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Construction of Lyapunov Functionals for Stochastic Hereditary Systems: A Survey of Some Recent Results

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Abstract—It is well known that many processes in automatic regulation, physics, mechanics, biology, economy, ecology, etc., can be modelled by hereditary systems. Many stability results in the theory of hereditary systems and their applications were obtained by construction of appropriate Lyapunov functionals (see, for instance, [1–4]). The construction of every such functional was a long time an art of its author. In this paper, formal procedure for construction of Lyapunov functionals for stochastic difference and differential equations and some results on asymptotic mean square stability conditions are considered. More details on these results are presented in [5–52]. The bibliography does not contain works of other researchers since this paper is a short survey of the authors' works.

Keywords—Stochastic systems, Delay, Stability, Lyapunov functionals, Method of construction.

1. STATEMENT OF THE PROBLEM

1.1. Introduction

Let i be a discrete time, $i \in Z \cup Z_0$, $Z = \{0, 1, \dots\}$, $Z_0 = \{-h, \dots, 0\}$, h be a given nonnegative number, process $x_i \in R^n$ be a solution of the equation

$$x_{i+1} = F(i, x_{-h}, \dots, x_i) + \sum_{j=0}^{i} G(i, j, x_{-h}, \dots, x_j)\xi_j, \qquad i \in Z, \tag{1.1}$$

$$x_i = \varphi_i, \qquad i \in Z_0.$$

Here $F : Z * S \Rightarrow R^n$, $G : Z * Z * S \Rightarrow R^n$, S is a space of sequences with elements in R^n. It is assumed that $F(i, \dots)$ does not depend on x_j for $j > i$, $G(i, j, \dots)$ does not depend on x_k for $k > j$ and $F(i, 0, \dots, 0) = 0$, $G(i, j, 0, \dots, 0) = 0$.

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Let $\{\Omega, \sigma, \mathbf{P}\}$ be a basic probability space, $f_i \in \sigma$, $i \in Z$, be a sequence of σ-algebras, $f_i \subset f_j$ for $i < j$, ξ_i be a sequence of mutually independent f_{i+1}-adapted random variables, $\mathbf{E}\xi_i = 0$, $\mathbf{E}\xi_i^2 = 1$. Recall that f_{i+1}-adapted means that random variable ξ_i is f_{i+1}-measurable for each $i \in Z$.

DEFINITION 1.1. *The zero solution of equation (1.1) is called p-stable, $p > 0$, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|x_i|^p < \epsilon$, $i \in Z$, if $\|\varphi\|^p = \sup_{i \in Z_0} \mathbf{E}|\varphi_i|^p < \delta$. If, besides, $\lim_{i\to\infty} \mathbf{E}|x_i|^p = 0$ for every initial function φ, then the zero solution of equation (1.1) is called asymptotically p-stable. In particular, if $p = 2$ then the zero solution of equation (1.1) is called asymptotically mean square stable.*

THEOREM 1.1. *(See [15,27,37].) Let there exist a nonnegative functional $V(i, x_{-h}, \dots, x_i)$, which satisfies the conditions*

$$\mathbf{E}V(0, x_{-h}, \dots, x_0) \le c_1 \|\varphi\|^p,$$
$$\mathbf{E}\Delta V_i \le -c_2 \mathbf{E}|x_i|^p, \qquad i \in Z.$$

Here $c_1 > 0$, $c_2 > 0$, $p > 0$, and

$$\Delta V_i = V(i+1, x_{-h}, \dots, x_{i+1}) - V(i, x_{-h}, \dots, x_i). \tag{1.2}$$

Then the zero solution of equation (1.1) is asymptotically p-stable.

From Theorem 1.1 it follows that an investigation of stability of stochastic equations can be reduced to the construction of appropriate Lyapunov functionals. Below some formal procedure of Lyapunov functionals construction for equations of type (1.1) is proposed.

1.2. Formal Procedure of Lyapunov Functionals Construction

The proposed procedure of Lyapunov functionals construction consists of four steps.

Step 1. Represent the functions F and G at the right-hand side of equation (1.1) in the form

$$\begin{gathered} F(i, x_{-h}, \dots, x_i) = F_1(i, x_{i-\tau}, \dots, x_i) + F_2(i, x_{-h}, \dots, x_i) + \Delta F_3(i, x_{-h}, \dots, x_i), \\ F_1(i, 0, \dots, 0) \equiv F_2(i, 0, \dots, 0) \equiv F_3(i, 0, \dots, 0) \equiv 0, \\ G(i, j, x_{-h}, \dots, x_j) = G_1(i, j, x_{j-\tau}, \dots, x_j) + G_2(i, j, x_{-h}, \dots, x_j), \\ G_1(i, j, 0, \dots, 0) \equiv G_2(i, j, 0, \dots, 0) \equiv 0. \end{gathered} \tag{1.3}$$

Here $\tau \ge 0$ is a given integer, operator Δ is defined by (1.2).

Step 2. Suppose that the zero solution of the auxiliary difference equation

$$y_{i+1} = F_1(i, y_{i-\tau}, \dots, y_i) + \sum_{j=0}^{i} G_1(i, j, y_{j-\tau}, \dots, y_j)\xi_j, \qquad i \in Z, \tag{1.4}$$

is asymptotically mean square stable and there exists a Lyapunov function $v_i = v(i, y_{i-\tau}, \dots, y_i)$ for this equation which satisfies the conditions of Theorem 1.1.

Step 3. A Lyapunov functional V_i is constructed in the form $V_i = V_{1i} + V_{2i}$, where the main component is

$$V_{1i} = v(i, x_{i-\tau}, \dots, x_{i-1}, x_i - F_3(i, x_{-h}, \dots, x_i)).$$

Step 4. In order to satisfy the conditions of Theorem 1.1 it is necessary to calculate $\mathbf{E}\Delta V_{1i}$ and in a reasonable way to estimate it. After that the additional component V_{2i} is chosen in a standard way.

Consider some peculiarities of this procedure.

It is clear that representation (1.3) in the first step is not unique. Hence, for different representations (1.3) it is possible to construct different Lyapunov functionals, and therefore, get different stability conditions.

In the second step for one auxiliary equation (1.4) it is possible to choose different Lyapunov functions v_i, and therefore, to construct different Lyapunov functionals for equation (1.1).

At last it is necessary to stress that choosing different ways of estimation of $\mathbf{E}\Delta V_{1i}$ it is possible to construct different Lyapunov functionals and as a result to obtain different stability conditions [39,52].

2. ILLUSTRATIVE EXAMPLE

Here it is shown that using different representations of the initial equation in form (1.3) it is possible to get different stability conditions.

Let us investigate a region of asymptotic mean square stability of the scalar equation with constant coefficients

$$x_{i+1} = a_0 x_i + a_1 x_{i-1} + \sigma x_{i-1}\xi_i, \qquad i \in Z. \tag{2.1}$$

2.1. First Way of Lyapunov Functional Construction

Using the four steps of the procedure described above, we obtain the following.

1. The right-hand side of equation (2.1) is represented already in form (1.3) with $\tau = 0$, $F_1(i, x_i) = a_0 x_i$, $F_2(i, x_{-1}, \dots, x_i) = a_1 x_{i-1}$, $F_3(i, x_{-1}, \dots, x_i) = G_1(i, j, x_j) = 0$, $G_2(i, j, x_{-1}, \dots, x_j) = 0$, $j = 0, \dots, i-1$, $G_2(i, i, x_{-1}, \dots, x_i) = \sigma x_{i-1}$.
2. Auxiliary equation (1.4) in this case has the form $y_{i+1} = a_0 y_i$. The function $v_i = y_i^2$ is a Lyapunov function for this equation if $|a_0| < 1$, since $\Delta v_i = (a_0^2 - 1) y_i^2$.
3. The main part V_{1i} of the Lyapunov functional $V_i = V_{1i} + V_{2i}$ must be chosen in the form $V_{1i} = x_i^2$.
4. Estimating $\mathbf{E}\Delta V_{1i}$ for equation (2.1), it is possible to show that

$$\mathbf{E}\Delta V_{1i} \le \left(a_0^2 - 1 + |a_0 a_1|\right) \mathbf{E}x_i^2 + A\mathbf{E}x_{i-1}^2,$$

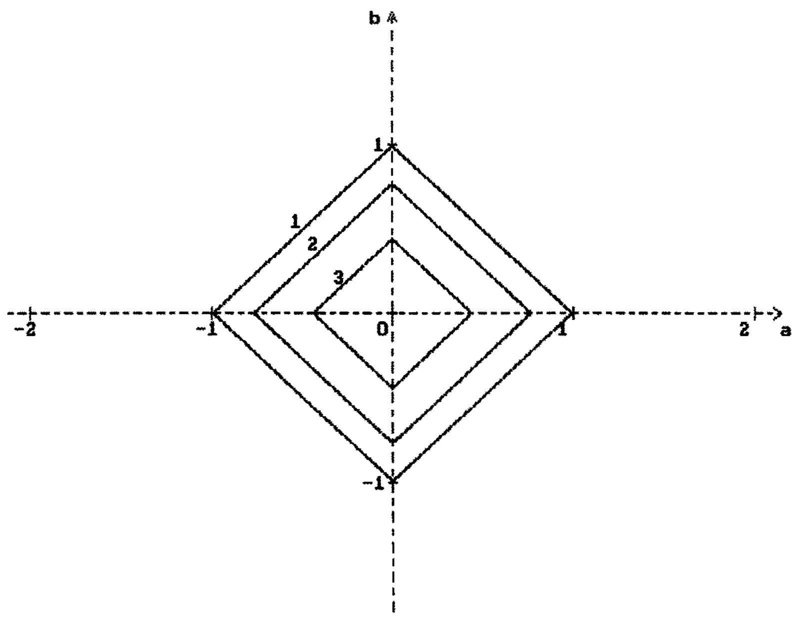

Figure 1.

where $A = a_1^2 + |a_0 a_1| + \sigma^2$. Put $V_2 = A x_{i-1}^2$. Then $\Delta V_2 = A(x_i^2 - x_{i-1}^2)$ and for $V_i = V_{1i} + V_{2i}$ we have $\mathbf{E}\Delta V_i \le ((|a_0| + |a_1|)^2 + \sigma^2 - 1)\mathbf{E}x_i^2$. Therefore, under the condition

$$|a_0| + |a_1| < \sqrt{1 - \sigma^2} \tag{2.2}$$

the functional V_i satisfies the conditions of Theorem 1.1 and the zero solution of equation (2.1) is asymptotically mean square stable.

The stability regions for equation (2.1), given by inequality (2.2), are shown on Figure 1 (with $a = a_0$, $b = a_1$) for different values of σ^2:

(1) $\sigma^2 = 0$;
(2) $\sigma^2 = 0.4$;
(3) $\sigma^2 = 0.8$.

2.2. Second Way of Lyapunov Functional Construction

Use now another representation of equation (2.1).

1. Represent the right-hand side of equation (2.1) in form (1.3) with $\tau = 0$, $F_1(i, x_i) = (a_0 + a_1)x_i$, $F_2(i, x_{-1}, \dots, x_i) = 0$, $F_3(i, x_{-1}, \dots, x_i) = -a_1 x_{i-1}$, and $G_1(i, j, x_j), G_2(i, j, x_{-1}, \dots, x_j)$ as before.
2. Auxiliary equation (2.2) in this case is $y_{i+1} = (a_0 + a_1)y_i$. The function $v_i = y_i^2$ is a Lyapunov function for this equation if $|a_0 + a_1| < 1$, since $\Delta v_i = ((a_0 + a_1)^2 - 1)y_i^2$.
3. The main part V_{1i} of the Lyapunov functional $V_i = V_{1i} + V_{2i}$ must be chosen in the form

$$V_{1i} = (x_i + a_1 x_{i-1})^2. \tag{2.3}$$

4. Estimating $\mathbf{E}\Delta V_{1i}$ by virtue of (2.1),(2.3), we can show that

$$\mathbf{E}\Delta V_{1i} \le \left((a_0 + a_1)^2 - 1 + |a_1(a_0 + a_1 - 1)|\right)\mathbf{E}x_i^2 + B\mathbf{E}x_{i-1}^2,$$

where $B = \sigma^2 + |a_1(a_0 + a_1 - 1)|$. Put $V_{2i} = Bx_{i-1}^2$. Then for the functional $V_i = V_{1i} + V_{2i}$ we have

$$\mathbf{E}\Delta V_i \le \left((a_0 + a_1)^2 - 1 + 2|a_1(a_0 + a_1 - 1)| + \sigma^2\right)\mathbf{E}x_i^2.$$

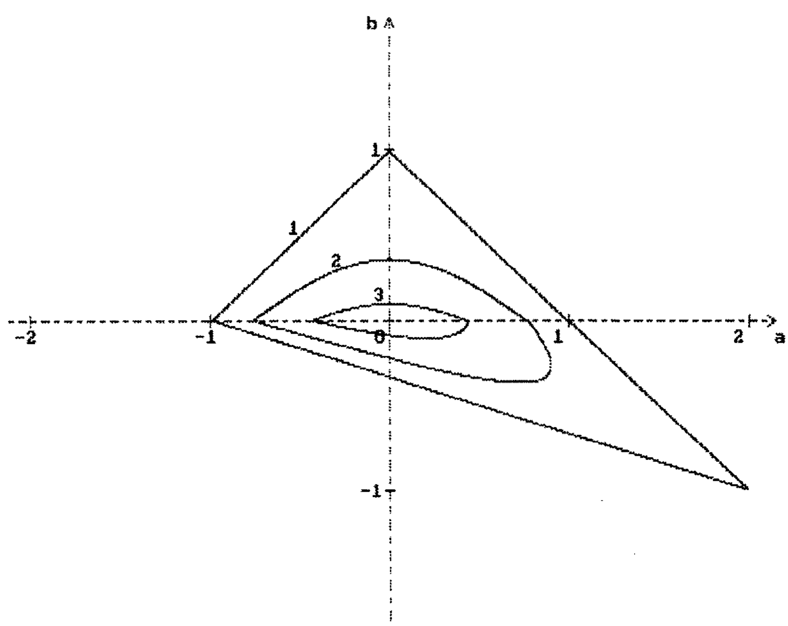

Figure 2.

Therefore, if the condition

$$(a_0 + a_1)^2 + 2|a_1(a_0 + a_1 - 1)| + \sigma^2 < 1 \tag{2.4}$$

holds, then the functional V_i satisfies the conditions of Theorem 1.1 and the zero solution of equation (2.1) is asymptotically mean square stable.

Note that condition (2.4) can be written in the form

$$|a_0 + a_1| < \sqrt{1 - \sigma^2}, \qquad \sigma^2 < (1 - a_0 - a_1)(1 + a_0 + a_1 - 2|a_1|). \tag{2.5}$$

The stability regions, defined by condition (2.5), are shown on Figure 2 (with $a = a_0$, $b = a_1$) for different values of σ^2:

(1) $\sigma^2 = 0$;
(2) $\sigma^2 = 0.4$;
(3) $\sigma^2 = 0.8$.

2.3. Third Way of Lyapunov Functional Construction

In some cases, the auxiliary equation can be obtained by iterating right-hand side of equation (2.1). For example, from equation (2.1) we get

$$\begin{aligned} x_{i+1} &= a_0(a_0 x_{i-1} + a_1 x_{i-2} + \sigma x_{i-2}\xi_{i-1}) + a_1 x_{i-1} + \sigma x_{i-1}\xi_i \\ &= \left(a_0^2 + a_1\right) x_{i-1} + a_0 a_1 x_{i-2} + a_0 \sigma x_{i-2}\xi_{i-1} + \sigma x_{i-1}\xi_i. \end{aligned} \tag{2.6}$$

1. Here representation (1.3) is used with $\tau = 0$, $F_1(i, x_i) = F_3(i, x_{-1}, \dots, x_i) = 0$, $F_2(i, x_{-1}, \dots, x_i) = (a_0^2 + a_1)x_{i-1} + a_0 a_1 x_{i-2}$, $G_1(i, j, x_j) = 0$, $j = 0, \dots, i$, $G_2(i, j, x_{-1}, \dots, x_j) = 0$, $j = 0, \dots, i-2$, $G_2(i, i-1, x_{-1}, \dots, x_{i-1}) = a_0\sigma x_{i-1}$, $G_2(i, i, x_{-1}, \dots, x_i) = \sigma x_{i-1}$.
2. The auxiliary equation is $y_{i+1} = 0$, $i \in Z$. The function $v_i = y_i^2$ is a Lyapunov function for this equation since $\Delta v_i = y_{i+1}^2 - y_i^2 = -y_i^2$.
3. The main part V_{1i} of the Lyapunov functional $V_i = V_{1i} + V_{2i}$ must be chosen in the form $V_{1i} = x_i^2$.
4. Estimating $\mathbf{E}\Delta V_{1i}$ by virtue of (2.6), we obtain

$$\mathbf{E}\Delta V_{1i} \leq -\mathbf{E}x_i^2 + A_1\mathbf{E}x_{i-1}^2 + A_2\mathbf{E}x_{i-2}^2,$$

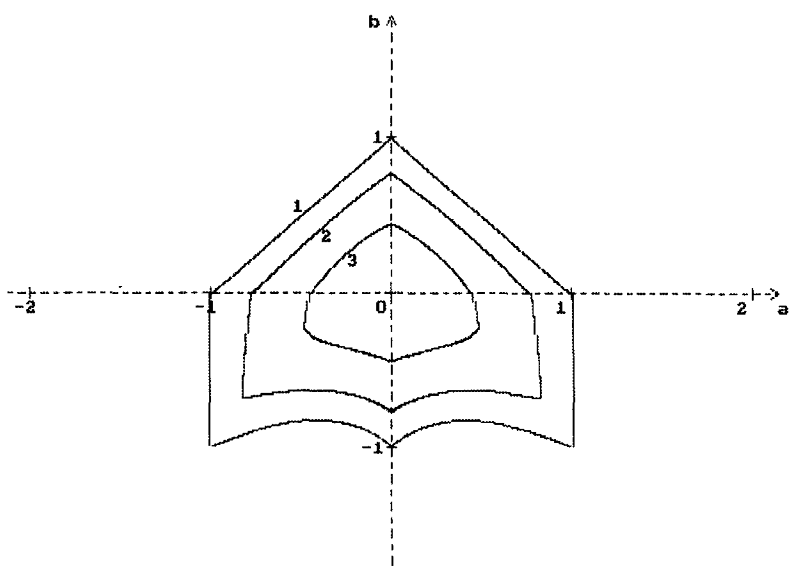

Figure 3.

where $A_1 = \sigma^2 + |a_0 a_1||a_0^2 + a_1| + (a_0^2 + a_1)^2$, $A_2 = \sigma^2 a_0^2 + |a_0 a_1||a_0^2 + a_1| + a_0^2 a_1^2$. Put $V_{2i} = (A_1 + A_2)x_{i-1}^2 + A_2 x_{i-2}^2$. Then for $V_i = V_{1i} + V_{2i}$ we have $\mathbf{E}\Delta V_i \le -(1 - A_1 - A_2)\mathbf{E}x_i^2$. Therefore, the condition of asymptotic mean square stability of the zero solution of equation (2.6) has the form $A_1 + A_2 < 1$ or

$$|a_0 a_1| + \left|a_0^2 + a_1\right| < \sqrt{1 - \sigma^2\left(1 + a_0^2\right)}. \tag{2.7}$$

The stability regions, defined by condition (2.7), are shown on Figure 3 (with $a = a_0$, $b = a_1$) for different values of σ^2:

(1) $\sigma^2 = 0$;
(2) $\sigma^2 = 0.4$;
(3) $\sigma^2 = 0.8$.

2.4. Fourth Way of Lyapunov Functional Construction

Consider now the case $\tau = 1$.

1. Represent equation (2.1) in form (1.3) with $F_1(i, x_{i-1}, x_i) = a_0 x_i + a_1 x_{i-1}$, $G_2(i, i, x_{-1}, \ldots, x_i) = \sigma x_{i-1}$, $F_2(i, x_{-1}, \ldots, x_i) = F_3(i, x_{-1}, \ldots, x_i) = 0$, $G_1(i, j, x_j) = 0$, $j = 0, \ldots, i$, $G_2(i, j, x_{-1}, \ldots, x_j) = 0$, $j = 0, \ldots, i-1$.
2. In this case, the auxiliary equation is

$$y_{i+1} = a_0 y_i + a_1 y_{i-1}. \tag{2.8}$$

Using the vector $y(i) = (y_{i-1}, y_i)'$, equation (2.8) can be written in the form

$$y(i+1) = Ay(i), \qquad A = \begin{pmatrix} 0 & 1 \\ a_1 & a_0 \end{pmatrix}. \tag{2.9}$$

Let C be an arbitrary nonnegative definite matrix. If the equation

$$A'DA - D = -C \tag{2.10}$$

has a positive definite solution D, then the function $v_i = y'(i)Dy(i)$ is a Lyapunov function for equation (2.9). In particular, if

$$C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}, \qquad c_1 \ge 0, \quad c_2 > 0, \tag{2.11}$$

then the solution D of equation (2.10) has the elements d_{ij}, such that

$$d_{11} = c_1 + a_1^2 d_{22}, \qquad d_{12} = \frac{a_0 a_1}{1 - a_1} d_{22},$$
$$d_{22} = \frac{(c_1 + c_2)(1 - a_1)}{(1 + a_1)\left[(1 - a_1)^2 - a_0^2\right]}. \tag{2.12}$$

It is easy to see that the matrix D is a positive definite matrix by the conditions

$$|a_1| < 1, \qquad |a_0| < 1 - a_1. \tag{2.13}$$

3. The functional V_{1i} must be chosen in the form $V_{1i} = x'(i)Dx(i)$.
4. Estimating $\mathbf{E}\Delta V_{1i}$ by virtue of (2.10),(2.11), we obtain

$$\Delta V_{1i} = -c_2 \mathbf{E}x_i^2 + \left(\sigma^2 d_{22} - c_1\right) \mathbf{E}x_{i-1}^2.$$

Therefore, if $\sigma^2 d_{22} \le c_1$, then $V_{2i} = 0$ and for the functional $V_i = V_{1i}$ we have $\mathbf{E}\Delta V_i \le -c_2 \mathbf{E}x_i^2$. If $\sigma^2 d_{22} > c_1$, then $V_{2i} = (\sigma^2 d_{22} - c_1)x_{i-1}^2$ and for the functional $V_i = V_{1i} + V_{2i}$ we have $\mathbf{E}\Delta V_i = -(c_1 + c_2 - \sigma^2 d_{22})\mathbf{E}x_i^2$.

Thus, if condition (2.13) and $\sigma^2 d_{22} < c_1 + c_2$, or otherwise

$$\frac{\sigma^2(1 - a_1)}{(1 + a_1)\left[(1 - a_1)^2 - a_0^2\right]} < 1 \tag{2.14}$$

hold, then the zero solution of equation (2.1) is asymptotically mean square stable.

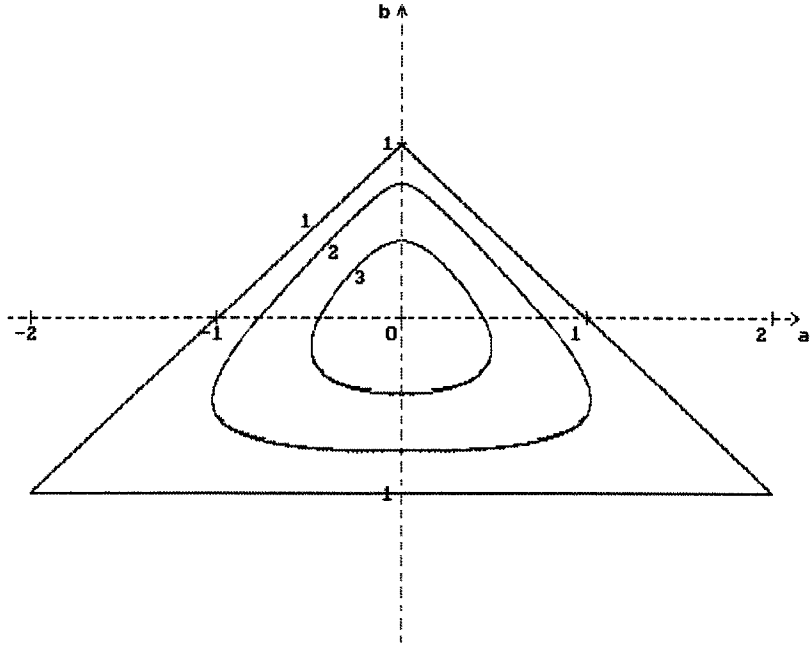

Figure 4.

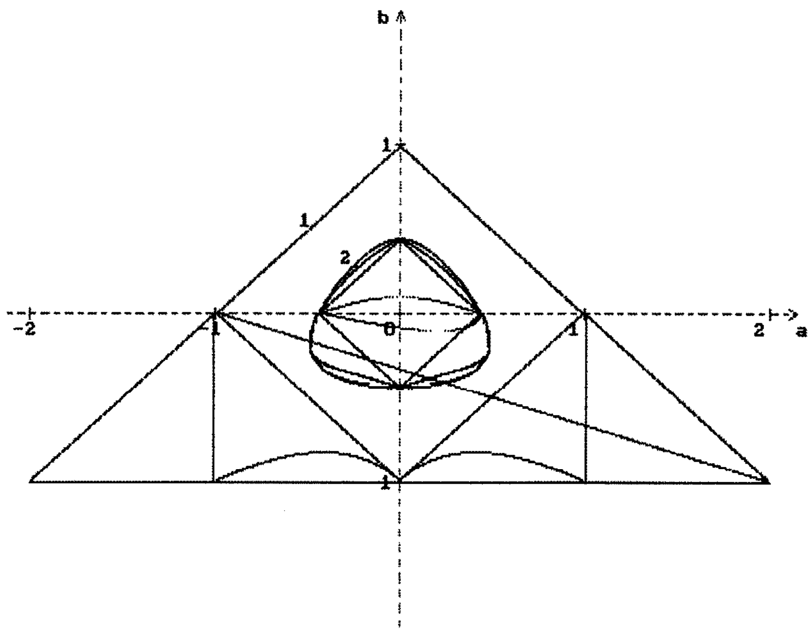

Figure 5.

The stability regions, defined by conditions (2.13),(2.14), are shown on Figure 4 (with $a = a_0$, $b = a_1$) for different values of σ^2:

(1) $\sigma^2 = 0$;
(2) $\sigma^2 = 0.4$;
(3) $\sigma^2 = 0.8$.

On Figure 5 (with $a = a_0$, $b = a_1$), a comparison of the stability regions, which are obtained by conditions (2.2), (2.5), (2.7), and (2.13),(2.14), is shown for two values of σ^2:

(1) $\sigma^2 = 0$,
(2) $\sigma^2 = 0.8$.

REMARK 2.1. Note that if $\sigma^2 d_{22} \ge c_1 + c_2$, then there exists the functional

$$V_i = x'(i)Dx(i) + \left(\sigma^2 d_{22} - c_1 - c_2\right) x_{i-1}^2,$$

for which $\mathbf{E}\Delta V_i = (\sigma^2 d_{22} - c_1 - c_2)\mathbf{E}x_i^2 \ge 0$. It means that if $\sigma^2 d_{22} \ge c_1 + c_2$ then the zero solution of equation (2.1) do not asymptotically mean square stable. Therefore, conditions (2.13),(2.14) are necessary and sufficient conditions of asymptotic mean square stability of the zero solution of equation (2.1). For example, the inequality $\sigma^2 < 1$ is a necessary and sufficient condition of asymptotic mean square stability of the zero solution of the equation $x_{i+1} = \sigma x_{i-1}\xi_i$. Really, in this case we have $\mathbf{E}x_{2m-k}^2 = \sigma^{2m}\mathbf{E}x_{-k}^2$, $k = 0, 1$, $m = 0, 1, \dots$.

In more detail, construction of necessary and sufficient conditions of asymptotic mean square stability is discussed in [26].

3. LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Now the proposed procedure of Lyapunov functionals construction is applied to the equation

$$x_{i+1} = \sum_{l=-h}^{i} a_{i-l}x_l + \sum_{j=0}^{i}\sum_{l=-h}^{j} \sigma_{j-l}^{i-j} x_l \xi_j, \qquad i \in Z, \tag{3.1}$$

$$x_i = \varphi_i, \qquad i \in Z_0.$$

Here a_i and σ_j^i are given constants. Below the following symbols are used also:

$$S_0 = \sum_{p=0}^{\infty}\left(\sum_{l=0}^{\infty}|\sigma_l^p|\right)^2, \qquad S_k = \sum_{p=k}^{\infty}\sum_{l=0}^{\infty}|\sigma_l^p|, \quad k = 1, 2, 3.$$

3.1. First Way of Lyapunov Functional Construction

1. Represent the right-hand side of equation (3.1) in form (1.3) with $\tau = 0$, $F_1(i, x_i) = a_0 x_i$, $F_3(i, x_{-h}, \dots, x_i) = G_1(i, j, x_j) = 0$,

$$F_2(i, x_{-h}, \dots, x_i) = \sum_{l=-h}^{i-1} a_{i-l}x_l, \qquad G_2(i, j, x_{-h}, \dots, x_j) = \sum_{l=-h}^{j} \sigma_{j-l}^{i-j} x_l.$$

2. Auxiliary difference equation (1.4) in this case is $y_{i+1} = a_0 y_i$. The function $v_i = y_i^2$ can be taken as a Lyapunov function for this equation if $|a_0| < 1$, since $\Delta v_i = (a_0^2 - 1)y_i^2$.
3. The main part V_{1i} of the Lyapunov functional $V_i = V_{1i} + V_{2i}$ must be chosen in the form $V_{1i} = x_i^2$.
4. Estimating $\mathbf{E}\Delta V_{1i}$ by virtue of (3.1), it is possible to show that

$$\mathbf{E}\Delta V_{1i} \le -\mathbf{E}x_i^2 + \sum_{k=-h}^{i} A_{ik}\mathbf{E}x_k^2,$$

where

$$A_{ik} = (\alpha_0 + S_1)|a_{i-k}| + \alpha_0 \sum_{p=1}^{i-k_m} |\sigma_{i-k-p}^p| + \sum_{p=0}^{i-k_m} |\sigma_{i-k-p}^p| \sum_{l=0}^{\infty} |\sigma_l^p|,$$

$$\alpha_0 = \sum_{l=0}^{\infty} |a_l|, \qquad k_m = \max(0, k).$$

Putting

$$V_{2i}=\sum_{l=-h}^{i-1}x_l^2\sum_{j=i-l}^{\infty}A_{j+l,l}$$

and calculating $\mathbf{E}\Delta V_{2i}$, we obtain

$$\mathbf{E}\Delta V_{2i}=\mathbf{E}x_i^2\sum_{j=1}^{\infty}A_{j+i,i}-\sum_{k=-h}^{i-1}A_{ik}\mathbf{E}x_k^2.$$

As a result, for the functional $V_i=V_{1i}+V_{2i}$ we get

$$\mathbf{E}\Delta V_i\le-\mathbf{E}x_i^2\left(1-\sum_{j=0}^{\infty}A_{j+i,i}\right).$$

It is shown [15] that $\sum_{j=0}^{\infty}A_{j+i,i}\le\alpha_0^2+2\alpha_0S_1+S_0$. Therefore, the condition of asymptotic mean square stability of the zero solution of equation (3.1) is

$$\alpha_0^2+2\alpha_0S_1+S_0<1. \tag{3.2}$$

In particular, for equation (2.1) we have $\alpha_0=|a_0|+|a_1|$, $S_0=\sigma^2$, $S_1=0$, and from (3.2) condition (2.2) follows.

3.2. Second Way of Lyapunov Functional Construction

1. Represent the right-hand side of equation (3.1) in form (1.3) with $\tau=0$, $F_1(i,x_i)=\beta x_i$, $F_2(i,x_{-h},\dots,x_i)=G_1(i,j,x_j)=0$,

$$\beta=\sum_{j=0}^{\infty}a_j,\qquad F_3(i,x_{-h},\dots,x_i)=-\sum_{l=-h}^{i-1}x_l\sum_{j=i-l}^{\infty}a_j,$$

$$G_2(i,j,x_{-h},\dots,x_j)=\sum_{l=-h}^{j}\sigma_{j-l}^{i-j}x_l.$$

2. Auxiliary equation (1.4) in this case is $y_{i+1}=\beta y_i$. The function $v_i=y_i^2$ can be taken as a Lyapunov function for this equation if $|\beta|<1$, since $\Delta v_i=(\beta^2-1)y_i^2$.
3. The main part V_{1i} of the Lyapunov functional $V_i=V_{1i}+V_{2i}$ should be chosen in the form $V_{1i}=(x_i-F_3)^2$.
4. Using (3.1), it is possible to show [15], that

$$\mathbf{E}\Delta V_{1i}\le\left[\beta^2-1+|\beta|S_1+\left|\sigma_0^0\right|\sum_{l=0}^{\infty}\left|\sigma_l^0\right|+|\beta-1|\alpha\right]\mathbf{E}x_i^2+\sum_{k=-h}^{i-1}B_{ik}\mathbf{E}x_k^2,$$

where

$$B_{ik}=|\beta|\sum_{p=1}^{i-k_m}\left|\sigma_{i-k-p}^p\right|+\sum_{p=0}^{i-k_m}\left|\sigma_{i-k-p}^p\right|\sum_{l=0}^{\infty}|\sigma_l^p|+|\beta-1|\left|\sum_{j=i-k}^{\infty}a_j\right|$$
$$+\alpha\sum_{p=2}^{i-k_m}\left|\sigma_{i-k-p}^p\right|+S_2\left|\sum_{m=i-k}^{\infty}a_m\right|,\qquad\alpha=\sum_{l=1}^{\infty}\left|\sum_{m=l}^{\infty}a_m\right|.$$

Putting

$$V_{2i} = \sum_{k=-h}^{i-1} x_k^2 \sum_{j=i-k}^{\infty} B_{j+k,k},$$

as before it is possible to get [15] that

$$\mathbf{E}\Delta V_i \leq \left[\beta^2 - 1 + 2\alpha|\beta - 1| + S_0 + 2[|\beta|S_1 + \alpha S_2]\right] \mathbf{E}x_i^2.$$

Therefore, the zero solution of equation (3.1) is asymptotically mean square stable by condition $\beta^2 + 2\alpha|\beta - 1| + S_0 + 2[|\beta|S_1 + \alpha S_2] < 1$ or otherwise

$$|\beta| < 1, \qquad S_0 + 2\left[|\beta|S_1 + \alpha S_2\right] < (1-\beta)(1+\beta-2\alpha). \tag{3.3}$$

In particular, for equation (2.1) we have $\beta = a_0 + a_1$, $\alpha = |a_1|$, $S_0 = \sigma^2$, $S_1 = S_2 = 0$, and from (3.3) condition (2.4) follows.

3.3. Third Way of Lyapunov Functional Construction

1. Represent the right-hand side of equation (3.1) in form (1.3) with $\tau = 1$, $F_1(i, x_{i-1}, x_i) = a_0 x_i + a_1 x_{i-1}$, $F_3(i, x_{-h}, \dots, x_i) = G_1(i, j, x_j) = 0$,

$$F_2(i, x_{-h}, \dots, x_i) = \sum_{l=-h}^{i-2} a_{i-l} x_l, \qquad G_2(i, j, x_{-h}, \dots, x_j) = \sum_{l=-h}^{j} \sigma_{j-l}^{i-j} x_l.$$

2. In this case, the auxiliary equation has the form (2.8) (or (2.9)).
3. The functional V_{1i} is chosen in the form $V_{1i} = x'(i)Dx(i)$, where matrix D is defined by conditions (2.10)–(2.12). It is supposed also that conditions (2.13) hold.
4. Estimating $\mathbf{E}\Delta V_{1i}$ and choosing V_{2i} in a standard way, we obtain [15] that inequalities (2.13) and

$$\frac{\left(\alpha_2^2 + 2\alpha_2 S_3 + S_0 + |a_1|(2\alpha_2 + 2S_2)\right)(1 - a_1) + 2(\alpha_2 + S_1)|a_0|}{(1 + a_1)\left[(1 - a_1)^2 - a_0^2\right]} < 1, \tag{3.4}$$

where

$$\alpha_2 = \sum_{l=2}^{\infty} |a_l|,$$

are sufficient conditions of asymptotic mean square stability of the zero solution of equation (3.1).

In particular, for equation (2.1) we have $\alpha_2 = S_1 = S_2 = S_3 = 0$, $S_0 = \sigma^2$, and from (3.4) it follows condition (2.14).

Note that, using other representations of equation (3.1) in form (1.3) (for instance, with $\tau = 2$), it is possible to obtain other sufficient conditions of asymptotic mean square stability of the zero solution of equation (3.1).

EXAMPLE 3.1. Consider the scalar equation

$$\begin{aligned} x_{i+1} &= a_0 x_i + a_k x_{i-k} + \sigma x_{i-l}\xi_i, \qquad i \in Z, \\ x_i &= \varphi_i, \qquad i \in Z_0, \qquad k \geq 1, \qquad l \geq 0. \end{aligned} \tag{3.5}$$

In this case, $\alpha_0 = |a_0| + |a_k|$, $S_0 = \sigma^2$, $S_1 = 0$. From (3.2), we get the sufficient condition of asymptotic mean square stability of the zero solution of equation (3.5), which is more general than (2.2)

$$|a_0| + |a_k| < \sqrt{1 - \sigma^2}. \tag{3.6}$$

Since $\beta = a_0 + a_k$, $\alpha = k|a_k|$, $S_1 = S_2 = 0$, from (3.3) we obtain the sufficient condition of asymptotic mean square stability of the zero solution of equation (3.5), which is a generalization of condition (2.4)

$$(a_0 + a_k)^2 + 2k|a_k(a_0 + a_k - 1)| + \sigma^2 < 1. \tag{3.7}$$

If $k \geq 2$, then $a_1 = 0$, $\alpha_2 = |a_k|$, $S_1 = S_2 = S_3 = 0$. In this case, from (2.13),(3.4) the sufficient condition of asymptotic mean square stability of the zero solution of equation (3.5) follows in form (3.6) as well.

4. SYSTEMS WITH MONOTONE COEFFICIENTS

By virtue of construction of appropriate Lyapunov functionals stability conditions type of (3.2)–(3.4) were obtained [15] also for the equation with variable coefficients

$$x_{i+1} = \sum_{l=-h}^{i} a_{il} x_l + \sum_{j=0}^{i} \sum_{l=-h}^{j} \sigma_{jl}^{i} x_l \xi_j.$$

These stability conditions contain some assumptions about convergence of series from coefficients a_{ij}. Assumptions of such type sometimes are very limiting.

Stability conditions of another type were obtained [39] for linear equation of the form

$$x_{i+1} = -\sum_{j=0}^{i} a_{ij} x_j + \sum_{j=0}^{i} \sigma_{ij} x_j \xi_i. \tag{4.1}$$

THEOREM 4.1. *Let*

$$\begin{aligned} &a_{ij} \geq a_{i,j-1} \geq 0, \qquad j = 1, 2, \dots, i, \\ &a_{i+1,j-1} - a_{i+1,j} - a_{i,j-1} + a_{ij} \geq 0, \\ &\sup_{i \in Z} (a_{i+1,i+1} + a_{ii} - a_{i+1,i}) < 2\left(1 - \sigma^2\right), \end{aligned} \tag{4.2}$$

where

$$\sigma^2 = \sup_{i \in Z} \sum_{j=0}^{\infty} |\sigma_{j+i,i}| \sum_{k=0}^{j+i} |\sigma_{j+i,k}|.$$

Then the zero solution of equation (4.1) is asymptotically mean square stable.

These stability conditions were obtained without any assumptions about convergence of the series with coefficients a_{ij} by virtue of construction of special Lyapunov functional in the form $V_i = V_{1i} + V_{2i}$, where $V_{1i} = x_i^2$,

$$V_{2i} = \sum_{j=0}^{i} \alpha_{ij} \left(\sum_{k=j}^{i} x_k \right)^2 + (1 + \gamma) \sum_{k=0}^{i-1} x_k^2 \sum_{j=i-k}^{\infty} B_{j+k,k}.$$

Parameters α_{ij}, B_{ij}, and γ are defined by virtue of coefficients of system (4.1).

Note that in the case $a_{ij} = a_{i-j}$, $\sigma_{ij} = \sigma_{i-j}$, conditions (4.2) take the form

$$\begin{aligned} &a_i \geq a_{i+1} \geq 0, \qquad a_{i+2} - 2a_{i+1} + a_i \geq 0, \quad i = 0, 1, \dots, \\ &2a_0 - a_1 < 2\left(1 - \sigma^2\right), \qquad \sigma^2 = \left(\sum_{j=0}^{\infty} |\sigma_j| \right)^2. \end{aligned} \tag{4.3}$$

EXAMPLE 4.1. Consider the equation

$$x_{i+1} = -a x_i - b \sum_{j=1}^{i} x_{i-j} + \sigma x_{i-1} \xi_i. \tag{4.4}$$

It is easy to see, that conditions (3.2)–(3.4) cannot be used in this case. But from (4.3) it follows that the sufficient condition of asymptotic mean square stability of the zero solution of equation (4.4) has the form: $0 \le b \le a < b/2 + 1 - \sigma^2$.

Note that for $a = b$ this condition has the form $0 \le b < 2(1 - \sigma^2)$ and it is a necessary and sufficient condition of asymptotic mean square stability of the zero solution of equation (4.4).

EXAMPLE 4.2. Consider the equation

$$x_{i+1} = -ax_i - \sum_{j=1}^{i} b^j x_{i-j} + \sigma x_{i-1}\xi_i. \tag{4.5}$$

From (4.3), it follows that the sufficient condition of asymptotic mean square stability of the zero solution of equation (4.5) has the form: $0 \le b \le 1$, $2b - b^2 \le a < b/2 + 1 - \sigma^2$.

5. EQUATIONS WITH VARYING DELAYS

5.1. Systems with Nonincreasing Delays

Consider the equation

$$x_{i+1} = ax_i + bx_{i-k(i)} + \sigma x_{i-m(i)}\xi_i. \tag{5.1}$$

It is assumed that the delays $k(i)$ and $m(i)$ satisfy the conditions

$$k(i) \ge k(i+1) \ge 0, \qquad m(i) \ge m(i+1) \ge 0. \tag{5.2}$$

First, we construct a Lyapunov functional for equation (5.1) in the form $V_i = V_{1i} + V_{2i}$, where $V_{1i} = x_i^2$. Estimating $\mathbf{E}\Delta V_{1i}$, we obtain

$$\begin{aligned}\mathbf{E}\Delta V_{1i} &= \mathbf{E}\left[(ax_i + bx_{i-k(i)} + \sigma x_{i-m(i)}\xi_i)^2 - x_i^2\right] \\ &\le \left(a^2 - 1 + |ab|\right)\mathbf{E}x_i^2 + \left(b^2 + |ab|\right)\mathbf{E}x_{i-k(i)}^2 + \sigma^2\mathbf{E}x_{i-m(i)}^2.\end{aligned}$$

Choosing the functional V_{2i} in the form

$$V_{2i} = \left(b^2 + |ab|\right)\sum_{j=i-k(i)}^{i-1} x_j^2 + \sigma^2 \sum_{j=i-m(i)}^{i-1} x_j^2$$

and using (5.2), it is possible to show [35] that

$$\Delta V_{2i} \le \left(b^2 + |ab| + \sigma^2\right)x_i^2 - \left(b^2 + |ab|\right)x_{i-k(i)}^2 - \sigma^2 x_{i-m(i)}^2.$$

As a result for $V_i = V_{1i} + V_{2i}$ we have $\mathbf{E}\Delta V_i \le ((|a| + |b|)^2 + \sigma^2 - 1)\mathbf{E}x_i^2$. So, by conditions (5.2) and $(|a| + |b|)^2 + \sigma^2 < 1$ the zero solution of equation (5.1) is asymptotically mean square stable.

Consider now another way of Lyapunov functional construction. Following the general method of Lyapunov functionals construction, represent right-hand side of equation (5.1) in form (1.3) with $\tau = 0$, $F_{1i} = (a + b)x_i$,

$$F_{2i} = -b\sum_{j=i+1-k(i)}^{i-k(i+1)} x_j, \qquad F_{3i} = -b\sum_{j=i-k(i)}^{i-1} x_j, \qquad \Delta F_{3i} = -b\left(x_i - \sum_{j=i-k(i)}^{i-k(i+1)} x_j\right).$$

As a result we have $x_{i+1} = F_{1i} + F_{2i} + \Delta F_{3i} + \sigma x_{i-m(i)}\xi_i$. The functional V_{1i} must be chosen in the form

$$V_{1i} = (x_i - F_{3i})^2 = \left(x_i + b\sum_{j=i-k(i)}^{i-1} x_j\right)^2.$$

Calculating $\mathbf{E}\Delta V_{1i}$ and using the representations for x_{i+1} and F_{1i}, we have

$$\begin{aligned}\mathbf{E}\Delta V_{1i} &= \mathbf{E}\left[\left(x_{i+1} - F_{3(i+1)}\right)^2 - (x_i - F_{3i})^2\right] \\ &= \mathbf{E}\left(x_{i+1} - x_i - \Delta F_{3i}\right)\left(x_{i+1} + x_i - F_{3(i+1)} - F_{3i}\right) \\ &= \left((a+b)^2 - 1\right)\mathbf{E}x_i^2 + 2(a+b)\mathbf{E}x_iF_{2i} + \mathbf{E}F_{2i}^2 \\ &\quad - 2(a+b-1)\mathbf{E}x_iF_{3i} - 2\mathbf{E}F_{2i}F_{3i} + \sigma^2\mathbf{E}x_{i-m(i)}^2.\end{aligned}$$

Estimating x_iF_{2i}, F_{2i}^2, x_iF_{3i}, and $F_{2i}F_{3i}$, and using that $k(0) \ge k(i)$, we get [35]

$$\mathbf{E}\Delta V_{1i} \le (A-1)\mathbf{E}x_i^2 + B\sum_{j=i+1-k(0)}^{i-k_m}\mathbf{E}x_j^2 + C\sum_{j=i-k(0)}^{i-1}\mathbf{E}x_j^2 + \sigma^2\mathbf{E}x_{i-m(i)}^2,$$

where

$$\begin{gathered}A = (a+b)^2 + |b(a+b)|k_0 + |b(a+b-1)|k(0), \\ B = |b(a+b)| + b^2(k_0 + k(0)), \qquad C = |b(a+b-1)| + b^2k_0, \\ k_0 = \sup_{i\in Z}(k(i) - k(i+1)), \qquad k_m = \inf_{i\in Z} k(i).\end{gathered}$$

Let

$$\begin{aligned}V_{2i} = B&\left(\sum_{l=i}^{i+k_m-2}\ \sum_{j=l+1-k(0)}^{l-k_m} x_j^2 + \sum_{j=i-k(0)+k_m}^{i-1}(j-i+1+k(0)-k_m)x_j^2\right) \\ &+ C\sum_{j=i-k(0)}^{i-1}(j-i+1+k(0))x_j^2 + \sigma^2\sum_{j=i-m(i)}^{i-1}x_j^2.\end{aligned}$$

For the functional $V_i = V_{1i} + V_{2i}$ it is shown [35] that

$$\mathbf{E}\Delta V_i \le \left(A + B(k(0) - k_m) + Ck(0) + \sigma^2 - 1\right)\mathbf{E}x_i^2.$$

Thus, by conditions (5.2) and

$$\begin{gathered}(a+b)^2 + 2k(0)|b(a+b-1)| + |b(a+b)|(k_0 + k(0) - k_m) + \\ + b^2(k_0k(0) + (k_0 + k(0))(k(0) - k_m)) + \sigma^2 < 1\end{gathered} \tag{5.3}$$

the zero solution of equation (5.1) is asymptotically mean square stable.

Note that if $k(i) = k = \text{const}$, then condition (3.7) follows from (5.3) with $a = a_0$, $b = a_k$.

5.2. Systems With Unbounded Delays

Consider the equation

$$x_{i+1} = \sum_{j=0}^{k(i)} a_j x_{i-j} + \sum_{j=0}^{m(i)} \sigma_j x_{i-j}\xi_i. \tag{5.4}$$

Here it is supposed that the delays $k(i)$ and $m(i)$ satisfy the conditions

$$\begin{gathered}k(i+1) - k(i) \le 1, \qquad m(i+1) - m(i) \le 1, \\ \hat{k} = \sup_{i\in Z} k(i) \le \infty, \qquad \hat{m} = \sup_{i\in Z} m(i) \le \infty.\end{gathered} \tag{5.5}$$

Note that from (5.5) it follows that $k(i) \le k(0) + i$, $m(i) \le m(0) + i$.

Using the Lyapunov functional $V_i = V_{1i} + V_{2i}$, where $V_{1i} = x_i^2$,

$$V_{2i} = \sum_{j=1}^{k(i)} x_{i-j}^2 \sum_{l=j}^{\hat{k}} A_l + \sum_{j=1}^{m(i)} x_{i-j}^2 \sum_{l=j}^{\hat{m}} B_l,$$

$$A_l = a|a_l|, \qquad B_l = \sigma|\sigma_l|, \qquad a = \sum_{j=0}^{\hat{k}} |a_j|, \qquad \sigma = \sum_{j=0}^{\hat{m}} |\sigma_j|,$$

it is shown [35] that $\mathbf{E}\Delta V_i \le (a^2 + \sigma^2 - 1)\mathbf{E}x_i^2$. Thus, by conditions (5.5) and $a^2 + \sigma^2 < 1$ the zero solution of equation (5.4) is asymptotically mean square stable.

Other stability conditions for difference equations with varying delays were obtained in [37].

6. VOLTERRA EQUATIONS OF THE SECOND TYPE

6.1. Problem Statement

Let $\{\Omega, \mathbf{P}, \sigma\}$ be a probability space, $f_i \in \sigma$, $i \in Z$, be a sequence of σ-algebras, $f_i \subset f_j$ for $i < j$, H_p, $p > 0$, be a space of sequences $x = \{x_i, i \in Z\}$ of f_i-adapted random variables $x_i \in R^n$ with norm $\|x\|^p = \sup_{i \in Z} \mathbf{E}|x_i|^p$.

Consider the stochastic difference equation in the form

$$x_{i+1} = \eta_{i+1} + F(i, x_0, \dots, x_i), \qquad i \in Z, \quad x_0 = \eta_0. \tag{6.1}$$

Here it is assumed that $\eta \in H_p$, the functional F is such that $F : Z * H_p \Rightarrow R^n$ and $F(i, \cdot)$ does not depend on x_j for $j > i$, $F(i, 0, \dots, 0) = 0$.

DEFINITION 6.1. *A sequence x_i from H_p is called uniformly p-bounded if $\|x\|^p < \infty$, asymptotically p-trivial if $\lim_{i\to\infty} \mathbf{E}|x_i|^p = 0$, p-summable if $\sum_{i=0}^{\infty} \mathbf{E}|x_i|^p < \infty$.*

Note that if the sequence x_i is p-summable, then it is uniformly p-bounded and asymptotically p-trivial.

THEOREM 6.1. *(See [21].) Let there exist a nonnegative functional $V_i = V(i, x_0, \dots, x_i)$ and a sequence of nonnegative numbers γ_i, such that*

$$\mathbf{E}V(0, x_0) < \infty, \qquad \sum_{i=0}^{\infty} \gamma_i < \infty,$$
$$\mathbf{E}\Delta V_i \le -c\mathbf{E}|x_i|^p + \gamma_i, \qquad i \in Z, \quad c > 0.$$

Then the solution of equation (6.1) is p-summable.

6.2. Illustrative Example

Using the procedure of Lyapunov functionals construction, described above, let us investigate the asymptotic behavior of the scalar equation with constant coefficients

$$\begin{gathered} x_0 = \eta_0, \qquad x_1 = \eta_1 + a_0\eta_0, \\ x_{i+1} = \eta_{i+1} + a_0 x_i + a_1 x_{i-1}, \qquad i \ge 1. \end{gathered} \tag{6.2}$$

The right-hand side of equation (6.2) is represented already in form (1.3) with $\tau = 0$, $F_1(i, x_i) = a_0 x_i$, $F_2(i, x_0, \dots, x_i) = a_1 x_{i-1}$, $F_3(i, x_0, \dots, x_i) = 0$. The auxiliary equation (1.4) in this case is $y_{i+1} = a_0 y_i$. The function $v_i = y_i^2$ is a Lyapunov function for this equation if $|a_0| < 1$, since $v_i = (a_0^2 - 1)y_i^2$.

The main part V_{1i} of the Lyapunov functional $V_i=V_{1i}+V_{2i}$ must be chosen in the form $V_{1i}=x_i^2$. Estimating $\mathbf{E}V_{1i}$ for equation (6.2), we have

$$\begin{aligned}\mathbf{E}\Delta V_{1i} &= \mathbf{E}\left[(\eta_{i+1}+a_0x_i+a_1x_{i-1})^2-x_i^2\right]\\ &\le \left[1+\alpha^{-1}(|a_0|+|a_1|)\right]\mathbf{E}\eta_{i+1}^2+\left(a_0^2-1+|a_0a_1|+\alpha|a_0|\right)\mathbf{E}x_i^2+A\mathbf{E}x_{i-1}^2,\end{aligned}$$

where $A=a_1^2+|a_0a_1|+\alpha|a_1|$, $\alpha>0$.

Let $V_{2i}=Ax_{i-1}^2$. Then $\Delta V_{2i}=A(x_i^2-x_{i-1}^2)$ and for $V_i=V_{1i}+V_{2i}$ we get

$$\mathbf{E}\Delta V_i \le \left[1+\alpha^{-1}(|a_0|+|a_1|)\right]\mathbf{E}\eta_{i+1}^2+\left[(|a_0|+|a_1|)^2+\alpha(|a_0|+|a_1|)-1\right]\mathbf{E}x_i^2.$$

If $|a_0|+|a_1|<1$, then there exists a small $\alpha>0$ that $(|a_0|+|a_1|)^2+\alpha(|a_0|+|a_1|)<1$. So, under the condition $|a_0|+|a_1|<1$ the functional V_i satisfies the conditions of Theorem 6.1 with $p=2$, and therefore, the solution of equation (6.2) is mean square summable.

Similar to previous in [21,27] the summability conditions type of (2.4) and (2.13),(2.14) were obtained for equation (6.2). Summability conditions, which are similar to (4.2), were obtained also for the Volterra equation of type

$$x_{i+1}=\eta_{i+1}+\sum_{j=0}^{i}a_{ij}x_j.$$

To get another type of conditions consider the nonlinear Volterra equation

$$x_{i+1}=\eta_{i+1}+\sum_{j=0}^{i}a_{ij}g(x_j),\qquad |g(x)|\le|x|. \tag{6.3}$$

THEOREM 6.2. *Let the sequence η_i be p-summable and the kernel a_{ij} satisfy the condition $\alpha\beta^{p-1}<1$, $p\ge1$, where*

$$\alpha=\sup_{i\in Z}\sum_{l=0}^{\infty}|a_{l+i,i}|<1,\qquad \beta=\sup_{i\in Z}\sum_{l=0}^{i}|a_{il}|.$$

Then the solution of equation (6.3) is p-summable.

For proving Theorem 6.2 it was shown [48,50] that the functional V_i, $i\in Z$, where

$$V_0=|x_0|^p,\qquad V_i=\sum_{j=0}^{i-1}\sum_{l=i-j-1}^{\infty}|a_{l+j,j}|\,|g(x_j)|^p,\qquad i>0,$$

satisfies the conditions of Theorem (6.2).

Note that if in equation (6.3) $a_{ij}=a_{i-j}$, then $\alpha=\beta=\sum_{j=0}^{\infty}|a_j|$ and the inequality $\alpha<1$ is a sufficient condition of p-summability with $p\ge1$.

EXAMPLE 6.1. Let in equation (6.3) $a_{ij}=a_{i-j}$ and $a_i=\lambda q^i$, $i\in Z$, $|q|<1$. Then $\alpha=|\lambda|(1-|q|)^{-1}$ and the sufficient condition of p-summability with $p\ge1$ has the form $|\lambda|+|q|<1$.

6.3. Nonlinear System with Monotone Coefficients

In some cases, for some systems of special type it is possible to get summability conditions using special characteristics of the system under consideration. Consider, for instance, the nonlinear system

$$x_{i+1}=\eta_{i+1}-\sum_{j=0}^{i}a_{ij}g(x_j), \tag{6.4}$$

where the function $g(x)$ satisfies the condition

$$0<c_1\le\frac{g(x)}{x}\le c_2,\qquad x\ne0.$$

THEOREM 6.3. *(See [50].) Let the coefficients a_{ij}, $i \in Z$, $j = 0, \dots, i$, satisfy the conditions*

$$a_{ij} \geq a_{i,j-1} \geq 0,$$
$$a_{i+1,j-1} - a_{i+1,j} - a_{i,j-1} + a_{ij} \geq 0,$$
$$a = \sup_{i \in Z}(a_{i+1,i+1} + a_{ii} - a_{i+1,i}) < \frac{2}{c_2},$$
$$\alpha = \sup_{i \in Z} \sum_{l=0}^{\infty} a_{l+i,i} < \infty, \qquad \beta = \sup_{i \in Z} \sum_{j=0}^{i} a_{ij} < \infty.$$

Then the solution of equation (6.4) is mean square summable.

For proving this theorem the Lyapunov functional

$$V_i = x_i g(x_i) + \sum_{j=0}^{i} \alpha_{ij} \left(\sum_{k=j}^{i} g(x_k) \right)^2 + \sum_{k=0}^{i-1} x_k^2 \sum_{j=i-k}^{\infty} Q_{j+k,k}$$

was constructed. Here the numbers α_{ij} and Q_{ij} are defined by virtue of parameters of the initial system (6.4).

Note that in the case $a_{ij} = a_{i-j}$ the conditions of Theorem 6.3 have the form

$$a_i \geq a_{i+1} \geq 0, \qquad a_{i+2} - 2a_{i+1} + a_i \geq 0, \qquad i \in Z,$$
$$2a_0 - a_1 < \frac{2}{c_2}, \qquad \alpha = \beta = \sum_{j=0}^{\infty} a_j < \infty.$$

EXAMPLE 6.2. Let in equation (6.4) $a_{ij} = a_{i-j}$ and $a_i = \lambda q^i$, $i \in Z$, $\lambda > 0$, $0 < q < 1$. From Theorem 6.2, using Example 6.1, we obtain the sufficient condition of mean square summability in the form $\lambda c_2 + q < 1$. Theorem 6.3 gives us another sufficient condition of mean square summability $\lambda c_2 < 2(2-q)^{-1}$. It is easy to see that the second condition is weaker than first one, i.e., $1 - q < 2(2-q)^{-1}$.

EXAMPLE 6.3. Consider equation (6.4) with $a_{ij} = a_{i-j}$ and $a_i = \lambda(i+1)^{-\gamma}$, $\lambda > 0$, $\gamma > 1$, $i \in Z$. In this case, $\alpha = \beta = \lambda\zeta(\gamma)$, where $\zeta(\gamma)$ is the Riemann function $\zeta(\gamma) = \sum_{i=1}^{\infty} i^{-\gamma} < \infty$. From Theorem 6.2, we obtain a sufficient condition of mean square summability in the form $\lambda c_2 < \zeta^{-1}(\gamma)$. Theorem 6.3 gives us another summability condition $\lambda c_2 < 2(2 - 2^{-\gamma})^{-1}$. It is easy to see that $\zeta^{-1}(\gamma) < 1$, but $2(2-2^{-\gamma})^{-1} > 1$. Thus, the second condition is weaker than the first one. For instance, for $\gamma = 2$ these conditions take the forms $\lambda c_2 < \zeta^{-1}(2) = 1.645^{-1} = 0.608$ and $\lambda c_2 < 2(2 - 2^{-2})^{-1} = 1.143$.

EXAMPLE 6.4. Consider equation (6.4) with $a_{ij} = \lambda j^{\gamma}(i+1)^{-(1+\gamma)}$, $0 \leq j \leq i$, $\lambda > 0$, $\gamma > 0$. It is shown [50] that in this case $\alpha \leq \lambda\gamma^{-1}$, $\beta \leq \lambda(1+\gamma)^{-1}$. Thus, Theorem 6.2 gives us with $p = 2$ the sufficient condition of mean square summability in the form $\lambda c_2 < \sqrt{\gamma(1+\gamma)}$. Using Theorem 6.3 and the estimate $a \leq 2\lambda(1+\gamma)^{-1}$, we obtain the condition of mean square summability in the form $\lambda c_2 < 1 + \gamma$. In spite of the fact that the estimate of a is rough enough, the last condition is weaker than previous one. In fact, for concrete $\gamma > 0$ it is possible to get an estimate of a which is essentially better than we used above. For instance, for $\gamma = 1$ it is easy to show that $a \leq 13\lambda/36$ and the summability conditions take the forms $\lambda c_2 < \sqrt{2} = 1.414$, $\lambda c_2 < 72/13 = 5.538$. If $\gamma = 2$, then $a \leq 17\lambda/72$ and the summability conditions take the forms $\lambda c_2 < \sqrt{6} = 2.449$, $\lambda c_2 < 144/17 = 8.471$.

7. DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

Here, the general method of Lyapunov functionals construction is demonstrated for hereditary systems with continuous time. Consider the stochastic differential equation of neutral type

$$\begin{aligned} &d(x(t) - G(t, x_t)) = a_1(t, x_t)\,dt + a_2(t, x_t)\,d\xi(t), \\ &t \geq 0, \qquad x(t) \in R^n, \qquad x(s) = \varphi(s), \qquad s \leq 0. \end{aligned} \tag{7.1}$$

Here, $x_t = x(t+s)$, $s \leq 0$, $\xi(t) \in R^m$ is a standard Wiener process, $a_i(t, 0) = 0$, $i = 1, 2$,

$$|G(t, \varphi)| \leq \int_0^\infty |\varphi(-s)|\,dK(s), \qquad \int_0^\infty dK(s) < 1. \tag{7.2}$$

DEFINITION 7.1. *The zero solution of equation (7.1) is called mean square stable if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|x(t)|^2 < \epsilon$, $t \geq 0$, if $\|\varphi\|^2 = \sup_{s \leq 0} \mathbf{E}|\varphi(s)|^2 < \delta$. If, besides, $\lim_{t\to\infty} \mathbf{E}|x(t)|^2 = 0$ for every initial function $\varphi(s)$ then the zero solution of equation (7.1) is called asymptotically mean square stable.*

THEOREM 7.1. *(See [14].) Let condition (7.2) hold and there exist the functional*

$$V(t, \varphi) = W(t, \varphi) + |\varphi(0) - G(t, \varphi)|^2,$$

such that

$$0 \leq \mathbf{E}W(t, x_t) \leq c_1 \|x_t\|^2,$$

$$\mathbf{E}LV(t, x_t) \leq -c_2 \mathbf{E}|x(t)|^2,$$

where $c_i > 0$, $i = 1, 2$, L is the generator of equation (7.1). Then the zero solution of equation (7.1) is asymptotically mean square stable.

As before the proposed procedure of Lyapunov functionals construction consists of four steps.

Step 1. Transform equation (7.1) to the form

$$dz(t, x_t) = (b_1(t, x(t)) + c_1(t, x_t))\,dt + (b_2(t, x(t)) + c_2(t, x_t))\,d\xi(t), \tag{7.3}$$

where $z(t, x_t)$ is some functional on x_t, $z(t, 0) = 0$, functionals b_i, $i = 1, 2$, depend on t and $x(t)$ only and do not depend on the previous values $x(t+s)$, $s < 0$, of the solution, $b_i(t, 0) = 0$.

Step 2. Assume that the zero solution of the auxiliary equation without memory

$$dy(t) = b_1(t, y(t))\,dt + b_2(t, y(t))\,d\xi(t). \tag{7.4}$$

is asymptotically mean square stable, and therefore, there exists a Lyapunov function $v(t, y)$, for which the condition $L_0 v(t, y) \leq -|y|^2$ holds. Here, L_0 is the generator of equation (7.4).

Step 3. A Lyapunov functional $V(t, x_t)$ is constructed in the form $V = V_1 + V_2$, where $V_1(t, x_t) = v(t, z(t, x_t))$. Here the argument y of the function $v(t, y)$ is replaced on the functional $z(t, x_t)$ from the left-hand part of equation (7.3).

Step 4. Usually, the functional V_1 almost satisfies the conditions of Theorem 7.1. In order to satisfy these conditions completely, it is necessary to calculate LV_1 and estimate it. Then the additional component V_2 can be easily chosen in a standard way.

Note that representation (7.3) is not unique. This fact allows us, using different representations (7.3), to construct different Lyapunov functionals and, as a result, get different sufficient conditions of asymptotic mean square stability.

EXAMPLE 7.1. Using the proposed procedure, it is simple enough to construct conditions of asymptotic mean square stability for the scalar equation of neutral type

$$\dot{x}(t) + ax(t) + bx(t-h) + c\dot{x}(t-h) + \sigma x(t-\tau)\dot{\xi}(t) = 0, \qquad |c| < 1. \tag{7.5}$$

Note that conditions of asymptotic mean square stability for equation (7.5) were obtained already in [5]. But conditions, constructed here, give us greater stability region.

Following Step 1, rewrite equation (7.5) in the form

$$\dot{z}(t) = -ax(t) - bx(t-h) - \sigma x(t-\tau)\dot{\xi}(t),$$

where $z(t) = x(t) + cx(t-h)$. Suppose that $a > 0$. Then the function $v = y^2(t)$ is a Lyapunov function for the auxiliary equation $\dot{y}(t) = -ay(t)$, since $\dot{v} = -2ay^2(t)$. Thus, the zero solution of the auxiliary equation is asymptotically stable. Put $V_1 = z^2(t)$. Then

$$\begin{aligned} LV_1 &= 2z(t)(-ax(t) - bx(t-h)) + \sigma^2 x^2(t-\tau) \\ &= -2ax^2(t) - 2bcx^2(t-h) - 2(ac+b)x(t)x(t-h) + \sigma^2 x^2(t-\tau) \\ &\le (-2a + |ac+b|)x^2(t) + \rho x^2(t-h) + \sigma^2 x^2(t-\tau), \end{aligned}$$

where $\rho = |ac+b| - 2bc$ if $|ac+b| > 2bc$ and $\rho = 0$ if $|ac+b| \le 2bc$.

Let

$$V_2 = \rho \int_{t-h}^{t} x^2(s)\,ds + \sigma^2 \int_{t-\tau}^{t} x^2(s)\,ds,$$

then for the functional $V = V_1 + V_2$, we obtain

$$LV \le \left(-2a + |ac+b| + \rho + \sigma^2\right) x^2(t).$$

So, if $|ac+b| + \rho + \sigma^2 < 2a$, then the zero solution of equation (7.5) is asymptotically mean square stable. Using two representations for ρ, we obtain two stability conditions

$$2bc \ge |ac+b|, \qquad \sigma^2 + |ac+b| < 2a \tag{7.6}$$

and

$$2bc < |ac+b|, \qquad p + |ac+b| - bc < a, \qquad p = \frac{\sigma^2}{2}. \tag{7.7}$$

From (7.6) and $a > 0$, we have $bc = |bc|$ and $|ac+b| = a|c| + |b|$. So, inequalities (7.6) take the form $2|bc| \ge a|c| + |b|$ and $\sigma^2 + a|c| + |b| < 2a$. The first from these inequalities is impossible if $2|c| < 1$. Suppose that $2|c| \ge 1$. Then

$$\frac{\sigma^2 + |b|}{2 - |c|} < a \le \left(2 - \frac{1}{|c|}\right)|b|. \tag{7.8}$$

It is easy to see, that these inequalities are incompatible. Really, from (7.8) the impossible inequality $\sigma^2|c| + 2|b|(1-|c|)^2 < 0$ follows. Thus, condition (7.6) is impossible.

Consider condition (7.7). Suppose that $bc \ge 0$. From here and $a > 0$ we have $bc = |bc|$, $|ac+b| = a|c| + |b|$ and condition (7.7) takes the form

$$2|bc| < a|c| + |b|, \qquad a > |b| + \frac{p}{1-|c|}.$$

If $2|c| < 1$, then the first inequality holds for all a and b. If $2|c| \geq 1$, then the second inequality implies the first one. So, if $bc \geq 0$, then from condition (7.7) we have

$$bc \geq 0, \qquad a > |b| + \frac{p}{1-|c|}. \tag{7.9}$$

Let $bc < 0$. Then the first inequality (7.7) holds and condition (7.7) takes the form

$$bc < 0, \qquad p + |ac+b| - bc < a. \tag{7.10}$$

Since $bc < 0$, then $|ac+b| = |a|c| - |b||$. So, if $a|c| \geq |b|$ then from (7.10) we have

$$\frac{p}{1-|c|} - a < |b| \leq a|c|. \tag{7.11}$$

If $a|c| < |b|$, then

$$a|c| < |b| < a - \frac{p}{1+|c|}. \tag{7.12}$$

Combining (7.11) and (7.12), we obtain

$$bc < 0, \qquad \frac{p}{1-|c|} - a < |b| < a - \frac{p}{1+|c|}. \tag{7.13}$$

Note that the system

$$|b| = \frac{p}{1-|c|} - a, \qquad |b| = a - \frac{p}{1+|c|},$$

by $bc < 0$, has the solution

$$a = \frac{p}{1-c^2}, \qquad b = -\frac{pc}{1-c^2}. \tag{7.14}$$

So, combining (7.9), (7.13), (7.14), we obtain the stability conditions in the form

$$\begin{aligned} a &> \frac{p}{1-c} + b, \qquad b > -\frac{pc}{1-c^2}, \\ a &> \frac{p}{1+c} - b, \qquad b \leq -\frac{pc}{1-c^2}. \end{aligned} \tag{7.15}$$

Thus, if the conditions $|c| < 1$ and (7.15) hold, then the zero solution of equation (7.5) is asymptotically mean square stable.

The stability regions for equation (7.5), given by stability conditions (7.15), are shown on Figure 6 for $c = -0.5$, $h = 1$ and different values of p:

(1) $p = 0$;
(2) $p = 0.5$;
(3) $p = 1$,
(4) $p = 1.5$.

In Figure 7, the stability regions are shown for $c = 0.5$ and the same values of other parameters.

To get another stability condition represent equation (7.5) in the form

$$\dot{z}(t) = -(a+b)x(t) - \sigma x(t-\tau)\dot{\xi}(t),$$

where

$$z(t) = x(t) + cx(t-h) - b\int_{t-h}^{t} x(s)\,ds.$$

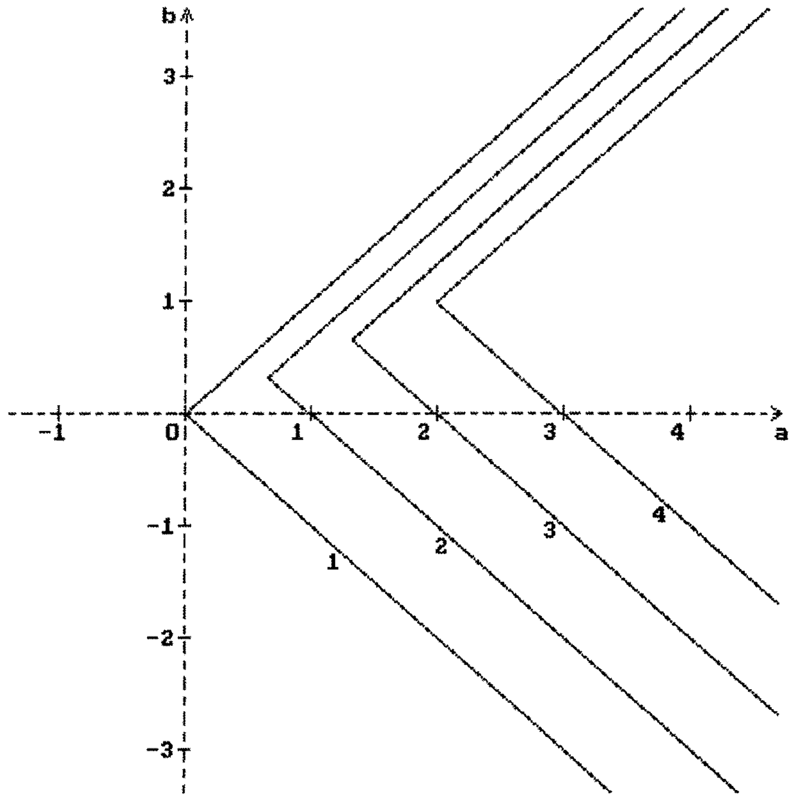

Figure 6.

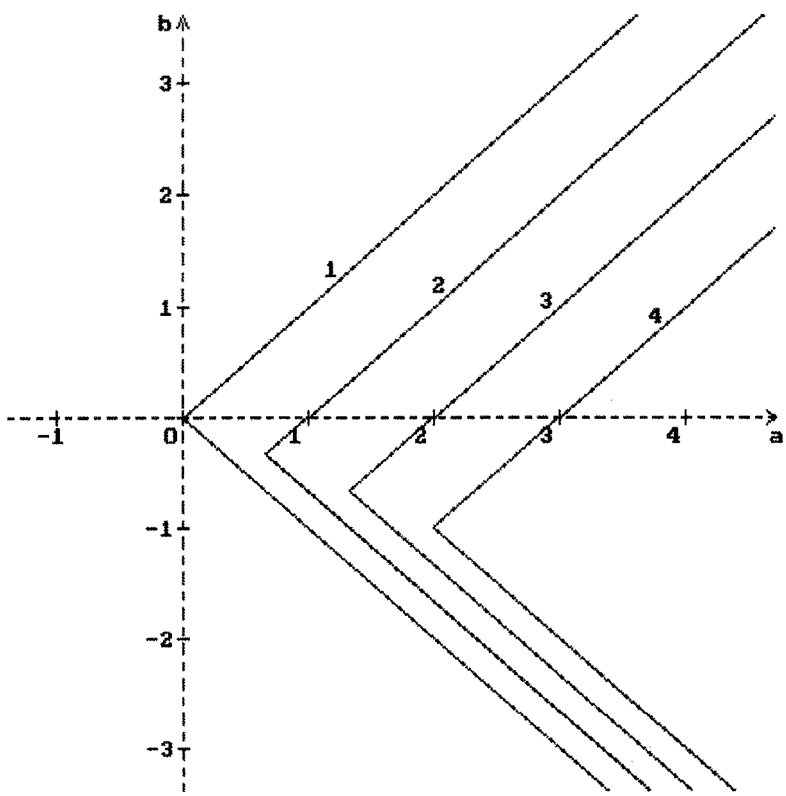

Figure 7.

Suppose that $a+b>0$. Then the function $v=y^2(t)$ is a Lyapunov function for the auxiliary equation $\dot{y}(t)=-(a+b)y(t)$, since $\dot{v}=-2(a+b)y^2(t)$. Thus, the zero solution of the auxiliary equation is asymptotically stable. Put $V_1=z^2(t)$. Then

$$\begin{aligned}LV_1 &= -2(a+b)x(t)z(t)+\sigma^2x^2(t-\tau)\\ &= -2(a+b)x^2(t)-2(a+b)cx(t)x(t-h)+2(a+b)b\int_{t-h}^{t}x(t)x(s)\,ds+\sigma^2x^2(t-\tau)\\ &\le (a+b)(-2+|c|+|b|h)x^2(t)+\sigma^2x^2(t-\tau)+(a+b)\left(|c|x^2(t-h)+|b|\int_{t-h}^{t}x^2(s)\,ds\right).\end{aligned}$$

Let

$$V_2=(a+b)\int_{t-h}^{t}[|c|+|b|(s-t+h)]x^2(s)\,ds+\sigma^2\int_{t-\tau}^{t}x^2(s)\,ds.$$

Then for the functional $V=V_1+V_2$ we obtain

$$LV\le\left[-2(a+b)(1-|c|-|b|h)+\sigma^2\right]x^2(t).$$

Thus, the stability condition has the form $p<(a+b)(1-|c|-|b|h)$, $|c|+|b|h<1$ or

$$a>\frac{p}{1-|c|-|b|h}-b,\qquad |b|<\frac{1-|c|}{h}. \tag{7.16}$$

The stability regions for equation (7.5), given by stability condition (7.16), are shown in Figure 8 for $|c|=0.5$, $h=0.2$ and different values of p:

(1) $p=0.2$;
(2) $p=0.6$;
(3) $p=1$;
(4) $p=1.4$;

and on Figure 9 for $|c|=0.5$, $p=0.4$ and different values of h:

(1) $h=0.1$;
(2) $h=0.15$;
(3) $h=0.2$;
(4) $h=0.25$.

It is easy to see, that for $b\le 0$ conditions (7.15) are better than (7.16). So, condition (7.16) it is better to use for $b>0$ only in the form

$$a>\frac{p}{1-|c|-bh}-b,\qquad 0<b<\frac{1-|c|}{h}. \tag{7.17}$$

For $h\to 0$ condition (7.17) takes the form

$$a>\frac{p}{1-|c|}-b,\qquad b>0. \tag{7.18}$$

Note that for $h=0$ we have $LV_1=-2(a+b)(1+c)x^2(t)+\sigma^2x^2(t-\tau)$ and $LV=[-2(a+b)(1+c)+\sigma^2]x^2(t)$. So, for $h=0$ the necessary and sufficient condition of asymptotic mean square stability has the form

$$a>\frac{p}{1+c}-b. \tag{7.19}$$

For $b>0$ and $c>0$, condition (7.18) is essentially worse than (7.19). But for $b>0$ and $c\le 0$ conditions (7.18) and (7.19) coincide. The second condition of (7.15) coincides with condition (7.19) as well.

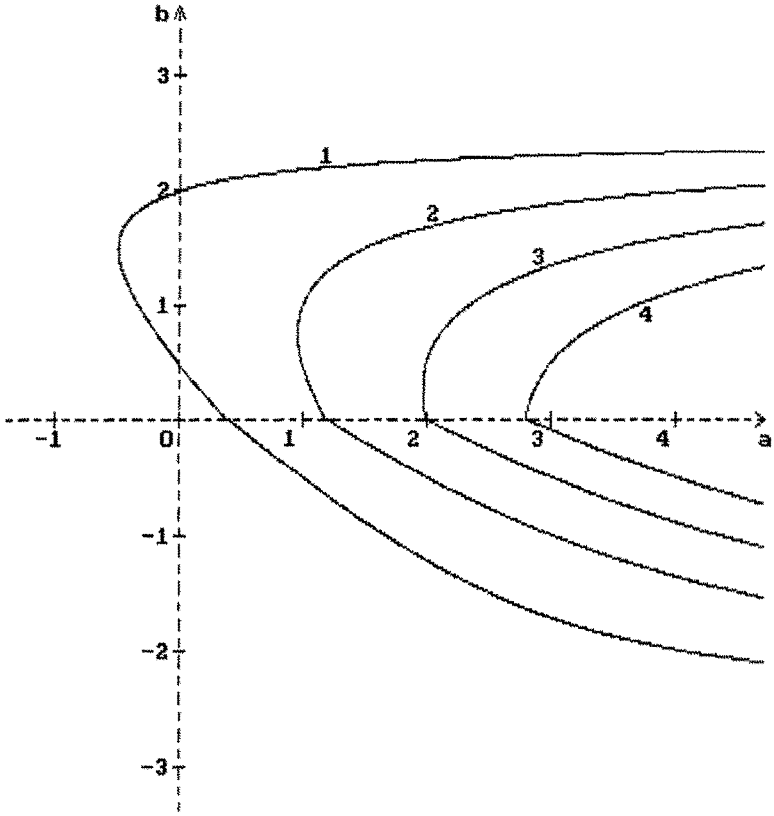

Figure 8.

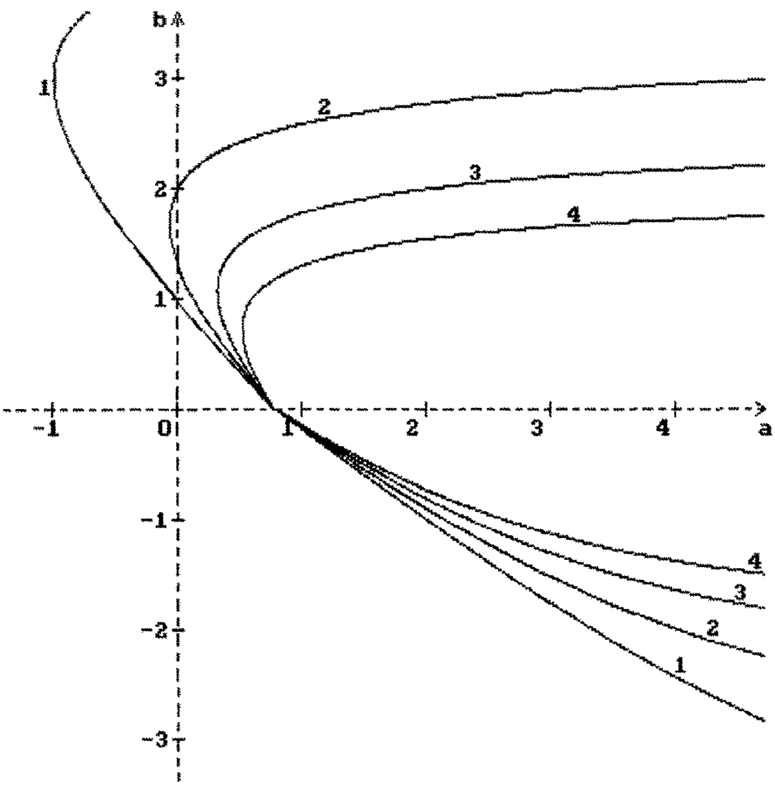

Figure 9.

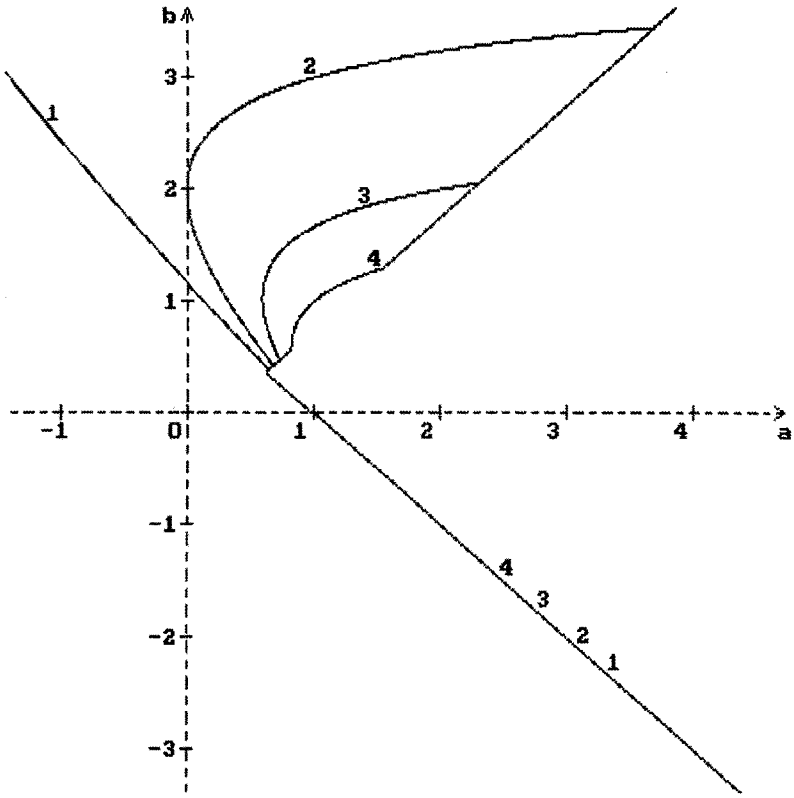

Figure 10.

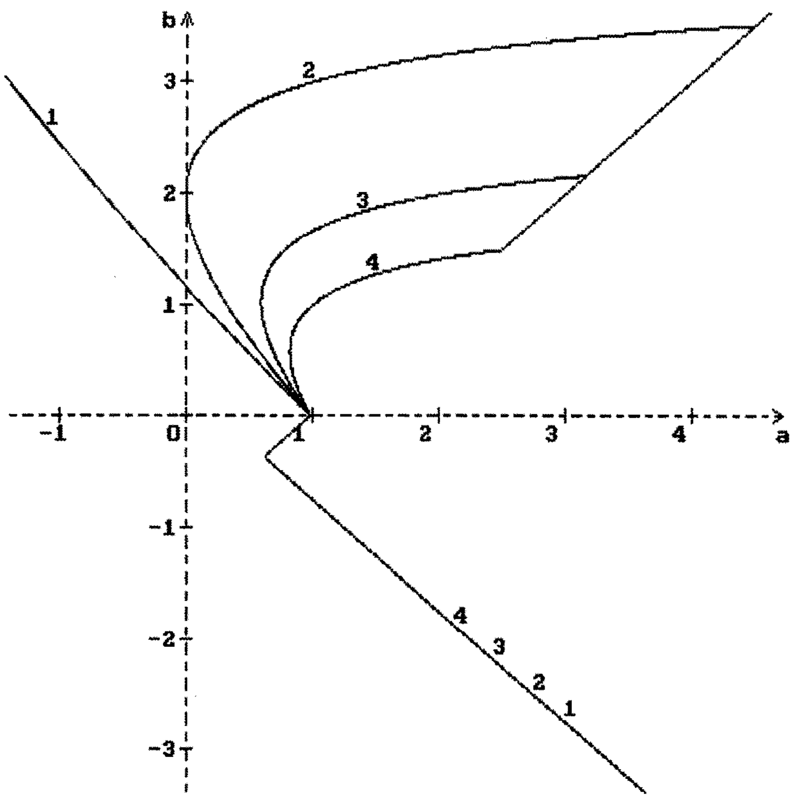

Figure 11.

The stability regions for equation (7.5), given by stability conditions (7.15) and (7.17) together, are shown in Figure 10 for $c = -0.6$, $p = 0.4$ and different values of h:

(1) $h = 0.05$;
(2) $h = 0.1$;
(3) $h = 0.15$;
(4) $h = 0.2$.

In Figure 11, the stability regions are shown for $c = 0.6$ and the same values of other parameters.

Other examples of stability conditions for stochastic differential equations are in [5–9,14,22].

8. CONCLUSIONS

Besides problems described above, many other stability problems, which were solved by general method of Lyapunov functionals construction, are considered in [5–52]. For instance, some peculiarities of this method are considered in [39,52], stability in probability for nonlinear differential and difference equations is considered in [17,18,40,49], stability of systems with Markov switching is considered in [22,25,46,47], investigation of numerical approximations of nonlinear integro-differential equations is considered in [23,44,45], stability of hereditary systems with varying and distributed delays is considered in [24,33], a comparison of delay-dependent stability criteria for stochastic delay differential equations, which were obtained here, with similar results, obtained by other methods, is considered in [46,47], applications for medical, ecological, and mechanical problems are considered in [29,36,41,42,51].

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