality $\#(I_N^n)$ must tend to $+\infty$ as $n \to +\infty$. Then, for all $\omega \in \Omega_1$ and for all $n > \max\{N(\omega), n_0\}$, we have

$$\begin{aligned}
A_n^{(2)}(\omega) &= \sum_{i=1}^{n} \left(\varepsilon_i^{(1)} x_i \Phi(x_i) + \varepsilon_i^{(2)} \Phi^2(x_i)\right) \\
&= \sum_{i=1}^{N} \left(\varepsilon_i^{(1)} x_i \Phi(x_i) + \varepsilon_i^{(2)} \Phi(x_i)\right) + \sum_{i=N}^{n} \left(\varepsilon_i^{(1)} x_i \Phi(x_i) + \varepsilon_i^{(2)} \Phi^2(x_i)\right) \\
&\ge \sum_{i=N}^{n} \left(\varepsilon_i^{(1)} x_i \Phi(x_i) + \varepsilon_i^{(2)} \Phi^2(x_i)\right) \ \ge \sum_{i=N, i\in I_N^n}^{n} \left(\varepsilon_i^{(1)} x_i \Phi(x_i) + \varepsilon_i^{(2)} \Phi^2(x_i)\right) \\
&\ge c_1(\omega) \sum_{i=N, i\in I_N^n}^{n} \left(\varepsilon_i^{(1)} + \varepsilon_i^{(2)}\right) \ge c_1(\omega)\varepsilon\#(I_N^n) \ \to\ \infty
\end{aligned}$$

as $n \to \infty$, due to hypothesis (H4). Therefore, $\limsup_{n\to\infty} A_n^{(2)} = \lim_{n\to\infty} A_n^{(2)} = \infty$. This result contradicts the finiteness of $\lim_{n\to\infty} A_n^{(2)}$ as claimed by Lemma 2.2. Therefore, we must have that $\lim_{n\to\infty} x_n = 0$ holds (a.s.), independently of initial values $\{x_l\}_{l\le 0}$. This observation obviously confirms the conclusion of Theorem 3.1. □

Remark 2. We note that Remark 1 implies that Theorem 3.1 holds true if there exists non-random $N_0 \in \mathbb{N}$ such that conditions (11), (16), (17), (19) are fulfilled for all $n \ge N_0$.

Theorem 3.2. *Let ξ_n, $n \in \mathbb{N}$, be square-integrable, independent random variables with $\mathbb{E}\,\xi_n = 0$, $\mathbb{E}\,\xi_n^2 = \eta_n$ and hypotheses (H0)-(H3), (H5) be fulfilled and*

$$\sum_{n=1}^{\infty} \left[\epsilon_n^{(1)} + \epsilon_n^{(2)}\right] = \infty. \tag{36}$$

Suppose, in addition, that the coefficients $\gamma_{i,n}^{(l)}$, $l = 0, 1, 2$ from conditions (13) and (14) possess the following property. There exists $k \in \mathbb{N}$ such that

$$\gamma_{i,n}^{(l)} = 0, \quad \text{for} \quad i > k, \quad \text{and} \quad \lim_{n\to\infty} \gamma_{i,n}^{(l)} = 0, \quad \text{for} \quad i = 1, 2, \ldots, k. \tag{37}$$

Assume also that one of the following conditions holds:

- *(i)* $\lim_{n\to\infty} a_n = a = 0$
- *(ii)* $\lim_{n\to\infty} a_n = a > 0$, *function $\Phi(u)$ does not decrease and $\Phi(-u) = -\Phi(u)$ for all $u \in \mathbb{R}$.*

Then, for every $(\mathcal{F}_0, \mathcal{B}(\mathbb{R}^1))$-measurable initial condition $\{x_l\}_{l\le 0}$, the solution process $\{x_n\}_{n\in\mathbb{N}}$ of equation (1) has the property that $\lim_{n\to\infty} x_n = 0$ almost surely. If additionally Φ, f_n, g_n and σ_n have 0 as its trivial equilibrium then 0 is a globally a.s. asymptotically stable equilibrium for equation (1).

Remark 3. Note that the function $\Phi(u) = \sum_{i=1}^{m} \alpha_i u^{2i-1}$, $\alpha_i \in \mathbb{R}$, $m \in \mathbb{N}$, $m \ge 1$, in particular, $\Phi(u) = u^{2m-1}$, satisfies the conditions stated in (ii).

Proof. In the same way as in Theorem 3.1, we prove that $\lim_{n\to\infty} V_n$ exists. Then for some a.s. finite random variable $H(\omega) > 0$ and all $n \in \mathbb{N}$ we have

$$x_n \Phi(x_n), \Phi^2(x_n) \le H.$$

We show that $\lim_{n\to\infty} V_{1,n}$ exists. In order to do it we note that

$$\begin{aligned} V_{2,n} &= \sum_{i=1}^{\infty} \alpha_{i,n-1}^{(1)} x_{n-i}\Phi(x_{n-i}) + \sum_{i=1}^{\infty} \alpha_{i,n-1}^{(2)} \Phi^2(x_{n-i}) \\ &= \sum_{i=1}^{\infty}\sum_{j=i}^{k} \beta_{j,n-1}^{(1)} x_{n-i}\Phi(x_{n-i}) + \sum_{i=1}^{\infty}\sum_{j=i}^{k} \beta_{j,n-1}^{(2)} \Phi^2(x_{n-i}) \\ &\le H\left(\sum_{i=1}^{k}\sum_{j=i}^{k} \beta_{j,n-1}^{(1)} + \sum_{i=1}^{k}\sum_{j=i}^{k} \beta_{j,n-1}^{(2)}\right), \end{aligned}$$

where $\beta_{j,n}^{(l)}$, $l = 1, 2$, are defined in (33). Therefore, due to property (37), we get

$$V_{2,n} \le 2Hk^2 \max_{j=1,\ldots,k} \{\beta_{j,n-1}^{(1)} + \beta_{j,n-1}^{(2)}\} \to 0, \quad \text{when} \quad n \to \infty.$$

The above relation implies that $\lim_{n\to\infty} V_{1,n} = 0$. To prove that $\lim_{n\to\infty} x_n^2$ exists, we suppose the opposite, i.e. there are numbers $\bar{l}$, $\underline{l}$ with $|\bar{l}| > |\underline{l}|$ and sequences $\{n_k\}_{k\in\mathbb{N}}$, $\{n_i\}_{i\in\mathbb{N}}$, such that

$$\lim_{k\to\infty} x_{n_k} = \bar{l}, \quad \lim_{i\to\infty} x_{n_i} = \underline{l}.$$

By substituting n_k and n_i instead of n in the expression $V_{1,n} = (x_n + a_n\Phi(x_n))^2$ and passing to the limit twice, we arrive at

$$(\bar{l} + a\Phi(\bar{l}))^2 = (\underline{l} + a\Phi(\underline{l}))^2. \tag{38}$$

In case (i) equality (38) is reduced to $\bar{l}^2 - \underline{l}^2 = 0$ which contradicts the assumption $|\bar{l}| > |\underline{l}|$.

We show now that in case (ii) equality (38) is also impossible. Note that $\Psi(u) = u + a\Phi(u)$ is strictly increasing function. When $sign\,\bar{l} = sign\,\underline{l}$, equality (38) implies that $\Psi(\bar{l}) = \Psi(\underline{l})$, which is impossible since $\bar{l} \neq \underline{l}$. Suppose now that $\bar{l} \geq 0 \geq \underline{l}$. Then

$$\bar{l} + a\Phi(\bar{l}) = -\underline{l} - a\Phi(\underline{l}).$$

Since $-\underline{l} - a\Phi(\underline{l}) = -\underline{l} + a\Phi(-\underline{l}) = \Psi(-\underline{l})$, we arrive at

$$\Psi(-\underline{l}) = -\underline{l} + a\Phi(-\underline{l}) = \bar{l} + a\Phi(\bar{l}) = \Psi(\bar{l}),$$

which is also impossible, since $-\underline{l}$ cannot be equal to $\bar{l}$. Case $\bar{l} \leq 0 \leq \underline{l}$ can be treated in the same way.

Thus $\lim_{n\to\infty} x_n^2$ exists. In order to prove that $\lim_{n\to\infty} x_n = 0$, we act in the similar way as in the proof of Theorem 3.1. $\square$

4. **Stability of continuous time equations.** For abbreviation, define $X_t = \{X(s) : s \leq t\}$. Consider initial value problems for Itô-type equation

$$\begin{aligned} dX(t) &= [-\alpha(t)\Phi(X(t)) + \beta_0(t)f(X(t)) + \beta_1(t)g(X_t) + \lambda(t)]dt \\ &\quad + \sigma(t, X_t)dW(t), \quad t \geq 0, \quad X(s) = \varphi(s), \quad s \in (-\infty, 0], \end{aligned} \tag{39}$$

driven by the one-dimensional Wiener process $W = \{W(t)\}_{t\geq 0}$. Suppose that the following set of hypotheses is satisfied.

(A0) m is a positive integer, $a_0(t) \geq 0$, $b_0(t) \geq 0$, $\gamma(t) \geq 0$ for all $t \geq 0$, $a_i \geq 0$, $b_i \geq 0$ $(i = 0, 1, 2, ...)$; $\beta_j : \mathbb{R}^1_+ \to \mathbb{R}^1_+$ with $\lim_{t\to\infty} \beta_j(t) = 0$ $(j = 0, 1, 2)$ are non-negative, non-increasing, continuous functions, $\tau_0 = 0$, $\tau_i \geq 0$ for $i = 1, ..., k$, $\tau = \max_{0\leq i\leq k} \tau_i$.

(A1) $\Phi, f : \mathbb{R}^1 \to \mathbb{R}^1$ are continuous functions with

$$\forall x \in \mathbb{R}^1 \qquad xf(x) \leq c_1 x\Phi(x), \quad c_1 > 0, \tag{40}$$

$$x\Phi(x) \geq c_2 x^{2m}, \quad c_2 > 0. \tag{41}$$

(A2) $g, \sigma(t, .) : C^0([-\tau, t) \to \mathbb{R}^1$ are continuous functionals such that, for all $t \geq 0$, we have

$$|g(X_t)| \leq \int_0^\tau a_0(s)|X(t-s)|^{2m-1}ds + \sum_{i=0}^k a_i|X(t-\tau_i)|^{2m-1}, \tag{42}$$

$$\begin{aligned} \sigma^2(t, X_t) &\leq \beta_2(t)\int_0^\tau b_0(s)X(t-s)\Phi(X(t-s))ds \\ &\quad + \beta_2(t)\sum_{i=0}^k b_i X(t-\tau_i)\Phi(X(t-\tau_i)) + \gamma(t). \end{aligned} \tag{43}$$

(A3) $\mu_0(t)$ and $\mu_1(t)$ satisfy the conditions

$$\forall t \geq 0 \;:\; \mu_0(t) \geq 0, \qquad \int_0^\infty \mu_0(t)dt = \infty, \tag{44}$$

$$\int_0^\infty \mu_1(t)dt < \infty, \tag{45}$$

where μ_0, μ_1 are defined by

$$\begin{aligned} \mu_0(t) &= 2\alpha(t) - 2c_1\beta_0(t) - \frac{2A}{c_2}\beta_1(t) - B\beta_2(t) - \frac{1}{mc_2}\lambda^{2m\epsilon}(t), \\ A &= \int_0^\tau a_0(s)ds + \sum_{i=0}^k a_i, \quad B = \int_0^\tau b_0(s)ds + \sum_{i=0}^k b_i, \qquad (46) \\ \mu_1(t) &= \frac{2m-1}{m}\lambda^{\frac{2m(1-\epsilon)}{2m-1}}(t) + \gamma(t) \qquad (47) \end{aligned}$$

for some $\epsilon \in [0,1)$.

Theorem 4.1. *Assume that hypotheses (A0)-(A3) are satisfied for some $\epsilon \in [0,1)$. Then $\lim_{t\to\infty} X(t) = 0$ (a.s.) for any solution of equation (39) started at initial values $\varphi \in C^0([-\tau, 0] \to \mathbb{R}^1)$ which are independent of the σ-algebra $\sigma(\{W(s) : 0 \leq s < \infty\})$.*

Proof. Let L be the infinitesimal generator of equation (39) and $V(t) = V(t, X_t)$ be some functional on the trajectories of equation (39) for which the generator L is defined. Let us construct some functional $V(t)$ for which an estimation of $LV(t)$ depends on $X(t)$ only and does not depend on $X(s)$ for $s < t$. Following the general method of Lyapunov functionals construction [6], [7] we shall construct such functional $V(t)$ in the form $V(t) = V_1(t) + V_2(t)$, where $V_1(t) = X^2(t)$. The additional part $V_2(t)$ of the functional $V(t)$ will be chosen as stated below. Calculating $LV_1(t)$ while using (41) yields that

$$\begin{aligned} LV_1(t) &= 2X(t)[-\alpha(t)\Phi(X(t)) + \beta_0(t)f(X(t)) + \beta_1(t)g(X_t) + \lambda(t)] + \sigma^2(t, X_t) \\ &\leq [-2\alpha(t) + 2c_1\beta_0(t)]X(t)\Phi(X(t)) + 2\beta_1(t)|X(t)g(X_t)| + 2|X(t)\lambda(t)| + \sigma^2(t, X_t). \end{aligned}$$

Applying inequality (6), we have

$$\begin{aligned} |X(t)g(X_t)| &\leq \int_0^\tau a_0(s)|X(t)||X(t-s)|^{2m-1}ds + \sum_{i=0}^k a_i|X(t)||X(t-\tau_i)|^{2m-1} \\ &= \int_0^\tau a_0(s)\left[\frac{X^{2m}(t)}{2m} + \frac{(2m-1)X^{2m}(t-s)}{2m}\right] + \sum_{i=0}^k a_i\left[\frac{X^{2m}(t)}{2m} + \frac{(2m-1)X^{2m}(t-\tau_i)}{2m}\right] \\ &= \frac{X^{2m}(t)}{2m}\left(\int_0^\tau a_0(s)ds + \sum_{i=0}^k a_i\right) \\ &\quad + \frac{(2m-1)}{2m}\left(\int_0^\tau X^{2m}(t-s)a_0(s)ds + \sum_{i=0}^k a_iX^{2m}(t-\tau_i)\right) \\ &= \frac{AX^{2m}(t)}{2m} + \frac{(2m-1)}{2m}\left(\int_0^\tau a_0(s)X^{2m}(t-s)ds + \sum_{i=0}^k a_iX^{2m}(t-\tau_i)\right), \end{aligned}$$

and

$$\begin{aligned}
|X(t)\lambda(t)| &= |X(t)\lambda^{\epsilon}(t)\lambda^{1-\epsilon}(t)| \\
&\leq \frac{\lambda^{2m\epsilon}(t)X^{2m}(t)}{2m} + \frac{(2m-1)\lambda^{\frac{2m(1-\epsilon)}{2m-1}}(t)}{2m}.
\end{aligned}$$

So, assuming (40), (42), (43) with (47), we obtain that

$$\begin{aligned}
LV_1(t) &\leq \left(-2\alpha(t) + 2c_1\beta_0(t) + \frac{A\beta_1(t) + \lambda^{2m\epsilon}(t)}{mc_2}\right) X(t)\Phi(X(t)) + \mu_1(t) \\
&\quad + \frac{2m-1}{m}\beta_1(t)\int_0^{\tau} a_0(s)X^{2m}(t-s)ds \\
&\quad + \frac{2m-1}{m}\beta_1(t)\sum_{i=0}^{k} a_i X^{2m}(t-\tau_i) \\
&\quad + \beta_2(t)\int_0^{\tau} b_0(s)X(t-s)\Phi(X(t-s))ds \\
&\quad + \beta_2(t)\sum_{i=0}^{k} b_i X(t-\tau_i)\Phi(X(t-\tau_i)).
\end{aligned}$$

Recalling that β_1 and β_2 are non-increasing functions by assumption (A0) and applying condition (41), we may estimate

$$\begin{aligned}
LV_1(t) &\leq \left(-2\alpha(t) + 2c_1\beta_0(t) + \frac{A\beta_1(t) + \lambda^{2m\epsilon}(t)}{mc_2}\right) X(t)\Phi(X(t)) + \mu_1(t) \\
&\quad + \frac{2m-1}{mc_2}\left(\int_0^{\tau} a_0(s)\beta_1(t-s)X(t-s)\Phi(X(t-s))ds\right. \\
&\quad \left. + \sum_{i=0}^{k} a_i\beta_1(t-\tau_i)X(t-\tau_i)\Phi(X(t-\tau_i))\right) \\
&\quad + \int_0^{\tau} b_0(s)\beta_2(t-s)X(t-s)\Phi(X(t-s))ds \\
&\quad + \sum_{i=0}^{k} b_i\beta_2(t-\tau_i)X(t-\tau_i)\Phi(X(t-\tau_i)) \\
&= \left(-2\alpha(t) + 2c_1\beta_0(t) + \frac{A\beta_1(t) + \lambda^{2m\epsilon}(t)}{mc_2}\right) X(t)\Phi(X(t)) + \mu_1(t) \\
&\quad + \int_0^{\tau}\left(\frac{2m-1}{mc_2}a_0(s)\beta_1(t-s) + b_0(s)\beta_2(t-s)\right) X(t-s)\Phi(X(t-s))ds \\
&\quad + \sum_{i=0}^{k}\left(\frac{2m-1}{mc_2}a_i\beta_1(t-\tau_i) + b_i\beta_2(t-\tau_i)\right) X(t-\tau_i)\Phi(X(t-\tau_i)).
\end{aligned}$$

The terms with delays in this above estimation process can be neutralized in some standard way. For this purpose, we make use of the functional

$$\begin{aligned}
V_2(t) &= \int_0^{\tau}\int_{t-s}^{t}\left(\frac{2m-1}{mc_2}a_0(s)\beta_1(\theta) + b_0(s)\beta_2(\theta)\right) X(\theta)\Phi(X(\theta))d\theta ds \\
&\quad + \sum_{i=1}^{k}\int_{t-\tau_i}^{t}\left(\frac{2m-1}{mc_2}a_i\beta_1(s) + b_i\beta_2(s)\right) X(s)\Phi(X(s))ds.
\end{aligned}$$

Indeed, calculate that

$$\begin{aligned} LV_2(t) &= \Big(\frac{2m-1}{mc_2}\beta_1(t)\int_0^\tau a_0(s)ds + \beta_2(t)\int_0^\tau b_0(s)ds\Big) X(t)\Phi(X(t)) \\ &- \int_0^\tau\Big(\frac{2m-1}{mc_2}a_0(s)\beta_1(t-s) + b_0(s)\beta_2(t-s)\Big) X(t-s)\Phi(X(t-s))ds \\ &+ \Big(\frac{2m-1}{mc_2}\beta_1(t)\sum_{i=1}^k a_i + \beta_2(t)\sum_{i=1}^k b_i\Big) X(t)\Phi(X(t)) \\ &- \sum_{i=1}^k\Big(\frac{2m-1}{mc_2}a_i\beta_1(t-\tau_i) + b_i\beta_2(t-\tau_i)\Big) X(t-\tau_i)\Phi(X(t-\tau_i)). \end{aligned}$$

So, using (46) for the functional $V(t) = V_1(t) + V_2(t)$, we arrive at the required estimation

$$LV(t) \le -\mu_0(t)X(t)\Phi(X(t)) + \mu_1(t).$$

Applying now Itô formula to $V(t)$ and using the latter estimation leads to

$$\begin{aligned} dV(t) &= LV(t)dt + 2X(t)\sigma(t, X_t)dW(t) \\ &\le (-\mu_0(t)X(t)\Phi(X(t)) + \mu_1(t))dt + 2X(t)\sigma(t, X_t)dW(t). \end{aligned}$$

Therefore,

$$V(t) \le V(0) + A_1(t) - A_2(t) + M(t),$$

where

$$\begin{aligned} A_1(t) &= \int_0^t \mu_1(s)ds, \quad A_2(t) = \int_0^t \mu_0(s)X(s)\Phi(X(s))ds, \\ M(t) &= 2\int_0^t X(s)\sigma(s, X_s)dW(s). \end{aligned}$$

It is easy to see that $A_1 = \{A_1(t)\}_{t\ge 0}$ and $A_2 = \{A_2(t)\}_{t\ge 0}$ are continuous non-decreasing $(\mathcal{F}_t, \mathcal{B}(\mathbb{R}^1))$-measurable processes and $M = \{M(t)\}_{t\ge 0}$ is a continuous $\{\mathcal{F}_t\}_{t\ge 0}$-martingale. Thanks to condition (45), we know that $A_1(\infty) < \infty$. Therefore, we may apply Lemma 2.3 to the process $Z = V$ in order to conclude that a.s. $\lim_{t\to\infty} V(t) < \infty$ and $\lim_{t\to\infty} A_2(t) < \infty$ or, in another words, we have

$$\{\omega \in \Omega : A_1(\infty)(\omega) < \infty\} \subseteq \{\omega \in \Omega : V(\omega) \to\} \cap \{\omega \in \Omega : A_2(\infty)(\omega) < \infty\} \ a.s.$$

From this observation, we obtain that $V(t)$ and, therefore also $X^2(t)$, are a.s. bounded. Furthermore, we have also that $X(t)\Phi(X(t)) \le C$ for some a.s. finite random variable C. Now, verify that

$$
\begin{aligned}
V_2(t) &= \int_0^\tau \int_{t-s}^t \left(\frac{2m-1}{mc_2} a_0(s)\beta_1(\theta) + b_0(s)\beta_2(\theta) \right) x(\theta)\Phi(x(\theta)) d\theta ds \\
&\quad + \sum_{i=1}^k \int_{t-\tau_i}^t \left(\frac{2m-1}{mc_2} a_i \beta_1(s) + b_i \beta_2(s) \right) x(s)\Phi(x(s)) ds \\
&\le C \int_0^\tau \int_{t-s}^t \left(\frac{2m-1}{mc_2} a_0(s)\beta_1(\theta) + b_0(s)\beta_2(\theta) \right) d\theta ds \\
&\quad + C \sum_{i=1}^k \int_{t-\tau_i}^t \left(\frac{2m-1}{mc_2} a_i \beta_1(s) + b_i \beta_2(s) \right) ds \\
&\le C(A+B)\tau \max_{j=1,2} \beta_j(t-\tau) \to 0,
\end{aligned}
$$

via (46) and $\beta_j(t) \to 0$ as $t \to \infty$. Note that $X^2(t) = V(t) - V_2(t)$. Thus, the limit $\lim_{t\to\infty} X^2(t)$ must exist and is finite (a.s.). To prove that $\lim_{t\to\infty} X^2(t) = 0$ we use the indirect proof technique. Suppose the opposite, that is that $\lim_{t\to\infty} X^2(t) > 0$ on a set of non-zero probability. Then, there exists $\Omega_1 \subset \Omega$, $\mathbb{P}(\Omega_1) > 0$, $T = T(\omega)$ and $\zeta = \zeta(\omega) > 0$ such that $X(t,\omega)\Phi(X(t,\omega)) \ge \zeta(\omega)$ for $t \ge T(\omega)$. Consequently, we arrive at

$$
\begin{aligned}
A_2(t) &= \int_0^t \mu_0(s) X(s)\Phi(X(s)) ds \\
&= \int_0^T \mu_0(s) X(s)\Phi(X(s)) ds + \int_T^t \mu_0(s) X(s)\Phi(X(s)) ds \\
&\ge \int_T^t \mu_0(s) X(s)\Phi(X(s)) ds \ \ge\ \zeta(\omega) \int_T^t \mu_0(s) ds \to \infty
\end{aligned}
$$

as $t \to \infty$ on Ω_1 (a.s.), due to condition (44). Thus, we obtain a contradiction to the conclusion of Lemma 2.3 which proves Theorem 4.1. □

Example 4. Take any positive real constants $\lambda_j > 0$, $j = 0, 1, 2$. Put

$$
\Phi(x) = x^{2m-1}(\lambda_1 |\sin x| + \lambda_2 |\cos x|), \qquad f(x) = \lambda_0 \Phi(x) |\sin x|,
$$
$$
\alpha(t) = \frac{p}{(t+1)^{\nu_0}} > 0, \qquad \beta_j(t) = \frac{q_j}{(t+1)^{\nu_0}} \ge 0, \quad j = 0, 1, 2, \quad \nu_0 \le 1,
$$
$$
\gamma(t) = \frac{r_1}{(t+1)^{\nu_1}} \ge 0, \quad \lambda(t) = \frac{r_2}{(t+1)^{\nu_1}} \ge 0, \quad \nu_1 > 1.
$$

It is easy to see that conditions (40) with $c_1 = \lambda_0$, (41) with $c_2 = \sqrt{2}\min(\lambda_1, \lambda_2)$, (44), (45) are fulfilled if additionally

$$
p > q_0 \lambda_0 + \frac{1}{2m} \left(\frac{q_1 + r_2^{2m\varepsilon}}{c_2} + (2m-1) A q_1 \right) + \frac{1}{2} B q_2
$$

and

$$
2m\epsilon\nu_1 > 1, \qquad \frac{2m(1-\epsilon)\nu_1}{2m-1} > 1
$$

are satisfied, where A and B are defined as in (46). In fact, the latter two inequalities hold if

$$
0 < \frac{1}{2m\nu_1} < \epsilon < 1 - \frac{2m-1}{2m\nu_1} < 1.
$$

Theorem 4.1 implies that all solutions $X(t)$ of (39) have the limit $\lim_{t\to+\infty} X(t) = 0$ (a.s.).

Remark 4. Let equation (39) do not contain delays, i.e. $\beta_1(t) = 0$, $\lambda(t) = 0$, $\gamma(t) = 0$, $b_0(t) = 0$, $n = 1$, $b_1 = 1$, $\tau_1 = 0$, $\tau_0 = 0$, and, also, $\lambda(t) \equiv \gamma(t) \equiv 0$. Then $\mu_0(t) = 2\alpha(t) - 2c_1\beta_0(t) - \beta_2(t)$, $\mu_1(t) = 0$ and, instead of condition (40), it is sufficient to require that $x\Phi(x) > 0$ for all $x \neq 0$.

5. **Application to discretization of SDEs.** Here, we construct a new non-standard numerical method to illustrate the non-trivial applicability of previous results from Sections 3 and 4. Consider the Itô-type equation (39) with $\lambda(t) = 0$ discretized by the **partial nonlinear drift-implicit stochastic θ-method**

$$\begin{aligned} Y_{n+1} \;=\; & Y_n - \Big(\theta_n\alpha(t_{n+1})\Phi(Y_{n+1}) + (1-\theta_n)\alpha(t_n)\Phi(Y_n)\Big)\Delta_n + \\ & +\Big(\beta_0(t_n)f(Y_n) + \beta_1(t_n)g(\{Y_l\}_{l\leq n})\Big)\Delta_n + \sigma(t_n, \{Y_l\}_{l\leq n})\Delta W_n \end{aligned} \tag{48}$$

along non-random partitions $0 \leq t_0 < t_1 < t_2 < ... < t_n < t_{n+1} < ... < t_{n_T} \leq T$ with current step sizes $\Delta_n = t_{n+1} - t_n$ and increments $\Delta W_n = \xi_{n+1}$, where ξ_n are independent random variables with finite 2nd moments $\mathbb{E}\,\xi_{n+1}^2 = \Delta_n$ and $\mathbb{E}\,\xi_n = 0$. Such class of partial-implicit methods with non-random implicitness parameters $\theta_n \in \mathbb{R}^1$ are due to ideas of [18]. θ-method is relatively easy implementable and superior compared to the qualitative behavior of many other known standard numerical methods (e.g. with respect to numerical stability, see [21] for an overview, and [16], [17], [19], [20], [22] for more specific stability aspects). In fact, it contains backward Euler-Maruyama (i.e. $\theta_n \equiv 1$), trapezoidal rule (i.e. $\theta_n \equiv \frac{1}{2}$), and forward Euler-Maruyama method (i.e. $\theta_n \equiv 0$) - as the most commonly used stochastic Runge-Kutta methods. θ-method is used also in [31] for investigation of stability analogue of stochastic Volterra integro-differential equation. Difference analogue of linear inverted pendulum is studied in [30]. In passing, we note that method (48) is locally consistent in mean and mean square sense. Global consistency (global convergence) and its rates still need to be verified. This is left to the future due the enormous complexity of such a study. It may be noted that stability investigations play a central role in any sophisticated analysis with respect to numerical convergence (see [23], for general principles of numerical approximation theory).

In order to illustrate the fruitful applicability of our analysis and the ability to reproduce almost sure stability by method (48), we apply discretization (48) to the Itô delay equation with polynomial nonlinearity

$$\begin{aligned} dX(t) \;=\; & \Big[-\alpha(t)X^{2m-1}(t) + \beta_0(t)r(X(t))X^{2m-1}(t) \\ & +\beta_1(t)\int_0^\tau K_1(s)X^{2m-1}(t-s)ds\Big]dt + \sigma(t, X_t)dW(t), \end{aligned} \tag{49}$$

where $m \in \mathbb{N}$, $m \geq 1$, $|r(x)| \leq r_0$, $\tau > 0$, $\alpha(t) \geq 0$, $\beta_0(t) \geq 0$, $\beta_1(t) \geq 0$, $X(0) = \varphi(0)$ and

$$\sigma^2(t, x_t) = \sigma_0^2(t) + \sigma_1^2(t)x^{2m}(t) + \sigma_2^2(t)\int_0^\tau K_2(s)x^{2m}(t-s)ds.$$

To satisfy hypothesis (H1), assumptions (A1) and (A2), we set

$$\Phi(x) = x^{2m-1}, \quad f(x) = r(x)x^{2m-1}, \quad g(x_t) = \int_0^\tau K_1(s)x^{2m-1}(t-s)ds,$$

$$c = c_2 = 1, \quad c_1 = r_0, \quad \beta_2(t) = \max\{\sigma_1^2(t), \sigma_2^2(t)\},$$

$$
\begin{gathered}
k_i = \int_0^\tau |K_i(s)|ds, \quad i = 1, 2, \\
a_0(s) = |K_1(s)|, \quad a_i = 0, \ \ i = 0, 1, ..., k, \\
b_0(s) = |K_2(s)|, \quad b_0 = 1, \ \ b_i = 0, \ \ i = 1, ..., k, \\
A = k_1, \quad B = 1 + k_2, \quad \mu_1(t) = \gamma(t) = \sigma_0^2(t), \\
\mu_0(t) = 2\left[\alpha(t) - r_0\beta_0(t) - k_1\beta_1(t)\right] - (1 + k_2)\beta_2(t).
\end{gathered}
$$

If

$$
\mu_0(t) \geq 0, \quad \int_0^\infty \mu_0(t)dt = \infty, \quad \int_0^\infty \sigma_0^2(t)dt < \infty,
$$

then all conditions of Theorem 4.1 are fulfilled.

After discretizing equation (49) by partial nonlinear drift-implicit stochastic θ-method (48), we arrive at the following stochastic difference equation

$$
\begin{aligned}
Y_{n+1} \quad = \quad & Y_n - \Big(\theta_n\alpha(t_{n+1})Y_{n+1}^{2m-1} + (1-\theta_n)\alpha(t_n)Y_n^{2m-1}\Big)\Delta_n + \\
& + \left(\beta_0(t_n)r(Y_n)Y_n^{2m-1} + \beta_1(t_n)\sum_{i=0}^{r-1} K_1(t_i)Y_{n-i}^{2m-1}\Delta_i\right)\Delta_n + \\
& + \sigma(t_n, \{Y_l\}_{l\leq n})\Delta W_n,
\end{aligned} \tag{50}
$$

with $\tau = \sum_{l=0}^{r-1}\Delta_l$ and

$$
\sigma^2(t_n, \{Y_l\}_{l\leq n}) = \sigma_0^2(t_n) + \sigma_1^2(t_n)Y_n^{2m} + \sigma_2^2(t_n)\sum_{i=0}^{r-1} K_2(t_i)Y_{n-i}^{2m}\Delta_i.
$$

We set

$$
\xi_{n+1} = \frac{W(t_{n+1}) - W(t_n)}{\sqrt{\Delta_n}}, \tag{51}
$$

then $\{\xi_n\}_{n\geq 0}$ is a sequence of standardized normally distributed random variables. Therefore, for all $n \in \mathbb{N}$, we have

$$
\eta_n = \mathbb{E}\left[\xi_n^2\right] = 1.
$$

Equation (50) takes the form

$$
\begin{aligned}
Y_{n+1} \quad = \quad & Y_n - \Delta_n\Big(\theta_n\alpha(t_{n+1})Y_{n+1}^{2m-1} + (1-\theta_n)\alpha(t_n)Y_n^{2m-1}\Big) \\
& + \Delta_n\left(\beta_0(t_n)r(Y_n)Y_n^{2m-1} + \beta_1(t_n)\sum_{i=0}^{r-1}\Delta_i K_1(t_i)Y_{n-i}^{2m-1}\right) \\
& + \sqrt{\Delta_n}\sigma(t_n, \{Y_l\}_{l\leq n})\xi_{n+1}.
\end{aligned} \tag{52}
$$

We identify

$$
\begin{gathered}
a_n = \theta_n\alpha(t_{n+1})\Delta_n, \quad \kappa_n = -(1-\theta_n)\alpha(t_n)\Delta_n, \\
c = 1, \quad \nu_n = \sigma_0^2(t_n)\Delta_n, \quad \delta_n = r_0^2\beta_0^2(t_n)\Delta_n^2,
\end{gathered}
$$

$$
\gamma_{i,n}^{(0)} = \begin{cases} \beta_1(t_n)\Delta_n|K_1(t_i)|\Delta_i, & i < r, \\ 0, & i \geq r, \end{cases} \qquad \gamma_{i,n}^{(2)} = 0, \quad i \geq 0,
$$

$$
\gamma_{i,n}^{(2)} = \begin{cases} \sigma_1^2(t_n)\Delta_n + \sigma_2^2(t_n)\Delta_n|K_2(t_0)|\Delta_0, & i = 0, \\ \sigma_2^2(t_n)\Delta_n|K_2(t_i)|\Delta_i, & 0 < i < r, \\ 0, & i \geq r. \end{cases}
$$

5.1. **Discussion on condition (H0).** Suppose that $\alpha : \mathbb{R}^1 \to \mathbb{R}^1_+$ is a positive continuous function.

$$a_n = \theta_n \alpha(t_{n+1})\Delta_n, \quad \kappa_n = -(1-\theta_n)\alpha(t_n)\Delta_n \tag{53}$$

for $n \in \mathbb{N}$. In order to apply Theorem 3.1 (in conjunction with Remark 2), we need to require

$$\theta_{n-1}\alpha(t_n)\Delta_{n-1} \geq \theta_n \alpha(t_{n+1})\Delta_n > |1-\theta_n|\alpha(t_n)\Delta_n \tag{54}$$

for sufficiently large $n \geq N_0$ (where N_0 is non-random). From here it follows that, for sufficiently large $n \geq N_0$, we have

$$\frac{\theta_{n-1}\Delta_{n-1}}{\theta_n \Delta_n} \geq \frac{\alpha(t_{n+1})}{\alpha(t_n)} \tag{55}$$

and

$$\theta_n + \theta_{n-1}\frac{\Delta_{n-1}}{\Delta_n} > 1 > \theta_n - \theta_{n-1}\frac{\Delta_{n-1}}{\Delta_n}. \tag{56}$$

Condition (56) holds if (55) and the requirement

$$\theta_n \left(1 + \frac{\alpha(t_{n+1})}{\alpha(t_n)}\right) > 1 > \theta_n \left(1 - \frac{\alpha(t_{n+1})}{\alpha(t_n)}\right) \tag{57}$$

are satisfied.

Let us suppose that the limit

$$q_0 = \lim_{n\to\infty} \frac{\alpha(t_{n+1})}{\alpha(t_n)}$$

exists and consider the following five cases:

- (i) $\alpha(t) \equiv \alpha_0$;
- (ii) for all $n \in \mathbb{N}$ the sequence $\alpha(t_n)$ is non-increasing one and $q_0 = 1$;
- (iii) for all $n \in \mathbb{N}$ the sequence $\alpha(t_n)$ is non-increasing one $q_0 < 1$;
- (iv) for all $n \in \mathbb{N}$ the sequence $\alpha(t_n)$ is non-decreasing one and $q_0 = 1$;
- (v) for all $n \in \mathbb{N}$ the sequence $\alpha(t_n)$ is non-decreasing one and $q_0 > 1$.

In conjunction with Remark 2, in the rest of this subsection we consider only sufficiently large n, i.e. the situation when there exists a non-random number N such that all relations, discussed below, hold for $n \geq N$.

In the case (i) (55) holds when $\{\Delta_n \theta_n\}_{n\in\mathbf{N}}$ is a non-increasing sequence and, for sufficiently large $n \geq N_0$, we have

$$\theta_n > \frac{1}{2}. \tag{58}$$

In particular, θ_n can be a constant, i.e. $\theta_n = \theta > \frac{1}{2}$ and $\{\Delta_n\}_{n\in\mathbf{N}}$ can be arbitrary non-increasing sequence.

In the case (ii), condition (57) holds when $\theta_n > \frac{1}{2}$. Condition (55) and the right-hand-side of (56) are satisfied when the sequence $\{\Delta_n\theta_n\}_{n\in\mathbf{N}}$ is non-increasing. The left-hand-side of (56) holds when, in addition to (58), the sequence $\{\theta_n\}_{n\in\mathbf{N}}$ is non-increasing. In particular θ_n and Δ_n can be constants, $\theta_n = \theta > \frac{1}{2}$.

In the case (iii) condition (57) holds if

$$\frac{1}{1+q_0} < \theta_n < \frac{1}{1-q_0}. \tag{59}$$

In particular, when $q_0 = \frac{1}{2}$, θ_n can be a constant such that $\theta_n \equiv \theta \in \left(\frac{2}{3}, 2\right)$ and Δ_n has to satisfy the following condition

$$\frac{\Delta_{n-1}}{\Delta_n} > \max\left\{\frac{1}{\theta} - 1,\, q_0\right\}. \tag{60}$$

Since the right-hand-side of (60) is less than 1, condition (60) holds for any non-increasing sequence $\{\Delta_n\}_{n\in\mathbf{N}}$, and therefore, for constant $\Delta_n \equiv \Delta > 0$.

In the case (iv) condition (56) holds when sequence $\{\Delta_n\theta_n\}_{n\in\mathbf{N}}$ is strictly decreasing and (58) is fulfilled. More details on this case are given by Example 6 below.

In the case (v) conditions (55) and (57) imply that if $\Delta_n \equiv \Delta$, then θ_n satisfies two contradictory conditions

$$\lim_{n\to\infty} \frac{\theta_{n-1}}{\theta_n} \ge q_0 > 1, \qquad \lim_{n\to\infty} \theta_n \ge \frac{1}{1+q_0} > 0.$$

Thus case (v) is impossible.

Example 5. Let $\alpha(t) = \frac{\alpha_0}{(t+1)^q}$, $\alpha_0 > 0$, $q \in [0,1]$. Then

$$\alpha(t_n) = \frac{\alpha_0}{(\sum_{i=1}^n \Delta_i + 1)^q}.$$

When $q = 0$ we have the case (i). When $q > 0$ and either $\Delta_n \to 0$ or $\sum_{i=1}^n \Delta_i \to \infty$, as $n \to \infty$, we obtain the case (ii).

Example 6. Let $\alpha(t) = \alpha_0\left(1 - \frac{\upsilon_0}{t+1}\right)$, $\alpha_0 > 0$, $\upsilon_0 \in (0,1)$. Then

$$\alpha(t_n) = \alpha_0\left(1 - \frac{\upsilon_0}{\sum_{i=1}^n \Delta_i + 1}\right).$$

When $\Delta_n \to 0$ or $\sum_{i=1}^n \Delta_i \to \infty$, as $n \to \infty$, we have the case (iv).

Let $p < 1$, $\Delta_n = p^n\Delta_0$ and $\theta_n \equiv \theta$. Then $\Delta_n \downarrow 0$ and conditions (55) and (56) hold.

Example 7. Let $\alpha(t) = \alpha_0 e^{\upsilon_0 t}$, $\alpha_0 > 0$, $\upsilon_0 > 0$, $t_n = n\Delta$. Then

$$\alpha(t_n) = \alpha_0 e^{\upsilon_0 n\Delta}, \qquad q_0 = \alpha_0 e^{\upsilon_0\Delta} > 1.$$

Here we encounter the case (v).

5.2. **Discussion on hypothesis (H3) with conditions (16), (17).** Conditions (16), (17) for equation (48) take the form

$$\begin{aligned}
\frac{\epsilon_n^{(1)}}{\Delta_n} &= 2\left(\theta_n\alpha(t_{n+1}) + (1-\theta_n)\alpha(t_n) - r_0\beta_0(t_n) - \beta_1(t_n)\sum_{i=0}^{r-1}\Delta_i|K_1(t_i)|\right) \\
&\quad - \left(\sigma_1^2(t_n) + \sigma_2^2(t_n)\sum_{i=0}^{r-1}\Delta_i|K_2(t_i)|\right) \ge 0, \qquad (61)\\
\frac{\epsilon_n^{(2)}}{\Delta_n^2} &= \theta_n^2\alpha^2(t_{n+1}) - (1-\theta_n)^2\alpha^2(t_n) \\
&\quad -2|1-\theta_n|\alpha(t_n)\left[r_0\beta_0(t_n) + \beta_1(t_n)\sum_{i=0}^{r-1}\Delta_i|K_1(t_i)|\right] \\
&\quad - \left(r_0\beta_0(t_n) + \beta_1(t_n)\sum_{i=0}^{r-1}\Delta_i|K_1(t_i)|\right)^2 \ge 0. \qquad (62)
\end{aligned}$$

For the sake of briefness, we consider only the case (iv), i.e., for all $n \in \mathbb{N}$, the sequence $\alpha(t_n)$ is non-decreasing one and $q_0 = 1$. Other cases can be treated in a similar way.

Let us suppose that $\sup_{t\geq 0} \beta_i(t) = \beta_i$, $i = 0, 1$, $\sup_{t\geq 0} \sigma_i^2(t) = \sigma_i^2$, $i = 1, 2$, and also

$$\int_0^\tau |K_j(s)|ds < \infty, \quad j = 1, 2,$$

which implies that there are constants K_j and Δ^* such that for $\Delta_i < \Delta^*$

$$\sum_{i=0}^{r-1} \Delta_i |K_j(t_i)| \leq K_j.$$

Then, from (61) and (62), it follows that

$$\frac{\epsilon_n^{(1)}}{\Delta_n} \geq 2\left(\alpha(t_n) - B\right) - S,$$

$$\frac{\epsilon_n^{(2)}}{\Delta_n^2} \geq (2\theta_n - 1)\alpha^2(t_n) - 2|1 - \theta_n|\alpha(t_n)B - B^2,$$

where

$$B = r_0\beta_0 + K_1\beta_1, \qquad S = \sigma_1^2 + K_2\sigma_2^2.$$

It is easy to check that conditions (61), (62) hold if for all sufficiently large $n \geq N_0$

$$\alpha(t_n) \geq B + \frac{1}{2}S, \qquad \theta_n \geq \frac{1}{2}\left(1 + \frac{B}{\alpha(t_n)}\right) > \frac{1}{2}.$$

So we can clearly recognize that the theorems from previous sections are applicable to stochastic differential equations and their discretization with memory.

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