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STEADY-STATE SOLUTIONS OF NONLINEAR MODEL OF INVERTED PENDULUM

Nonlinear mathematical model of inverted pendulum with stochastic perturbations and nonclassical method of stabilization is considered. Nonzero steady-state solutions of this system are studied. Conditions by which equilibrium points are stable, unstable or one-sided stable are obtained. Theoretical results are illustrated by numerical examples.

1. Introduction

The problem of stabilization of the inverted pendulum is very popular among the researches (see, for instance [1-6]). The linear mathematical model of the controlled inverted pendulum is described by linear differential equation of second order

$$\ddot{x}(t) - ax(t) = u(t), \qquad a > 0, \qquad t \geq 0. \tag{1.1}$$

The classical way of stabilization of the system (1.1) uses the control $u(t)$ in the form

$$u(t) = -b_1 x(t) - b_2 \dot{x}(t), \qquad b_1 > a, \qquad b_2 > 0.$$

In contrast to the classical way another way of stabilization was proposed in [7-9]. It was supposed that only the trajectory of the pendulum is observed, control $u(t)$ depends on whole trajectory $x(s)$, $s \leq t$, and has the form

$$u(t) = \int_0^\infty dK(\tau) x(t-\tau). \tag{1.2}$$

The initial condition for the system (1.3), (1.2) has the form

$$x(s) = \varphi(s), \qquad \dot{x}(s) = \dot{\varphi}(s), \qquad s \leq 0, \tag{1.3}$$

where $\varphi(s)$ is a given continuously differentiable function.

Definition 1.1. The zero solution of the system (1.1)-(1.3) is called stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that $\max\{|x(t)|, |\dot{x}(t)|\} < \epsilon$ for all $t \geq 0$ if $\|\varphi\| = \sup_{s\leq 0}(|\varphi(s)| + |\dot{\varphi}(s)|) < \delta$. If, besides, $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} \dot{x}(t) = 0$ for every initial function φ, then the zero solution of the system (1.1)-(1.3) is called asymptotically stable in the whole.

Put

$$k_i = \int_0^\infty \tau^i dK(\tau), \quad i = 0, 1, \qquad k_2 = \int_0^\infty \tau^2 |dK(\tau)|,$$
$$a_1 = -a - k_0, \qquad p_1 = \frac{a_1 + 1}{k_1}. \tag{1.4}$$

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Theorem 1.1. *Let*

$$k_0 < -a, \qquad k_1 > 0, \qquad k_2 < \frac{4}{1+\sqrt{1+p_1^2}}. \tag{1.5}$$

Then the zero solution of the system (1.1)-(1.3) is asymptotically stable in the whole.

This theorem is proved in [9] using a special method of Lyapunov functionals construction [10-14].

Remark 1.1. In [9] it is shown that first and second conditions (1.5) are necessary for asymptotic stability of the zero solution of the system (1.1)-(1.3).

Consider the nonlinear model of inverted pendulum

$$\ddot{x}(t) - a\sin x(t) = u(t), \qquad a > 0, \qquad t \geq 0, \tag{1.6}$$

with control (1.2) and initial condition (1.3).

In [9] it is shown that the zero solution of the system (1.6), (1.2), (1.3) is asymptotically stable and some estimate of the region of attraction for the zero solution of this system is constructed.

It is considered also the linear

$$\ddot{x}(t) - (a+\sigma\dot{\xi}(t))x(t) = u(t), \qquad a > 0, \qquad t \geq 0, \tag{1.7}$$

and the nonlinear models of inverted pendulum

$$\ddot{x}(t) - (a+\sigma\dot{\xi}(t))\sin x(t) = u(t), \qquad a > 0, \qquad t \geq 0, \tag{1.8}$$

with control (1.2) and initial condition (1.3) by stochastic perturbations $\dot{\xi}(t)$ of white noise type.

Definition 1.2. The zero solution of the system (1.7), (1.2), (1.3) is called mean square stable if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|x(t)|^2 + \mathbf{E}|\dot{x}(t)|^2 < \epsilon$ for any $t \geq 0$ provided that $\sup_{s\leq 0}\{\mathbf{E}|\varphi(s)|^2 + \mathbf{E}|\dot{\varphi}(s)|^2\} < \delta$. If, besides, $\lim_{t\to\infty}\{\mathbf{E}|x(t)|^2 + \mathbf{E}|\dot{x}(t)|^2\} = 0$ for every initial function φ, then the zero solution of the system (1.3), (1.2) is called asymptotically mean square stable.

Definition 1.3. The zero solution of the system (1.8), (1.2), (1.3) is called stable in probability if for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$ there exists $\delta > 0$ such that the solution $(x_1(t), x_2(t)) = (x_1(t,\varphi), x_2(t,\varphi))$ of equation (4.7),(2.4) satisfies the condition $\mathbf{P}\{|x_1(t,\varphi)|+|x_2(t,\varphi)| > \epsilon_1\} < \epsilon_2$ for any initial function $\varphi(s)$ such that $\mathbf{P}\{\sup_{s\leq 0}(|\varphi(s)| + |\dot{\varphi}(s)|) \leq \delta\} = 1$.

In [9] it is shown that if the conditions (1.5) hold and

$$\sigma^2 < \frac{2a_1}{p_1}\Big(1 - \frac{k_2}{4}\Big(1+\sqrt{1+p_1^2}\Big)\Big) \tag{1.9}$$

then the zero solution of the system (1.7), (1.2), (1.3) is asymptotically mean square stable and the zero solution of the system (1.8), (1.2), (1.3) is stable in probability.

2. Nonzero steady-state solutions

Here we will study nonzero steady-state solutions of the nonlinear system (1.6), (1.2), (1.3).

Substituting (1.2) into (1.6) and putting $x_1(t) = x(t)$, $x_2(t) = \dot{x}(t)$ we represent the system (1.6), (1.2) in the form

$$\begin{aligned} \dot{x}_1(t) =& x_2(t), \\ \dot{x}_2(t) =& a\sin x_1(t) + \int_0^\infty dK(\tau)x_1(t-\tau). \end{aligned} \tag{2.1}$$

To get steady-state solutions of the system (2.1) let us suppose that $\dot{x}_1(t) = \dot{x}_2(t) = 0$. Therefore $x_2(t) = 0$ and $\hat{x} = x_1(t)$ is a root of the equation

$$a \sin \hat{x} + k_0 \hat{x} = 0. \tag{2.2}$$

Suppose that $\hat{x} \neq 0$ and rewrite (2.2) in the form

$$S(\hat{x}) = 0, \tag{2.3}$$

where

$$S(x) = \frac{\sin x}{x} + \frac{k_0}{a}. \tag{2.4}$$

The function $S(x)$ we will call "characteristic function of the system (2.1)".

Remark 2.1. The statements "$\hat{x}$ is a steady-state solution of the system (2.1)" and "$\hat{x}$ is a root of the equation (2.3)" are equivalent.

Remark 2.2. For all $x \neq 0$

$$-\alpha \leq \frac{\sin x}{x} < 1,$$

where $0.217233 < \alpha < 0.217234$. Therefore if

$$-\alpha \leq -\frac{k_0}{a} < 1$$

or

$$0 < a + k_0 \leq (1 + \alpha) a \tag{2.5}$$

then there exists at least one nonzero root of the equation (2.3).

Remark 2.3. Since the function $\varphi(x)$ is an even function then if $\hat{x}$ is a root of the equation (2.3) then $-\hat{x}$ is a root of the equation (2.3) too.

Remark 2.4. The codition (2.5) contradicts to necessary condition $a_1 = -a - k_0 > 0$ of asymptotic stability of the corresponding linear system. Therefore by condition (2.5) the zero solution of the corresponding linear system is not stable.

3. Stable, unstable and one-sided stable points of equilibrium

Here we will investigate a stability of the steady-state solutions of the system (2.1). Let $\hat{x}$ be a root of the equation (2.3). Put

$$x_1 = \hat{x} + y_1, \quad x_2 = y_2. \tag{3.1}$$

Subsituting (3.1) into (2.1) and using (2.2), we obtain

$$\begin{aligned} \dot{y}_1(t) &= y_2(t), \\ \dot{y}_2(t) &= a[\sin(\hat{x} + y_1(t)) - \sin \hat{x}] + \int_0^\infty dK(\tau) y_1(t - \tau). \end{aligned} \tag{3.2}$$

After elementary trigonometric transformation we have

$$\begin{aligned} \dot{y}_1(t) &= y_2(t), \\ \dot{y}_2(t) &= 2a \cos\left(\hat{x} + \frac{y_1(t)}{2}\right) \sin\left(\frac{y_1(t)}{2}\right) + \int_0^\infty dK(\tau) y_1(t - \tau). \end{aligned} \tag{3.3}$$

It is easy to see that linear approximation of the system (3.3) has the form

$$\begin{aligned} \dot{y}_1(t) &= y_2(t), \\ \dot{y}_2(t) &= a \cos \hat{x} y_1(t) + \int_0^\infty dK(\tau) y_1(t - \tau). \end{aligned} \tag{3.4}$$

The conditions (1.5) for the system (3.4) have the form

$$k_0 < -a\cos\hat{x}, \quad k_1 > 0, \quad k_2 < \frac{4}{1+\sqrt{1+p_2^2}}, \tag{3.5}$$

where

$$a_2 = -a\cos\hat{x} - k_0, \quad p_2 = \frac{a_2+1}{k_1}.$$

Theorem 3.1. *Let $\hat{x} > 0$ is a point of stable equilibrium of the system (2.1). Then*

$$\dot{S}(\hat{x}) < 0, \tag{3.6}$$

i.e. $\hat{x}$ is a point of decrease of the characteristic function $S(x)$.

Let $\hat{x} > 0$ is a point of increase of the characteristic function $S(x)$, i.e.

$$\dot{S}(\hat{x}) > 0. \tag{3.7}$$

Then $\hat{x}$ is a point of unstable equilibrium of the system (2.1).

Proof. Using derivative of the function (2.4)

$$\dot{S}(x) = \frac{1}{x}\Big(\cos x - \frac{\sin x}{x}\Big)$$

and the equation (2.3) we have

$$\dot{S}(\hat{x}) = \frac{1}{\hat{x}}\Big(\cos\hat{x} - \frac{\sin\hat{x}}{\hat{x}}\Big) = \frac{a\cos\hat{x}+k_0}{\hat{x}a}.$$

From the first inequality of the conditions (3.5) follows (3.6).

Let (3.7) hold. Then the first inequality of the conditions (3.5) don't hold and $\hat{x}$ cannot (see *Remark 1.1*) be a point of stable equilibrium. Theorem is proved.

Remark 3.1. Let $\hat{x}$ is a point of extremum of the characteristic function $S(x)$, i.e.

$$\dot{S}(\hat{x}) = 0. \tag{3.8}$$

Then $\hat{x}$ is a point of one-sided stable of equilibrium of the system (2.1). It means that if system stays in the point x from any enough small neighborhood of $\hat{x}$ and $\dot{S}(x) < 0$ then system go to $\hat{x}$. But if system stays in the point x from any enough small neighborhood of $\hat{x}$ and $\dot{S}(x) > 0$ then system go away from $\hat{x}$.

Remark 3.2. Consider the system (1.6) by stochastic perturbations of white noise type which are proportional to the deviation $\sin x(t) - \sin\hat{x}$, where $\hat{x}$ is a steady-state solution of the system (1.6). In this case the system (1.6) can be written as

$$\ddot{x}(t) - a\sin x(t) = \sigma\dot{\xi}(t)(\sin x(t) - \sin\hat{x}) + u(t) \tag{3.9}$$

and corresponding system (3.4) has the form

$$\begin{aligned} \dot{y}_1(t) &= y_2(t), \\ \dot{y}_2(t) &= (a+\sigma\dot{\xi}(t))\cos\hat{x}\, y_1(t) + \int_0^\infty dK(\tau)y_1(t-\tau). \end{aligned}$$

If the conditions (3.5) and

$$\sigma^2 < \frac{2a_2}{p_2\cos^2\hat{x}}\Big(1 - \frac{k_2}{4}\Big(1+\sqrt{1+p_2^2}\Big)\Big)$$

hold then the steady-state solution $\hat{x}$ of the system (3.9), (1.2) is stable in probability.

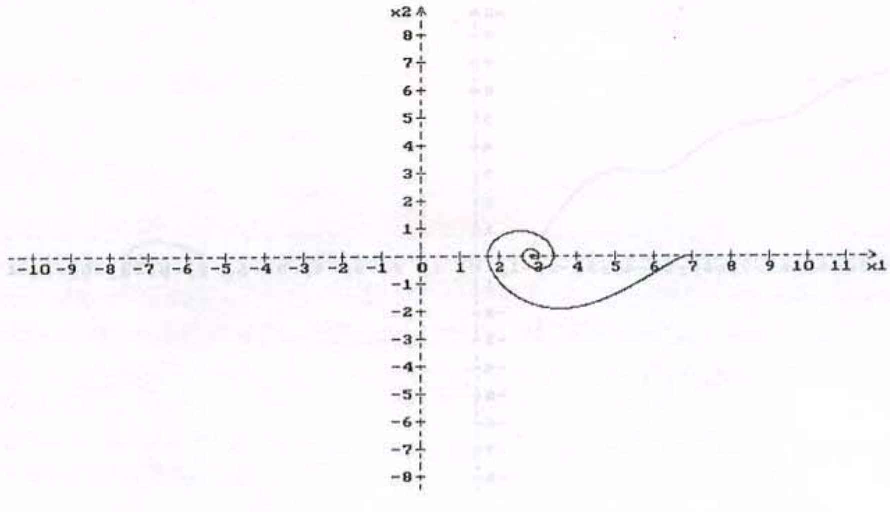

FIG. 4.1

4. NUMERICAL ILLUSTRATING EXAMPLES

Let in control (1.2) $dK(\tau) = (b_1\delta(\tau - h_1) + b_2\delta(\tau - h_2))d\tau$, $h_1, h_2 \geq 0$, $\delta(\tau)$ - Diraca function. In this case the system (1.6), (1.2) has the form

$$\ddot{x}(t) - a\sin x(t) = b_1 x(t - h_1) + b_2 x(t - h_2). \tag{4.1}$$

From (1.4) it follows that

$$k_0 = b_1 + b_2, \qquad k_1 = b_1 h_1 + b_2 h_2, \qquad k_2 = |b_1|h_1^2 + |b_2|h_2^2. \tag{4.2}$$

4.1. Let $a = 1$, $b_1 = 1$, $b_2 = -1.08$, $h_1 = 0.8$, $h_2 = 0.3$. By that $k_0 = -0.08$, $k_1 = 0.476$, $k_2 = 0.7372$. In this case the first condition from (1.5) don't hold. Therefore the zero solution of the equation (4.1) is unstable.

On the other hand the equation (2.3) has three positive roots $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$, such that $2.906892 < \hat{x}_1 < 2.906893$, $6.864548 < \hat{x}_2 < 6.864549$, $8.659471 < \hat{x}_3 < 8.659472$. Therefore (*Remark 2.1*) these points are the steady-state solutions of the system (4.1).

It is easy to check that

$$\dot{S}(\hat{x}_1) < 0, \qquad \dot{S}(\hat{x}_2) > 0, \qquad \dot{S}(\hat{x}_3) < 0.$$

It means that the points $\hat{x}_1$ and $\hat{x}_3$ are points of decrease of the function (2.4) and the point $\hat{x}_2$ is a point of increase of this function. Therefore the points $\hat{x}_1$ and $\hat{x}_3$ are points of stable equilibrium of the system (4.1) and the point $\hat{x}_2$ is a point of unstable equilibrium of the system (4.1).

Note that for the point of stable equilibrium $\hat{x}_3$ all conditions (3.5) hold but for the point of stable equilibrium $\hat{x}_1$ first and second conditions (3.5) hold only. For the point of unstable equilibrium $\hat{x}_2$ the first condition (3.5) don't hold.

Let $\varphi(s) = 6.864548$, $s \leq 0$, i.e. initial function close to $\hat{x}_2$ and less than $\hat{x}_2$. In this case the solution of the system (4.1) go away from point of unstable equilibrium $\hat{x}_2$ and go to point of stable equilibrium $\hat{x}_1$. This situation is shown on the Fig.4.1.

Let $\varphi(s) = 6.864549$, $s \leq 0$, i.e. initial function close to $\hat{x}_2$ and greater than $\hat{x}_2$. In this case the solution of the system (4.1) go away from point of unstable equilibrium $\hat{x}_2$ and go to point of stable equilibrium $\hat{x}_3$. This situation is shown on the Fig.4.2.

4.2. Let $a = 1$, $b_1 = 1$, $b_2 = -0.782766$, $h_1 = 0.8$, $h_2 = 0.3$. By that $k_0 = 0.217233$, $k_1 = 0.565170$, $k_2 = 0.710449$. In this case the system (4.1) has one positive steady-state solution $\hat{x} = 4.493409$ only. This point is a point of minimum of the function $S(x)$ and therefore the a point of one-sided stable equilibrium.

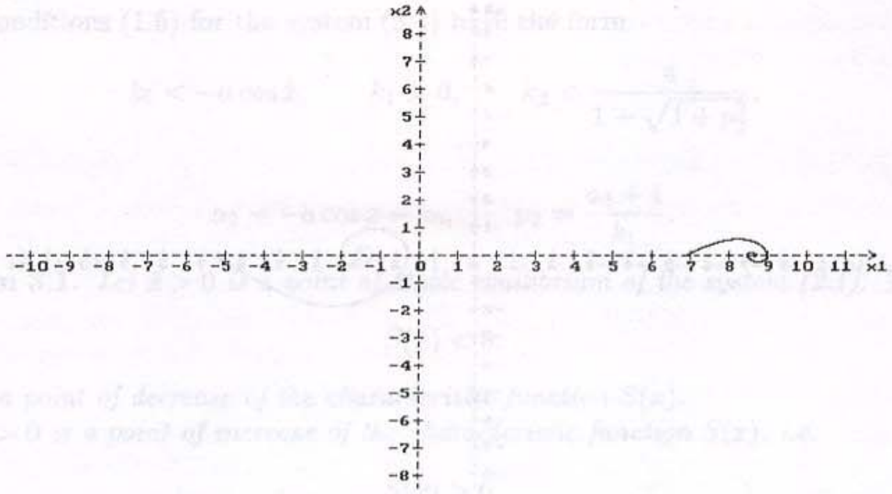

FIG. 4.2

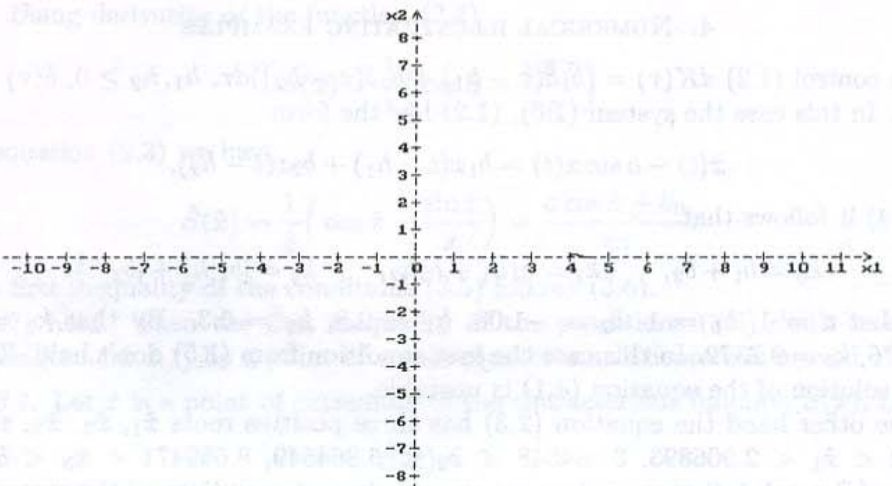

FIG. 4.3

Let $\varphi(s) = 4$, $s \leq 0$, i.e. initial function close to $\hat{x}$ and less than $\hat{x}$. In this case the solution of the system (4.1) go to the point $\hat{x}$. This situation is shown on the Fig.4.3.

Let $\varphi(s) = 4.5$, $s \leq 0$, i.e. initial function close to $\hat{x}$ and greater than $\hat{x}$. In this case the solution of the system (4.1) go away from the point $\hat{x}$ and go to infinity. This situation is shown on the Fig.4.4.

Remark 4.1. Since the function $S(x)$ is an even function then for negative roots of the equation (2.3) the pictures are symmetrical.

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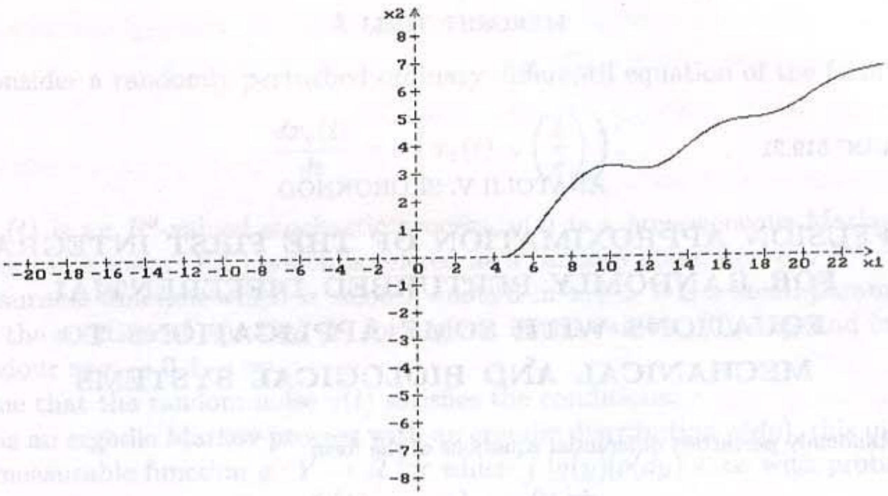

FIG. 4.4

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