STABILITY OF PREDATOR-PREY MODEL WITH AFTEREFFECT BY STOCHASTIC PERTURBATION

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Abstract. Many processes in automatic regulation, physics, mechanics, biology, economy, ecology etc. can be modelled by hereditary equations (see, e.g. [1-6]). One of the main problems for the theory of stochastic hereditary equations and their applications is connected with stability. Many stability results were obtained by the construction of appropriate Lyapunov functionals. In [7-15] the procedure is proposed, allowing, in some sense, to formalize the algorithm of the corresponding Lyapunov functionals construction for stochastic functional differential and stochastic difference equations. In this paper stability conditions are obtained by using this procedure for the mathematical model of the predator-prey system with aftereffect and stochastic perturbations. Similar problem for epidemic model with time delays influenced by stochastic perturbations was solved in [16].

Keywords. stability, eco-systems, stochastic systems, systems with aftereffects

1 Problem statement

One of the predator-prey mathematical model can be described by the equations

$$\dot{x_1}(t) = x_1(t)\left(a - \int_0^\infty f_1(s)x_1(t-s)ds - \int_0^\infty f_2(s)x_2(t-s)ds\right),$$

$$\dot{x_2}(t) = -bx_2(t) + \int_0^\infty g_1(s)x_1(t-s)ds \int_0^\infty g_2(s)x_2(t-s)ds. \tag{1}$$

Here $x_1(t)$ and $x_2(t)$ are densities of predator and prey populations, respectively. It is assumed that a, b are positive constants, $f_i(s), g_i(s)$, $i = 1, 2$, are nonnegative functions, such that

$$\alpha_i = \int_0^\infty f_i(s)ds < \infty, \qquad \beta_i = \int_0^\infty g_i(s)ds < \infty,$$

$$p_i = \int_0^\infty sf_i(s)ds < \infty, \qquad q_i = \int_0^\infty sg_i(s)ds < \infty.$$

It is easily to see that the positive point (x_1^*, x_2^*) of the system (1) equilibrium is given by

$$x_1^* = \frac{b}{\beta_1\beta_2}, \qquad x_2^* = \frac{a - \alpha_1 x_1^*}{\alpha_2} = \frac{a\beta_1\beta_2 - b\alpha_1}{\alpha_2\beta_1\beta_2}$$

provided that (everywhere below this condition it is assumed hold)

$$a\beta_1\beta_2 > b\alpha_1.$$

Note that some particular cases of the model (1) were often considered before. For example, let

$$f_i(s) = a_i\delta(s), \quad g_i(s) = b_i\delta(s-h), \quad i = 1,2, \quad h \geq 0,$$

where $\delta(s)$ is the Dirac function. Then the system (1) has the form [3]

$$\dot{x_1}(t) = x_1(t)(a - a_1x_1(t) - a_2x_2(t)),$$

$$\dot{x_2}(t) = -bx_2(t) + b_1b_2x_1(t-h)x_2(t-h).$$

If here $h = 0$ we get the classical Lotka-Volterra model

$$\dot{x_1}(t) = x_1(t)(a - a_1x_1(t) - a_2x_2(t)),$$

$$\dot{x_2}(t) = x_2(t)(-b + b_1b_2x_1(t)).$$

Let us assume that the system (1) is exposed by stochastic perturbations which are of white noise type and are directly proportional to distances $x_1(t), x_2(t)$ from values of x_1^*, x_2^*, influence on the $\dot{x_1}(t), \dot{x_2}(t)$ respectively. By this way, the system (1) will be reduced to the form

$$\dot{x_1}(t) = x_1(t)\left(a - \int_0^\infty f_1(s)x_1(t-s)ds - \int_0^\infty f_2(s)x_2(t-s)ds\right) + \sigma_1(x_1(t) - x_1^*)\dot{w}_1(t),$$

$$\dot{x_2}(t) = -bx_2(t) + \int_0^\infty g_1(s)x_1(t-s)ds \int_0^\infty g_2(s)x_2(t-s)ds + \sigma_2(x_2(t) - x_2^*)\dot{w}_2(t). \qquad (2)$$

Here σ_1, σ_2 are constants, $w_1(t), w_2(t)$ are independent from each other Wiener processes [17].

Let us centre the system (2) on the positive equilibrium by the change of variables $y_1 = x_1 - x_1^*$, $y_2 = x_2 - x_2^*$. By this way, we obtain

$$\dot{y}_1(t) = -(y_1(t) + x_1^*)\left(\int_0^\infty f_1(s)y_1(t-s)ds + \int_0^\infty f_2(s)y_2(t-s)ds\right) + \sigma_1y_1(t)\dot{w}_1(t),$$

$$\dot{y}_2(t) = -by_2(t) + x_2^*\int_0^\infty g_1(s)y_1(t-s)ds + x_1^*\int_0^\infty g_2(s)y_2(t-s)ds + \qquad (3)$$

$$+\int_0^\infty g_1(s)y_1(t-s)ds\int_0^\infty g_2(s)y_2(t-s)ds + \sigma_2y_2(t)\dot{w}_2(t).$$

It is easily to see that the stability of the system (2) equilibrium is equivalent to the stability of the zero solution of the system (3).

Below we will obtain the sufficient conditions for stability in probability of the zero solution of the system (3). Along with the system (3) we will consider the linear part of the system (3), i.e.

$$\dot{z}_1(t) = -x_1^*\left(\int_0^\infty f_1(s)z_1(t-s)ds + \int_0^\infty f_2(s)z_2(t-s)ds\right) + \sigma_1z_1(t)\dot{w}_1(t),$$

$$\dot{z}_2(t) = -bz_2(t) + x_2^*\int_0^\infty g_1(s)z_1(t-s)ds + x_1^*\int_0^\infty g_2(s)z_2(t-s)ds + \sigma_2z_2(t)\dot{w}_2(t). \qquad (4)$$

2 Definitions, auxiliary statements

Consider the stochastic differential equation of neutral type [1]

$$d(x(t) - G(t, x_t)) = a(t, x_t)dt + b(t, x_t)dw(t), \quad x_0 = \varphi \in H. \tag{5}$$

Let $\{\Omega, \sigma, \mathbf{P}\}$ be the probability space, $\{f_t, t \geq 0\}$ be the family of σ-algebras, $f_t \in \sigma$, H be the space of f_0-adapted functions $\varphi(s) \in R^n$, $s \leq 0$, $\|\varphi\|_0 = sup_{s\leq 0}|\varphi(s)|$, $\|\varphi\|_1^2 = sup_{s\leq 0}\mathbf{E}|\varphi(s)|^2$, $\mathbf{E}$ be the mathematical expectation, $x_t = x(t+s)$, $s \leq 0$, $w(t)$ be the m-dimensional f_t-adapted Wiener process, the n-dimensional vector $a(t, \varphi)$ and the $n*m$-dimensional matrix $b(t, \varphi)$ are defined for $t \geq 0$, $\varphi \in H$, $a(t, 0) = 0$, $b(t, 0) = 0$,

$$|G(t, \varphi)| \leq \int_0^\infty |\varphi(-s)|dK(s), \quad \int_0^\infty dK(s) < 1. \tag{6}$$

The generating operator L [3] of the equation (5) is defined by the formula

$$LV(t, \varphi) = lim_{\Delta\to 0}\frac{1}{\Delta}\left(\mathbf{E}_{t,\varphi}V(t+\Delta, y_{t+\Delta}) - V(t, \varphi)\right).$$

Here a scalar functional $V(t, \varphi)$ is defined for $t \geq 0$, $\varphi \in H$ and $y(s)$ is the solution of the equation (5) for $s \geq t$ with initial function $y_t = \varphi \in H$, $\mathbf{E}_{t,\varphi}$ is the conditional expectation.

Let us consider an arbitrary functional $V(t, \varphi)$, for $t \geq 0$, $\varphi \in H$, reduce it to the form $V(t, \varphi) = V(t, \varphi(0), \varphi(s))$, $s < 0$ and define the function

$$V_\varphi(t, x) = V(t, \varphi) = V(t, x_t) = V(t, x, x(t+s)),$$

$$s < 0, \qquad \varphi = x_t, \qquad x = \varphi(0) = x(t).$$

Let D be a class of functionals $V(t, \varphi)$ for which the function $V_\varphi(t, x)$ has two continuous derivatives with respect to x and one bounded derivative with respect to t for almost all $t \geq 0$. For functionals from D the generating operator L of the equation (5) is defined and can be calculated in a final form [3].

Definition 1. The zero solution of the equation (5) is called mean square stable if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|x(t)|^2 < \epsilon$ for any $t \geq 0$ provided that $\|\varphi\|_1^2 < \delta$.

Definition 2. The zero solution of the equation (5) is called asymptotically mean square stable if it is mean square stable and $\lim_{t\to\infty} \mathbf{E}|x(t)|^2 = 0$.

Definition 3. The zero solution of the equation (5) is called stable in probability if for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$ there exists such $\delta > 0$ that the solution $x(t) = x(t, \varphi)$ of the equation (5) satisfies the condition

$$\mathbf{P}\{|x(t, \varphi)| > \epsilon_1\} < \epsilon_2$$

for any initial function $\varphi \in H$ such that $\mathbf{P}\{\|\varphi\|_0 \leq \delta\} = 1$. Here $\mathbf{P}\{\cdot\}$ is the probability of the event enclosed in braces.

Theorem 1 *Let the condition (6) hold and there exists the functional*

$$V(t, \varphi) = W(t, \varphi) + |\varphi(0) - G(t, \varphi)|^2$$

such that $V(t, \varphi) \in D$,

$$0 \leq \mathbf{E}W(t, x_t) \leq c_1\|x_t\|_1^2,$$

$$\mathbf{E}LV(t, x_t) \leq -c_2\mathbf{E}|x(t)|^2,$$

$c_i > 0$, $i = 1, 2$. *Then the zero solution of the equation (5) is asymptotically mean square stable.*

Theorem 2 *Let there exists the functional* $V(t,\varphi) \in D$ *such that*

$$c_1|x(t)|^2 \leq V(t,x_t) \leq c_2\|x_t\|_0^2,$$

$$LV(t,x_t) \leq 0,$$

$c_i > 0$, *for any function* $\varphi \in H$ *such that* $\mathbf{P}\{\|\varphi\|_0 \leq \delta\} = 1$, *where* $\delta > 0$ *is sufficiently small. Then the zero solution of the equation (5) is stable in probability.*

The proofs of these theorems can be found in [1,2].

By this way, the construction of stability conditions is reduced to construction of Lyapunov functionals. Below we will use the general method of Lyapunov functionals construction [7-15] allowing to obtain stability conditions immediately in terms of parameters of a system under consideration.

3 Procedure of Lyapunov functionals construction

The proposed procedure of Lyapunov functionals construction consists of four steps.

1. Let us transform the equation (5) to the form

$$dz(t,x_t) = (b_1(t,x(t)) + c_1(t,x_t))dt + (b_2(t,x(t)) + c_2(t,x_t))dw(t), \tag{7}$$

   where $z(t,0) = b_i(t,0) = c_i(t,0) = 0$, $i = 1,2$, the functionals b_i, depend on t and $x(t)$ only and do not depend on the previous values $x(t+s)$, $s < 0$, of the solution.

2. Consider the equation without memory (ordinary stochastic differential equation)

$$dy(t) = b_1(t,y(t))dt + b_2(t,y(t))d\xi(t). \tag{8}$$

   Let us assume that the zero solution of the equation (8) is uniformly asymptotically mean square stable and there exists the Lyapunov function $v(t,y)$, for which the condition $L_0v(t,y) \leq -c|y|^2$, $c > 0$, hold. Here L_0 is the generating operator of the equation (8).

3. We'll construct the Lyapunov functional $V(t,x_t)$ in the form $V = V_1 + V_2$. Let us replace the argument y of the function $v(t,y)$ on the functional $z(t,x_t)$ from the left-hand part of the equation (7). As a result we obtain the main component $V_1(t,x_t) = v(t,z(t,x_t))$ of the functional $V(t,x_t)$.

4. Usually the functional V_1 almost satisfies the requirements of stability theorems for the equation (7). In order that these conditions would be completely satisfied the auxiliary component V_2 can be easily chosen by standard way.

Note that the representation (7) is not unique. This fact allows using different representations (7) to construct different Lyapunov functionals and as a result to get different sufficient stability conditions.

4 Asymptotic mean square stability

In this section we will construct the Lyapunov functional for getting sufficient conditions of asymptotic mean square stability of the linear system (4) zero solution.

Following the step 1 of the procedure let us represent the system (4) in the form:

$$\frac{d}{dt}\left(z_1(t) - x_1^*\int_0^\infty f_1(s)\int_{t-s}^t z_1(\theta)d\theta ds\right) =$$

$$= -\alpha_1 x_1^* z_1(t) - x_1^* \int_0^\infty f_2(s) z_2(t-s) ds + \sigma_1 z_1(t) \dot{w}_1(t),$$

$$\frac{d}{dt}\left(z_2(t) + x_1^* \int_0^\infty g_2(s) \int_{t-s}^t z_2(\theta) d\theta ds\right) = \tag{9}$$

$$= -(b - \beta_2 x_1^*) z_2(t) + x_2^* \int_0^\infty g_1(s) z_1(t-s) ds + \sigma_2 z_2(t) \dot{w}_2(t).$$

The auxiliary system (step 2) in this case has the form

$$\dot{u}_1(t) = -\alpha_1 x_1^* u_1(t) + \sigma_1 u_1(t) \dot{w}_1(t),$$

$$\dot{u}_2(t) = -(b - \beta_2 x_1^*) u_2(t) + \sigma_2 u_2(t) \dot{w}_2(t). \tag{10}$$

Lemma 1. *Let*

$$\sigma_1^2 < 2\alpha_1 x_1^*, \qquad \sigma_2^2 < 2(b - \beta_2 x_1^*). \tag{11}$$

Then the zero solution of the system (10) is asymptotically mean square stable.

Proof. Let us show that the function

$$v = u_1^2 + u_2^2$$

is a Lyapunov function for the system (10). Let L_0 be the generating operator of the system (10). Then

$$L_0 v = 2u_1(-\alpha_1 x_1^* u_1(t)) + 2u_2(-(b - \beta_2 x_1^*) u_2(t)) + \sigma_1^2 u_1^2 + \sigma_2^2 u_2^2 =$$

$$= -(2\alpha_1 x_1^* - \sigma_1^2) u_1^2(t) - (2(b - \beta_2 x_1^*) - \sigma_2^2) u_2^2(t).$$

Let

$$c = min[2\alpha_1 x_1^* - \sigma_1^2, 2(b - \beta_2 x_1^*) - \sigma_2^2].$$

Then

$$L_0 v \leq -c(u_1^2(t) + u_2^2(t)) = -c|u(t)|^2,$$

where $u = (u_1, u_2)$. From Theorem 1 it follows that the system (10) zero solution is asymptotic mean square stable. Lemma is proved.

Following the step 3 of procedure we will construct a Lyapunov functional for the system (4) in the form $V_1 + V_2$, where

$$V_1 = \left(z_1(t) - x_1^* \int_0^\infty f_1(s) \int_{t-s}^t z_1(\theta) d\theta ds\right)^2 +$$

$$+ \left(z_2(t) + x_1^* \int_0^\infty g_2(s) \int_{t-s}^t z_2(\theta) d\theta ds\right)^2. \tag{12}$$

Let L be the generating operator of the system (4). Calculating LV_1 for the system (4) using the representation (9), we obtain

$$LV_1 = 2\left(z_1(t) - x_1^* \int_0^\infty f_1(s) \int_{t-s}^t z_1(\theta) d\theta ds\right)\left(-\alpha_1 x_1^* z_1(t) - x_1^* \int_0^\infty f_2(s) z_2(t-s) ds\right) +$$

$$+2\left(z_2(t) + x_1^* \int_0^\infty g_2(s) \int_{t-s}^t z_2(\theta) d\theta ds\right)\left(-(b - \beta_2 x_1^*) z_2(t) + x_2^* \int_0^\infty g_1(s) z_1(t-s) ds\right) +$$

$$+\sigma_1^2 z_1^2 + \sigma_2^2 z_2^2 =$$

$$= -(2\alpha_1 x_1^* - \sigma_1^2) z_1^2(t) - (2(b - \beta_2 x_1^*) - \sigma_2^2) z_2^2(t) -$$

$$-2x_1^* \int_0^\infty f_2(s) z_1(t) z_2(t-s) ds + 2x_2^* \int_0^\infty g_1(s) z_2(t) z_1(t-s) ds +$$

$$+2\alpha_1(x_1^*)^2\int_0^\infty f_1(s)\int_{t-s}^t z_1(t)z_1(\theta)d\theta ds-$$

$$-2(b-\beta_2x_1^*)x_1^*\int_0^\infty g_2(s)\int_{t-s}^t z_2(t)z_2(\theta)d\theta ds+$$

$$+2(x_1^*)^2\int_0^\infty f_1(s)\int_0^\infty f_2(\tau)\int_{t-s}^t z_1(\theta)z_2(t-\tau)d\theta d\tau ds+$$

$$+2x_1^*x_2^*\int_0^\infty g_2(s)\int_0^\infty g_1(\tau)\int_{t-s}^t z_2(\theta)z_1(t-\tau)d\theta d\tau ds.$$

Let $\gamma_i>0$, $i=1,2,3,4$. Then

$$\begin{aligned}
2|z_1(t)z_2(t-s)| &\le \gamma_1 z_1^2(t)+\gamma_1^{-1}z_2^2(t-s),\\
2|z_1(t-s)z_2(t)| &\le \gamma_2 z_1^2(t-s)+\gamma_2^{-1}z_2^2(t),\\
2|z_1(\theta)z_2(t-\tau)| &\le \gamma_3 z_1^2(\theta)+\gamma_3^{-1}z_2^2(t-\tau),\\
2|z_1(t-\tau)z_2(\theta)| &\le \gamma_4 z_1(t-\tau)+\gamma_4^{-1}z_2(\theta).
\end{aligned} \tag{13}$$

Using these inequalities and taking into account (11) we obtain

$$LV_1\le -(2\alpha_1x_1^*-\sigma_1^2)z_1^2(t)-(2(b-\beta_2x_1^*)-\sigma_2^2)z_2(t)+$$

$$+x_1^*\int_0^\infty f_2(s)(\gamma_1z_1^2(t)+\gamma_1^{-1}z_2^2(t-s))ds+x_2^*\int_0^\infty g_1(s)(\gamma_2z_1^2(t-s)+\gamma_2^{-1}z_2^2(t))ds+$$

$$+\alpha_1(x_1^*)^2\int_0^\infty f_1(s)\int_{t-s}^t(z_1^2(t)+z_1^2(\theta))d\theta ds+$$

$$+(b-\beta_2x_1^*)x_1^*\int_0^\infty g_2(s)\int_{t-s}^t(z_2^2(t)+z_2^2(\theta))d\theta ds+$$

$$+(x_1^*)^2\int_0^\infty f_1(s)\int_0^\infty f_2(\tau)\int_{t-s}^t(\gamma_3z_1^2(\theta)+\gamma_3^{-1}z_2^2(t-\tau))d\theta d\tau ds+$$

$$+x_1^*x_2^*\int_0^\infty g_2(s)\int_0^\infty g_1(\tau)\int_{t-s}^t(\gamma_4z_1(t-\tau)+\gamma_4^{-1}z_2(\theta))d\theta d\tau ds=$$

$$=-(2\alpha_1x_1^*-\sigma_1^2-\alpha_1p_1(x_1^*)^2-\gamma_1\alpha_2x_1^*)z_1^2(t)-$$

$$-(2(b-\beta_2x_1^*)-\sigma_2^2-(b-\beta_2x_1^*)q_2x_1^*-\gamma_2^{-1}\beta_1x_2^*)z_2^2(t)+$$

$$+(\alpha_1+\gamma_3\alpha_2)(x_1^*)^2\int_0^\infty f_1(s)\int_{t-s}^t z_1^2(\theta)d\theta ds+$$

$$+(\gamma_1^{-1}+\gamma_3^{-1}p_1x_1^*)x_1^*\int_0^\infty f_2(s)z_2^2(t-s)ds+$$

$$+(\gamma_2+\gamma_4q_2x_1^*)x_2^*\int_0^\infty g_1(s)z_1^2(t-s)ds+$$

$$+(b-\beta_2x_1^*+\gamma_4^{-1}\beta_1x_2^*)x_1^*\int_0^\infty g_2(s)\int_{t-s}^t z_2^2(\theta)d\theta ds.$$

Choosing the functional V_2 in the form

$$V_2=(\alpha_1+\gamma_3\alpha_2)(x_1^*)^2\int_0^\infty f_1(s)\int_{t-s}^t(\theta-t+s)z_1^2(\theta)d\theta ds+$$

$$+(\gamma_1^{-1}+\gamma_3^{-1}p_1x_1^*)x_1^*\int_0^\infty f_2(s)\int_{t-s}^t z_2^2(\theta)d\theta ds+$$

$$+(\gamma_2+\gamma_4 q_2 x_1^*)x_2^*\int_0^\infty g_1(s)\int_{t-s}^t z_1^2(\theta)d\theta ds+$$

$$+(b-\beta_2 x_1^*+\gamma_4^{-1}\beta_1 x_2^*)x_1^*\int_0^\infty g_2(s)\int_{t-s}^t (\theta-t+s)z_2^2(\theta)d\theta ds,$$

for the functional $V=V_1+V_2$ we obtain

$$LV\le-[2\alpha_1 x_1^*(1-p_1x_1^*)-\sigma_1^2-(\gamma_1+\gamma_3 p_1 x_1^*)\alpha_2 x_1^*-(\gamma_2+\gamma_4 q_2 x_1^*)\beta_1 x_2^*]z_1^2(t)-$$

$$-[2(b-\beta_2 x_1^*)(1-q_2x_1^*)-\sigma_2^2-(\gamma_1^{-1}+\gamma_3^{-1}p_1x_1^*)\alpha_2 x_1^*-(\gamma_2^{-1}+\gamma_4^{-1}q_2x_1^*)\beta_1x_2^*]z_2^2(t).$$

Thus we proved the following

Theorem 3 *Let $\beta_1>1$ and there are positive constants γ_1, γ_2, γ_3, γ_4 such that*

$$2\alpha_1 x_1^*(1-p_1x_1^*)>\sigma_1^2+(\gamma_1+\gamma_3p_1x_1^*)\alpha_2x_1^*+(\gamma_2+\gamma_4q_2x_1^*)\beta_1x_2^*,$$

$$2(b-\beta_2x_1^*)(1-q_2x_1^*)>\sigma_2^2+(\gamma_1^{-1}+\gamma_3^{-1}p_1x_1^*)\alpha_2x_1^*+(\gamma_2^{-1}+\gamma_4^{-1}q_2x_1^*)\beta_1x_2^*. \quad (14)$$

Then the system (4) zero solution is asymptotically mean square stable.

For proof it is enough to show that

$$b-\beta_2x_1^*=b(1-\beta_1^{-1})>0.$$

Corollary 1*. Let the parameters of the system (4) satisfy the conditions*

$$\beta_1>1,\quad A>0,\quad AB>C^2, \quad (15)$$

where

$$A=2\alpha_1x_1^*(1-p_1x_1^*)-\sigma_1^2,\quad B=2(b-\beta_2x_1^*)(1-q_2x_1^*)-\sigma_2^2,$$

$$C=(1+p_1x_1^*)\alpha_2x_1^*+(1+q_2x_1^*)\beta_1x_2^*.$$

Then the system (4) zero solution is asymptotically mean square stable.

Proof. From (15) it follows that

$$\frac{A}{C}>\frac{C}{B}.$$

Therefore there exists a positive constant γ such that

$$\frac{A}{C}>\gamma>\frac{C}{B}.$$

Thus the conditions (14) hold by $\gamma_1=\gamma_2=\gamma_3=\gamma_4=\gamma$.

5 Stability in probability

It is known [15] that if the initial nonlinear system has a nonlinearity order more than one, then the conditions providing the asymptotic mean square stability of the linear part of the initial system, in the same time provide the stability in probability of the initial system. Show it for the zero solution of the nonlinear system (3).

Theorem 4 *Let the Theorem 3 conditions hold. Then the zero solution of the system (3) is stable in probability.*

Proof. Let us reduce the system (3) to the form

$$\frac{d}{dt}\left(y_1(t) - x_1^* \int_0^\infty f_1(s) \int_{t-s}^t y_1(\theta)d\theta ds\right) =$$
$$= -\alpha_1 x_1^* y_1(t) - x_1^* \int_0^\infty f_2(s) y_2(t-s)ds -$$
$$-y_1(t)\left(\int_0^\infty f_1(s)y_1(t-s)ds + \int_0^\infty f_2(s)y_2(t-s)ds\right) + \sigma_1 y_1(t)\dot{w}_1(t),$$
$$\frac{d}{dt}\left(y_2(t) + x_1^* \int_0^\infty g_2(s) \int_{t-s}^t y_2(\theta)d\theta ds\right) = \tag{16}$$
$$= -(b - \beta_2 x_1^*)y_2(t) + x_2^* \int_0^\infty g_1(s)y_1(t-s)ds +$$
$$+\int_0^\infty g_1(s)y_1(t-s)ds \int_0^\infty g_2(s)y_2(t-s)ds + \sigma_2 y_2(t)\dot{w}_2(t).$$

Let L be the generating operator of the system (3). Consider the functional

$$V_1 = \left(y_1(t) - x_1^* \int_0^\infty f_1(s) \int_{t-s}^t y_1(\theta)d\theta ds\right)^2 +$$
$$+\left(y_2(t) + x_1^* \int_0^\infty g_2(s) \int_{t-s}^t y_2(\theta)d\theta ds\right)^2.$$

Calculating LV_1 by virtue of the representation (16) we get

$$LV_1 = 2[y_1(t) - x_1^* \int_0^\infty f_1(s) \int_{t-s}^t y_1(\theta)d\theta ds][-\alpha_1 x_1^* y_1(t) - x_1^* \int_0^\infty f_2(s)y_2(t-s)ds -$$
$$-y_1(t)(\int_0^\infty f_1(s)y_1(t-s)ds + \int_0^\infty f_2(s)y_2(t-s)ds)] +$$
$$+2[y_2(t) + x_1^* \int_0^\infty g_2(s) \int_{t-s}^t y_2(\theta)d\theta ds][-(b-\beta_2 x_1^*)y_2(t) + x_2^* \int_0^\infty g_1(s)y_1(t-s)ds +$$
$$+\int_0^\infty g_1(s)y_1(t-s)ds \int_0^\infty g_2(s)y_2(t-s)ds] + \sigma_1^2 y_1^2 + \sigma_2^2 y_2^2 =$$
$$= [-2\alpha_1 x_1^* + \sigma_1^2]y_1^2(t) + [-2(b-\beta_2 x_1^*) + \sigma_2^2]y_2^2(t) +$$
$$+2\alpha_1 (x_1^*)^2 \int_0^\infty f_1(s) \int_{t-s}^t y_1(t)y_1(\theta)d\theta ds - 2x_1^* \int_0^\infty f_2(s)y_1(t)y_2(t-s)ds +$$
$$+2(x_1^*)^2 \int_0^\infty f_1(s) \int_0^\infty f_2(\tau) \int_{t-s}^t y_1(\theta)y_2(t-\tau)d\theta d\tau ds -$$
$$-2y_1^2(t)(\int_0^\infty f_1(s)y_1(t-s)ds + \int_0^\infty f_2(s)y_2(t-s)ds) +$$
$$+2x_1^* \int_0^\infty f_1(s) \int_{t-s}^t y_1(t)y_1(\theta)d\theta ds(\int_0^\infty f_1(\tau)y_1(t-\tau)d\tau + \int_0^\infty f_2(\tau)y_2(t-\tau)d\tau) -$$
$$-2(b-\beta_2 x_1^*)x_1^* \int_0^\infty g_2(s) \int_{t-s}^t y_2(t)y_2(\theta)d\theta ds + 2x_2^* \int_0^\infty g_1(s)y_1(t-s)y_2(t)ds +$$
$$+2x_1^* x_2^* \int_0^\infty g_2(s) \int_0^\infty g_1(\tau) \int_{t-s}^t y_1(t-\tau)y_2(\theta)d\theta d\tau ds +$$
$$+2y_2(t) \int_0^\infty g_1(s) \int_0^\infty g_2(\tau)y_1(t-s)y_1(t-\tau)d\tau ds +$$

$$+2x_1^*\int_0^\infty g_2(s)\int_{t-s}^t y_2(\theta)d\theta ds\int_0^\infty g_1(s)y_1(t-s)ds\int_0^\infty g_2(\tau)y_2(t-\tau)d\tau.$$

Supposing that $\sup_{s\leq t}|y_i(s)|<\delta$, $i=1,2$, and using the inequalities kind of (13) we obtain

$$LV_1\leq[-2\alpha_1x_1^*+\sigma_1^2+2\delta(\alpha_1+\alpha_2)]y_1^2(t)+[-2(b-\beta_2x_1^*)+\sigma_2^2]y_2^2(t)+$$

$$+\alpha_1(x_1^*)^2\int_0^\infty f_1(s)\int_{t-s}^t(y_1^2(t)+y_1^2(\theta))d\theta ds+$$

$$+x_1^*\int_0^\infty f_2(s)(\gamma_1y_1^2(t)+\gamma_1^{-1}y_2^2(t-s))ds+$$

$$+(x_1^*)^2\int_0^\infty f_1(s)\int_0^\infty f_2(\tau)\int_{t-s}^t(\gamma_3y_1^2(\theta)+\gamma_3^{-1}y_2^2(t-\tau))d\theta d\tau ds+$$

$$+\delta(\alpha_1+\alpha_2)x_1^*\int_0^\infty f_1(s)\int_{t-s}^t(y_1^2(t)+y_1^2(\theta))d\theta ds+$$

$$+(b-\beta_2x_1^*)x_1^*\int_0^\infty g_2(s)\int_{t-s}^t(y_2^2(t)+y_2^2(\theta))d\theta ds+$$

$$+x_2^*\int_0^\infty g_1(s)(\gamma_2y_1^2(t-s)+\gamma_2^{-1}y_2^2(t))ds+$$

$$+x_1^*x_2^*\int_0^\infty g_2(s)\int_0^\infty g_1(\tau)\int_{t-s}^t(\gamma_4y_1^2(t-\tau)+\gamma_4^{-1}y_2^2(\theta))d\theta d\tau ds+$$

$$+\delta(1+q_2x_1^*)\int_0^\infty g_1(s)\int_0^\infty g_2(\tau)(y_1^2(t-s)+y_2^2(t-\tau))d\tau ds=$$

$$=[-2\alpha_1x_1^*+\sigma_1^2+\alpha_1p_1(x_1^*)^2+\gamma_1\alpha_2x_1^*+\delta(\alpha_1+\alpha_2)(2+p_1x_1^*)]y_1^2(t)+$$

$$+[-2(b-\beta_2x_1^*)+\sigma_2^2+(b-\beta_2x_1^*)q_2x_1^*+\gamma_2^{-1}\beta_1x_2^*]y_2^2(t)+$$

$$+[(\alpha_1+\gamma_3\alpha_2)(x_1^*)^2+\delta(\alpha_1+\alpha_2)x_1^*]\int_0^\infty f_1(s)\int_{t-s}^t y_1^2(\theta)d\theta ds+$$

$$+(\gamma_1^{-1}+\gamma_3^{-1}p_1x_1^*)x_1^*\int_0^\infty f_2(s)y_2^2(t-s)ds+$$

$$+[(\gamma_2+\gamma_4q_2x_1^*)x_2^*+\delta\beta_2(1+q_2x_1^*)]\int_0^\infty g_1(s)y_1^2(t-s)ds+$$

$$+(b-\beta_2x_1^*+\gamma_4^{-1}\beta_1x_2^*)x_1^*\int_0^\infty g_2(s)\int_{t-s}^t z_2^2(\theta)d\theta ds+$$

$$+\delta\beta_1(1+q_2x_1^*)\int_0^\infty g_2(s)y_2^2(t-s)ds.$$

Choosing the functional V_2 in the form

$$V_2=[(\alpha_1+\gamma_3\alpha_2)(x_1^*)^2+\delta(\alpha_1+\alpha_2)x_1^*]\int_0^\infty f_1(s)\int_{t-s}^t(\theta-t+s)y_1^2(\theta)d\theta ds+$$

$$+(\gamma_1^{-1}+\gamma_3^{-1}p_1x_1^*)x_1^*\int_0^\infty f_2(s)\int_{t-s}^t y_2^2(\theta)d\theta ds+$$

$$+[(\gamma_2+\gamma_4q_2x_1^*)x_2^*+\delta\beta_2(1+q_2x_1^*)]\int_0^\infty g_1(s)\int_{t-s}^t y_1^2(\theta)d\theta ds+$$

$$+(b-\beta_2x_1^*+\gamma_4^{-1}\beta_1x_2^*)x_1^*\int_0^\infty g_2(s)\int_{t-s}^t(\theta-t+s)y_2^2(\theta)d\theta ds+$$

$$+\delta\beta_1(1+q_2x_1^*)\int_0^\infty g_2(s)\int_{t-s}^t y_2^2(\theta)d\theta ds,$$

as a result for the functional V_1+V_2 we get

$$L(V_1+V_2)\le -[2\alpha_1x_1^*(1-p_1x_1^*)-\sigma_1^2-(\gamma_1+\gamma_3p_1x_1^*)\alpha_2x_1^*-(\gamma_2+\gamma_4q_2x_1^*)\beta_1x_2^*-$$
$$-2\delta(\alpha_1+\alpha_2)(1+p_1x_1^*)-\delta\beta_2(1+q_2x_1^*)]y_1^2(t)-$$
$$-[2(b-\beta_2x_1^*)(1-q_2x_1^*)-\sigma_2^2-(\gamma_1^{-1}+\gamma_3^{-1}p_1x_1^*)\alpha_2x_1^*-$$
$$-(\gamma_2^{-1}+\gamma_4^{-1}q_2x_1^*)\beta_1x_2^*-\delta\beta_1\beta_2(1+q_2x_1^*)]y_2^2(t). \quad (17)$$

If the Theorem 3 conditions hold then for sufficiently small $\delta>0$ we obtain $L(V_1+V_2)\le 0$.

The functional V_1+V_2 is a nonnegative functional but it is not a positive definite functional. This is the reason that we cannot use now Theorem 2. For constructing a positive definite functional let us consider the functional

$$W_1=y_1^2(t)+y_2^2(t).$$

Calculate LW_1 using the equation (3). As a result we get

$$LW_1=-2y_1(t)(y_1(t)+x_1^*)[\int_0^\infty f_1(s)y_1(t-s)ds+\int_0^\infty f_2(s)y_2(t-s)ds]+$$
$$+2y_2(t)[-by_2(t)+x_2^*\int_0^\infty g_1(s)y_1(t-s)ds+x_1^*\int_0^\infty g_2(s)y_2(t-s)ds+$$
$$+\int_0^\infty g_1(s)y_1(t-s)ds\int_0^\infty g_2(s)y_2(t-s)ds]+\sigma_1^2y_1^2(t)+\sigma_2^2y_2^2(t)\le$$
$$\le\sigma_1^2y_1^2(t)+(\sigma_2^2-2b)y_2^2(t)+$$
$$+2y_1^2(t)[\int_0^\infty f_1(s)|y_1(t-s)|ds+\int_0^\infty f_2(s)|y_2(t-s)|ds]+$$
$$+x_1^*[\int_0^\infty f_1(s)(y_1^2(t)+y_1^2(t-s))ds+\int_0^\infty f_2(s)(y_1^2(t)+y_2^2(t-s))ds]+$$
$$+x_2^*\int_0^\infty g_1(s)(y_1^2(t-s)+y_2^2(t))ds+x_1^*\int_0^\infty g_2(s)(y_2^2(t)+y_2^2(t-s))ds+$$
$$+|y_2(t)|\int_0^\infty g_1(s)\int_0^\infty g_2(\tau)(y_1^2(t-s)+y_2^2(t-\tau))d\tau ds.$$

Supposing that $\sup_{s\le t}|y_i(s)|<\delta$, $i=1,2$, we obtain

$$LW_1\le[(2\delta+x_1^*)(\alpha_1+\alpha_2)+\sigma_1^2]y_1^2(t)+[-2b+\beta_1x_2^*+\beta_2x_1^*+\sigma_2^2]y_2^2(t)+$$
$$+x_1^*\int_0^\infty f_1(s)y_1^2(t-s)ds+x_1^*\int_0^\infty f_2(s)y_2^2(t-s)ds+$$
$$+(x_2^*+\delta\beta_2)\int_0^\infty g_1(s)y_1^2(t-s)ds+(x_1^*+\delta\beta_1)\int_0^\infty g_2(s)y_2^2(t-s)ds.$$

Choosing the functional W_2 in the form

$$W_2=x_1^*\int_0^\infty f_1(s)\int_{t-s}^t y_1^2(\theta)d\theta ds+x_1^*\int_0^\infty f_2(s)\int_{t-s}^t y_2^2(\theta)d\theta ds+$$
$$+(x_2^*+\delta\beta_2)\int_0^\infty g_1(s)\int_{t-s}^t y_1^2(\theta)d\theta ds+(x_1^*+\delta\beta_1)\int_0^\infty g_2(s)\int_{t-s}^t y_2^2(\theta)d\theta ds,$$

for the functional W_1+W_2 we get

$$L(W_1+W_1)\le[(2\delta+x_1^*)(\alpha_1+\alpha_2)+\sigma_1^2+\alpha_1x_1^*+(x_2^*+\delta\beta_2)\beta_1)]y_1^2(t)+$$

$$+[-2b + \beta_1 x_2^* + \beta_2 x_1^* + \sigma_2^2 + \alpha_2 x_1^* + (x_1^* + \delta\beta_1)\beta_2] y_2^2(t). \tag{18}$$

At last let us consider the functional $V = V_1 + V_2 + c_1(W_1 + W_2)$. From (17) and (18) it follows that for sufficiently small c_1 the functional V satisfies the Theorem 2 conditions. Therefore the system (3) zero solution is stable in probability. Theorem is proved.

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