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STABILITY IN PROBABILITY OF NONLINEAR STOCHASTIC SYSTEMS WITH DELAY

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We consider the nonlinear stochastic differential equation [1]

$$\begin{aligned}\dot{x}(t) &= Ax(t) + \sum_{i=1}^{r} B_i x(t-h_i) + \sum_{i=1}^{m} \sigma_i x(t-\tau_i)\dot{\xi}_i(t) \\ &+ g\big(t, x(t), x(t-\theta_1), \ldots, x(t-\theta_l)\big), \\ x_0 &= \varphi \in H_0.\end{aligned} \tag{1}$$

Using the method of Lyapunov functionals we get a new sufficient condition for the stability in probability of the trivial solution of (1). Analogous studies of the stability of stochastic systems with delay were also made in [2-7].

Let $\{\Omega, f, P\}$ be a probability space with flow of σ-algebras $f_t \subset f$. Let H_0 be the space of f_0-measurable functions $\varphi(s)$, $s \in [-h, 0]$, $h \in \max_i[h_i, \tau_i, \theta_i]$, with values in R^n and norm $\|\varphi\| = \sup_{-h\le s\le 0} |\varphi(s)|$ whose trajectories are bounded and right continuous for $s < 0$ and left continuous for $s = 0$ with probability 1, $\xi_1(t), \ldots, \xi_m(t)$ be mutually independent scalar Wiener processes, A, B_i, σ_i be constant $n \times n$ matrices, the function $g(t, x_0, x_1, \ldots, x_l)$ satisfy the condition

$$\begin{gathered}|g(t, x_0, x_1, \ldots, x_l)| \le \sum_{j=0}^{l} k_j |x_j|^{a_j}, \\ k_j \ge 0, \quad a_j > 1, \quad j = 0, 1, \ldots, l,\end{gathered} \tag{2}$$

$x(t)$ be a solution of (1) at time t, x_t be the trajectory of the process $x(s)$ for $s \le t$.

The trivial solution of (1) is said to be stable in probability if for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ one can find a $\delta > 0$ such that the solution $x(t) = x(t, \varphi)$ of (1) satisfies the inequality

$$P\{\sup_{t\ge 0} |x(t,\varphi)| > \varepsilon_1\} < \varepsilon_2$$

for any initial functions $\varphi \in H_0$ which are less than δ with probability 1, i.e.,

$$P\{\sup_{-h\le s\le 0} |\varphi(s)| < \delta\} = 1.$$

The square matrix R is said to be negative if the quadratic form $x'Rx$ is negative definite.

The complete infinitesimal L operator of (1) is defined by

$$LV(t,\varphi) = \overline{\lim_{\Delta\to+0}} \frac{1}{\Delta}[V(t+\Delta, y_{t+\Delta}) - V(t,\varphi)],$$

where $y(s)$ is a solution of (1) for $s \ge t$ with initial condition $y(t + s) = \varphi(s)$, $s \in [-h, 0]$.

We describe a class of functionals for which this operator can be computed.

We represent an arbitrary functional $V(t, \varphi)$ defined on $[0, \infty) \times H_0$ in the form $V(t, \varphi) = V(t, \varphi(0), \varphi(s))$, $s < 0$, and set

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$$V_\varphi(t,x) = V(t,\varphi) = V(t,x_t) = V(t,x,x(t+s)),$$
$$\varphi = x_t, \quad s < 0, \quad x = \varphi(0) = x(t).$$

Let D be the class of functionals $V(t, \varphi)$ for which the function $V_\varphi(t, x)$ is twice continuously differentiable in x and has bounded derivative in t for almost all $t \geq 0$. On functionals from D the complete infinitesimal operator L is defined and is equal to

$$LV(t,\varphi) = \frac{\partial}{\partial t}V_\varphi(t,x) + \Big(Ax(t) + \sum_{i=1}^{r} B_i x(t-h_i) + g\big(t, x(t), x(t-\theta_1), \dots, x(t-\theta_l)\big)\Big)' \frac{\partial}{\partial x}V_\varphi(t,x) + \frac{1}{2}\sum_{i=1}^{m} x'(t-\tau_i)\sigma_i' \frac{\partial^2}{\partial x^2}V_\varphi(t,x)\sigma_i x(t-\tau_i). \tag{3}$$

We consider the functional

$$V(t,x_t) = |x(t)|^2 + \nu\Big|x(t) + \sum_{i=1}^{r}\int_{t-h_i}^{t} B_i x(s)\,ds\Big|^2 + \nu\sum_{i=1}^{r}(p_i + \varepsilon_0(\delta)b_i)\int_{t-h_i}^{t}(s+h_i-t)\,|x(s)|^2\,ds + \sum_{i=1}^{r} b_i\int_{t-h_i}^{t}|x(s)|^2\,ds + (\nu+1)\sum_{i=1}^{m}\int_{t-\tau_i}^{t}|\sigma_i x(s)|^2\,ds + (\nu + 1 + \nu(b,h))\sum_{j=1}^{l} k_j\delta^{a_j-1}\int_{t-\theta_j}^{t}|x(s)|^2\,ds. \tag{4}$$

Here

$$\nu \geq 0, \quad b_i = |B_i|, \quad p_i = |(A+B)'B_i|, \quad B = \sum_{i=1}^{r} B_i,$$

$$(b,h) = \sum_{i=1}^{r} b_i h_i, \quad \varepsilon_i(\delta) = \sum_{j=i}^{l} k_j\delta^{a_j-1}, \quad \delta > 0, \quad i = 0,1.$$

In addition, let I be the identity matrix,

$$b = \sum_{i=1}^{r} b_i, \quad (p,h) = \sum_{i=1}^{r} p_i h_i, \quad Q = \frac{1}{2}\sum_{i=1}^{m}\sigma_i'\sigma_i.$$

THEOREM 1. If the matrix

$$R_0 = A + B + Q + \inf_{\nu\geq 0}\Big[\frac{bI - B + \nu(p,h)}{\nu+1}\Big] \tag{5}$$

is negative, then the functional (4) satisfies the condition

$$LV(t,x_t) \leq 0 \tag{6}$$

for all trajectories x_t such that

$$\|x_t\| < \delta, \tag{7}$$

where δ is a sufficiently small positive number.

Proof. From (3) and (4) we get

$$
\begin{aligned}
LV(t, x_t) = {} & 2x'(t)\Big(Ax(t) + \sum_{i=1}^{r} B_i x(t-h_i) \\
& + g\big(t, x(t), x(t-\theta_1), \dots, x(t-\theta_l)\big)\Big) \\
& + 2\nu\Big(x(t) + \sum_{i=1}^{r}\int_{t-h_i}^{t} B_i x(s)\,ds\Big)' \\
& \times \Big((A+B)x(t) + g\big(t, x(t), x(t-\theta_1), \dots, x(t-\theta_l)\big)\Big) \\
& + (\nu+1)\sum_{i=1}^{m} |\sigma_i x(t)|^2 + \nu\big((p,h) + \varepsilon_0(\delta)(b,h)\big)|x(t)|^2 \\
& - \nu\sum_{i=1}^{r}\big(p_i + \varepsilon_0(\delta)b_i\big)\int_{t-h_i}^{t} |x(s)|^2\,ds + b|x(t)|^2 \\
& + (\nu + 1 + \nu(b,h))\varepsilon_1(\delta)|x(t)|^2 - \sum_{i=1}^{r} b_i |x(t-h_i)|^2 \\
& - (\nu + 1 + \nu(b,h))\sum_{j=1}^{l} k_j \delta^{\alpha_j - 1}|x(t-\theta_j)|^2.
\end{aligned}
$$

Using (2) and (7) we get

$$LV(t, x_t) \le 2(\nu+1)x'(t)R_\delta x(t),$$

where

$$R_\delta = R_0 + \Big(1 + \frac{\nu(b,h)}{\nu+1}\Big)\varepsilon_0(\delta).$$

Since R_0 is negative, for sufficiently small δ the matrix R_δ is also negative. Consequently (6) holds. The theorem is proved.

Remark 1. Analogously to [7] and [8] one can show that from the existence of a functional satisfying the inequalities

$$V(t, \varphi) \ge |\varphi(0)|^2, \quad V(0, \varphi) \le c\|\varphi\|^2$$

and (6) and (7) the stability in probability of the trivial solution of (1) follows. Thus we have

THEOREM 2. If the matrix (5) is negative, then the trivial solution of (1) is stable in probability.

Remark 2. Let the nonlinearity in (1) be absent, i.e., $g \equiv 0$. Then it follows from the proof of Theorem 1 and the negativity of (5) that for some $c > 0$ one has

$$LV(t, x_t) \le 2(\nu+1)x'(t)R_0 x(t) \le -c|x(t)|^2.$$

Thus [6], the negativity of (5) not only guarantees the stability in probability of the trivial solution of (1) but also the mean-square asymptotic stability of the trivial solution of (1) for $g \equiv 0$.

Example. We consider the system of equations

$$
\begin{aligned}
\dot{x}_1(t) &= a_1 x_1(t) + b_1 x_1(t-h)x_2(t-h) \\
&\quad + c_1 x_1^{\alpha_1}(t) + \sigma_1 x_1(t-\tau_1)\dot{\xi}_1(t), \\
\dot{x}_2(t) &= a_2 x_2(t) + b_2 x_1(t-h)x_2(t-h) \\
&\quad + c_2 x_2^{\alpha_2}(t) + \sigma_2 x_2(t-\tau_2)\dot{\xi}_2(t).
\end{aligned} \tag{8}
$$

Here $\alpha_1 > 1$, $\alpha_2 > 1$, $\xi_1(t)$ and $\xi_2(t)$ are mutually independent Wiener processes. In this case the matrix R_0 has the form

$$R_0 = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}.$$

Thus if $2a_i + \sigma_i^2 < 0$, $i = 1, 2$, then the trivial solution of (8) is stable in probability.

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