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About Lyapunov Functionals Construction for Difference Equations with Continuous Time

L. SHAIKHET
Department of Higher Mathematics
Donetsk State University of Management
Chelyuskintsev str., 163-a, Donetsk 83015, Ukraine
leonid@dsum.ua leonid.shaikhet@usa.net

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Abstract—Stability investigation of hereditary systems is connected often with construction of Lyapunov functionals. One general method of Lyapunov functionals construction was proposed and developed in [1–9] both for differential equations with aftereffect and for difference equations with discrete time. Here, some modification of Lyapunov-type stability theorem is proposed, which allows one to use this method for difference equations with continuous time also.

Keywords—Difference equations with continuous time, Method of Lyapunov functionals construction, Stability.

1. STABILITY THEOREM

Consider the difference equation in the form

$$x(t+h_0) = F(t, x(t), x(t-h_1), x(t-h_2), \dots), \qquad t > t_0 - h_0, \tag{1.1}$$

with the initial condition

$$x(\theta) = \phi(\theta), \qquad \theta \in \Theta = \left[t_0 - h_0 - \max_{j\geq 1} h_j, t_0\right]. \tag{1.2}$$

Here $x \in \mathbf{R}^n$, $h_0, h_1, \dots$ are positive constants, the functional F satisfies the condition

$$|F(t, x_0, x_1, x_2, \dots)| \leq \sum_{j=0}^{\infty} a_j |x_j|, \qquad A = \sum_{j=0}^{\infty} a_j < \infty. \tag{1.3}$$

A solution of problem (1.1),(1.2) is a process $x(t) = x(t; t_0, \phi)$, which is equal to the initial function $\phi(t)$ from (1.2) for $t \leq t_0$ and is defined by equation (1.1) for $t > t_0$.

DEFINITION 1.1. *The trivial solution of equation (1.1),(1.2) is called stable if for any* $\epsilon > 0$ *and* $t_0 \geq 0$ *there exists a* $\delta = \delta(\epsilon, t_0) > 0$, *such that* $|x(t; t_0, \phi)| < \epsilon$, *for all* $t \geq t_0$ *if* $\|\phi\| = \sup_{\theta\in\Theta} |\phi(\theta)| < \delta$.

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DEFINITION 1.2. *The trivial solution of equation (1.1),(1.2) is called asymptotically stable if it is stable and* $\lim_{t\to\infty} x(t;t_0,\phi)=0$ *for all initial functions* ϕ.

DEFINITION 1.3. *The solution of equation (1.1) with initial condition (1.2) is called p-integrable,* $p>0$, *if*

$$\int_{t_0}^{\infty} |x(t;t_0,\phi)|^p\,dt<\infty. \tag{1.4}$$

In particular, if $p=2$ *then the solution* $x(t;t_0,\phi)$ *is called square integrable.*

THEOREM 1.1. *Let there exist a nonnegative functional* $V(t)=V(t,x(t),x(t-h_1),x(t-h_2),\dots)$ *and positive numbers* c_1, c_2, p, *such that*

$$V(t)\le c_1\sup_{s\le t}|x(s)|^p,\qquad t\in[t_0,t_0+h_0), \tag{1.5}$$

$$\Delta V(t)\le -c_2|x(t)|^p,\qquad t\ge t_0, \tag{1.6}$$

where

$$\Delta V(t)=V(t+h_0)-V(t). \tag{1.7}$$

Then the trivial solution of equation (1.1) is stable.

PROOF. From conditions (1.6),(1.7) it follows

$$c_2|x(t)|^p\le V(t),\qquad t\ge t_0. \tag{1.8}$$

On the other hand, using conditions (1.6),(1.7), we have

$$V(t)\le V(t-h_0)\le V(t-2h_0)\le\cdots\le V(s),\qquad t\ge t_0, \tag{1.9}$$

where $s=t-[(t-t_0)/(h_0)]h_0\in[t_0,t_0+h_0)$, $[t]$ is the integer part of a number t. From (1.5), it follows

$$\sup_{s\in[t_0,t_0+h_0)} V(s)\le c_1\sup_{t\le t_0+h_0}|x(t)|^p. \tag{1.10}$$

Using (1.1),(1.3),(1.2), for $t\le t_0+h_0$, we obtain

$$\begin{aligned}|x(t)|&=|F(t,x(t-h_0),x(t-h_0-h_1),x(t-h_0-h_2),\dots)|\\&\le a_0|\phi(t-h_0)|+\sum_{j=1}^{\infty}a_j|\phi(t-h_0-h_j)|\le\sum_{j=0}^{\infty}a_j\|\phi\|=A\|\phi\|.\end{aligned} \tag{1.11}$$

From (1.8)–(1.11), it follows $c_2|x(t)|^p\le c_1A^p\|\phi\|^p$ for $t\ge t_0$. It means that the trivial solution of equation (1.1),(1.2) is stable. The theorem is proven.

REMARK 1.1. If the conditions of Theorem 1.1 hold and $A<1$ (A is defined by (1.3)) then the trivial solution of equation (1.1),(1.2) is asymptotically stable. To prove this, it is enough similar to (1.11) to show that

$$|x(t)|\le A^{[(t-t_0)/h_0]+1}\|\phi\|,\qquad t\ge t_0.$$

REMARK 1.2. If the conditions of Theorem 1.1 hold, then the solution of equation (1.1) for each initial function (1.2) is p-integrable. Really, integrating (1.6) from $t=t_0$ to $t=T$, by virtue of (1.7) we have

$$\int_T^{T+h_0}V(t)\,dt-\int_{t_0}^{t_0+h_0}V(t)\,dt\le -c_2\int_{t_0}^{T}|x(t)|^p\,dt.$$

From here and (1.10),(1.11), it follows

$$c_2\int_{t_0}^{T}|x(t)|^p\,dt\le\int_{t_0}^{t_0+h_0}V(t)\,dt\le c_1A^p\|\phi\|^p h_0<\infty$$

and, by $T\to\infty$, we obtain (1.4).

COROLLARY 1.1. *Let there exist a functional* $V(t) = V(t, x(t), x(t-h_1), x(t-h_2), \dots)$ *and positive numbers* c_1, c_2, p, *such that conditions (1.5),(1.8) and* $\Delta V(t) \le 0$ *hold. Then the trivial solution of equation (1.1) is stable.*

From Theorem 1.1, Remarks 1.1 and 1.2, and Corollary 1.1, it follows that an investigation of behavior of the solution of equation (1.1) can be reduced to construction of appropriate Lyapunov functionals. Below some formal procedure of Lyapunov functionals construction for equations of type (1.1) is described.

2. FORMAL PROCEDURE OF LYAPUNOV FUNCTIONALS CONSTRUCTION

The proposed procedure of Lyapunov functionals construction consists of four steps.

STEP 1. Represent the right-hand side of equation (1.1) in the form

$$F(t) = F_1(t) + F_2(t) + \Delta F_3(t), \tag{2.1}$$

where $F_1(t) = F_1(t, x(t), x(t-h_1), \dots, x(t-h_k))$, $F_2(t) = F_2(t, x(t), x(t-h_1), x(t-h_2), \dots)$, $F_3(t) = F_3(t, x(t), x(t-h_1), x(t-h_2), \dots)$, $k \ge 0$ is a given integer, $F_j(t, 0, 0, \dots) = 0$, $j = 1, 2, 3$, the operator Δ is defined by (1.7).

STEP 2. Suppose that for the auxiliary equation

$$y(t+h_0) = F_1(t, y(t), y(t-h_1), \dots, y(t-h_k)), \qquad t > t_0 - h_0,$$

there exists a Lyapunov functional $v(t) = v(t, y(t), y(t-h_1), \dots, y(t-h_k))$, which satisfies the conditions of Theorem 1.1.

STEP 3. Consider Lyapunov functional $V(t)$ for equation (1.1) in the form $V(t) = V_1(t) + V_2(t)$, where the main component is $V_1(t) = v(t, x(t) - F_3(t), x(t-h_1), \dots, x(t-h_k))$. Calculate $\Delta V_1(t)$ and in a reasonable way, estimate it.

STEP 4. In order to satisfy the conditions of Theorem 1.1, the additional component $V_2(t)$ is chosen by some standard way.

3. LINEAR VOLTERRA EQUATIONS WITH CONSTANT COEFFICIENTS

Let us demonstrate the formal procedure of Lyapunov functionals construction described above for stability investigation of the scalar equation

$$x(t+1) = \sum_{j=0}^{[t]+r} a_j x(t-j), \quad t > -1, \qquad x(s) = \phi(s), \quad s \in [-r, 0], \tag{3.1}$$

where $r \ge 0$ is a given integer, a_j are known constants.

3.1. The First Way of Lyapunov Functional Construction

Following Step 1 of the procedure represent equation (3.1) in form (2.1) with $F_3(t) = 0$,

$$F_1(t) = \sum_{j=0}^{k} a_j x(t-j), \quad F_2(t) = \sum_{j=k+1}^{[t]+r} a_j x(t-j), \qquad k \ge 0, \tag{3.2}$$

and consider (Step 2) the auxiliary equation

$$\begin{aligned} y(t+1) &= \sum_{j=0}^{k} a_j y(t-j), \qquad t > -1, \quad k \ge 0, \\ y(s) &= \phi(s), \quad s \in [-r, 0], \qquad y(s) = 0, \quad s < -r. \end{aligned} \tag{3.3}$$

Introduce into consideration the vector $Y(t) = (y(t-k), \dots, y(t-1), y(t))'$ and represent the auxiliary equation (3.3) in the form

$$Y(t+1) = AY(t), \qquad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ a_k & a_{k-1} & a_{k-2} & \cdots & a_1 & a_0 \end{pmatrix}. \tag{3.4}$$

Consider the matrix equation

$$A'DA - D = -U, \qquad U = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{3.5}$$

and suppose that the solution D of this equation is a positive semidefinite symmetric matrix of dimension $k+1$ with $d_{k+1,k+1} > 0$. In this case, the function $v(t) = Y'(t)DY(t)$ is Lyapunov function for equation (3.4), i.e., it satisfies the condition of Theorem 1.1, in particular, condition (1.6) with $p = 2$. Really, using (3.4), we have $\Delta v(t) = Y'(t+1)DY(t+1) - Y'(t)Dy(t) = Y'(t)[A'DA - D]Y(t) = -Y'(t)UY(t) = -y^2(t)$.

Following Step 3 of the procedure, we will construct Lyapunov functional $V(t)$ for equation (3.1) in the form $V(t) = V_1(t) + V_2(t)$, where $V_1(t) = X'(t)DX(t)$, $X(t) = (x(t-k), \dots, x(t-1), x(t))'$. Rewrite equation (3.1) by virtue of representation (3.2) in the matrix form

$$X(t+1) = AX(t) + B(t), \tag{3.6}$$

where matrix A is defined by (3.4), $B(t) = (0, \dots, 0, F_2(t))'$. Calculating $\Delta V_1(t)$, by virtue of equation (3.6) we have

$$\begin{aligned} \Delta V_1(t) &= X'(t+1)DX(t+1) - X'(t)DX(t) \\ &= (AX(t) + B(t))'D(AX(t) + B(t)) - X'(t)DX(t) \\ &= -x^2(t) + B'(t)DB(t) + 2B'(t)DAX(t). \end{aligned} \tag{3.7}$$

Put

$$\alpha_l = \sum_{j=l}^{\infty} |a_j|, \qquad l = 0, 1, \dots. \tag{3.8}$$

Then, using the representation for $B(t)$, (3.2), (3.8), and (3.4), we obtain

$$\begin{aligned} B'(t)DB(t) = d_{k+1,k+1}F_2^2(t) &= d_{k+1,k+1} \left(\sum_{j=k+1}^{[t]+r} a_j x(t-j) \right)^2 \\ &\le d_{k+1,k+1}\alpha_{k+1} \sum_{j=k+1}^{[t]+r} |a_j| x^2(t-j) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} B'(t)DAX(t) &= F_2(t) \left[\sum_{l=1}^{k} d_{l,k+1} x(t-k+l) + d_{k+1,k+1} \sum_{m=0}^{k} a_m x(t-m) \right] \\ &= F_2(t) \left[\sum_{m=0}^{k-1} \left(a_m d_{k+1,k+1} + d_{k-m,k+1} \right) x(t-m) + a_k d_{k+1,k+1} x(t-k) \right] \\ &= F_2(t) d_{k+1,k+1} \sum_{m=0}^{k} Q_{km} x(t-m), \end{aligned} \tag{3.10}$$

where

$$Q_{km} = a_m + \frac{d_{k-m,k+1}}{d_{k+1,k+1}}, \quad m = 0, \dots, k-1, \qquad Q_{kk} = a_k. \tag{3.11}$$

Putting

$$\beta_k = \sum_{m=0}^{k} |Q_{km}| = |a_k| + \sum_{m=0}^{k-1} \left| a_m + \frac{d_{k-m,k+1}}{d_{k+1,k+1}} \right| \tag{3.12}$$

and using (3.10)–(3.12), (3.2), (3.8), we obtain

$$\begin{aligned} 2B'(t)DAX(t) &= 2d_{k+1,k+1} \sum_{m=0}^{k} \sum_{j=k+1}^{[t]+r} Q_{km} a_j x(t-m)x(t-j) \\ &\le d_{k+1,k+1} \sum_{m=0}^{k} \sum_{j=k+1}^{[t]+r} |Q_{km}|\,|a_j|(x^2(t-m) + x^2(t-j)) \\ &\le d_{k+1,k+1} \left(\alpha_{k+1} \sum_{m=0}^{k} |Q_{km}| x^2(t-m) + \beta_k \sum_{j=k+1}^{[t]+r} |a_j| x^2(t-j) \right). \end{aligned} \tag{3.13}$$

Now put

$$q_k = \alpha_{k+1} + \beta_k, \qquad R_{km} = \begin{cases} \alpha_{k+1}|Q_{km}|, & 0 \le m \le k, \\ q_k |a_m|, & m > k. \end{cases} \tag{3.14}$$

Then, from (3.7),(3.9),(3.13),(3.14), it follows

$$\Delta V_1(t) \le -x^2(t) + d_{k+1,k+1} \sum_{m=0}^{[t]+r} R_{km} x^2(t-m). \tag{3.15}$$

Now choose (Step 4) the functional $V_2(t)$ in the form

$$V_2(t) = d_{k+1,k+1} \sum_{m=1}^{[t]+r} \gamma_m x^2(t-m), \qquad \gamma_m = \sum_{j=m}^{\infty} R_{kj}. \tag{3.16}$$

Calculating $\Delta V_2(t)$, we obtain

$$\begin{aligned} \Delta V_2(t) &= d_{k+1,k+1} \left(\sum_{m=1}^{[t]+1+r} \gamma_m x^2(t+1-m) - \sum_{m=1}^{[t]+r} \gamma_m x^2(t-m) \right) \\ &= d_{k+1,k+1} \left(\sum_{m=0}^{[t]+r} \gamma_{m+1} x^2(t-m) - \sum_{m=1}^{[t]+r} \gamma_m x^2(t-m) \right) \\ &= d_{k+1,k+1} \left(\gamma_1 x^2(t) - \sum_{m=1}^{[t]+r} R_{km} x^2(t-m) \right). \end{aligned} \tag{3.17}$$

From (3.15),(3.17) for the functional $V(t) = V_1(t) + V_2(t)$, we have $\Delta V(t) \le -(1 - \gamma_0 d_{k+1,k+1}) x^2(t)$. If $\gamma_0 d_{k+1,k+1} < 1$, then the functional $V(t)$ satisfies the conditions of Theorem 1.1. If $\gamma_0 d_{k+1,k+1} = 1$, then the functional $V(t)$ satisfies the conditions of Corollary 1.1. So, if $\gamma_0 d_{k+1,k+1} \le 1$, then the trivial solution of equation (3.1) is stable. Using (3.16),(3.14),(3.12), transform γ_0 by the following way

$$\begin{aligned} \gamma_0 &= \sum_{j=0}^{\infty} R_{kj} = \sum_{j=0}^{k} R_{kj} + \sum_{j=k+1}^{\infty} R_{kj} = \alpha_{k+1} \sum_{j=0}^{k} |Q_{kj}| + q_k \sum_{j=k+1}^{\infty} |a_j| \\ &= \alpha_{k+1}\beta_k + (\alpha_{k+1} + \beta_k)\alpha_{k+1} = \alpha_{k+1}^2 + 2\alpha_{k+1}\beta_k. \end{aligned}$$

So, if

$$\alpha_{k+1}^2 + 2\alpha_{k+1}\beta_k \leq d_{k+1,k+1}^{-1}, \tag{3.18}$$

then, the trivial solution of equation (3.1) is stable. As follows from Remark 1.2 in the case of strict inequality (3.18), the solution of equation (3.1) is square integrable also.

REMARK 3.1. Suppose that in equation (3.1) $a_j = 0$ for $j > k$ and matrix equation (3.5) has a positive semidefinite solution with $d_{k+1,k+1} > 0$. Then the trivial solution of equation (3.1) is stable and square integrable. Really, in this case, $\alpha_{k+1} = 0$ and condition (3.18) strictly holds.

3.2 The Second Way of Lyapunov Functional Construction

Let us get another stability condition. Equation (3.1) can be represented (Step 1) in form (2.1) with $k = 0$, $F_2(t) = 0$,

$$F_1(t) = \beta x(t), \quad \beta = \sum_{j=0}^{\infty} a_j, \qquad F_3(t) = -\sum_{m=1}^{[t]+r} x(t-m) \sum_{j=m}^{\infty} a_j. \tag{3.19}$$

It is easy to check calculating $\Delta F_3(t)$ similar to (3.17).

In this case, the auxiliary equation (Step 2) has the form $y(t+1) = \beta y(t)$ and the function $v(t) = y^2(t)$ is Lyapunov function for this equation if $|\beta| < 1$. We will construct (Step 3) Lyapunov functional $V(t)$ for equation (3.1) in the form $V(t) = V_1(t) + V_2(t)$, where $V_1(t) = (x(t) - F_3(t))^2$. Calculating $\Delta V_1(t)$, by virtue of representation (3.19), we have

$$\Delta V_1(t) = (x(t+1) - F_3(t+1))^2 - (x(t) - F_3(t))^2 = (\beta^2 - 1)x^2(t) - Q(t),$$

where $Q(t) = 2(\beta - 1)x(t)F_3(t)$. Putting

$$\alpha = \sum_{m=1}^{\infty} \left| \sum_{j=m}^{\infty} a_j \right|, \qquad B_m = |\beta - 1| \left| \sum_{j=m}^{\infty} a_j \right|,$$

and using (3.19), we obtain $|Q(t)| \leq \alpha|\beta - 1|x^2(t) + \sum_{m=1}^{[t]+r} B_m x^2(t-m)$. As a result,

$$\Delta V_1(t) \leq \left[\beta^2 - 1 + \alpha|\beta - 1|\right] x^2(t) + \sum_{m=1}^{[t]+r} B_m x^2(t-m).$$

Now put (Step 4)

$$V_2(t) = \sum_{m=1}^{[t]+r} \delta_m x^2(t-m), \qquad \delta_m = \sum_{j=m}^{\infty} B_j.$$

Calculating $\Delta V_2(t)$ similar to (3.17) and using $\delta_1 = \alpha|\beta - 1|$, for the functional $V(t) = V_1(t) + V_2(t)$ we obtain $\Delta V(t) \leq [\beta^2 - 1 + 2\alpha|\beta - 1|]x^2(t)$. Thus, if

$$\beta^2 + 2\alpha|\beta - 1| < 1, \tag{3.20}$$

then the trivial solution of equation (3.1) is stable and (Remark 1.2) square integrable. It is easy to see that condition (3.20) can also be written in the form $2\alpha < 1 + \beta$, $|\beta| < 1$.

REMARK 3.2. Similar to [1–9] one can show that the method of Lyapunov functionals construction described above can be used also for stochastic difference equations with continuous time.

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