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STABILITY IN PROBABILITY OF NONLINEAR STOCHASTIC HEREDITARY SYSTEMS

L. Shaikhet[1]

[1]Department of Mathematics, Informatics and Computering
Donetsk State Academy of Management, Tshelyskintsev ul.
163-a, Donetsk 340015, Ukraine

ABSTRACT: It is shown that investigation of stability in probability of nonlinear stochastic hereditary systems can be reduced to the investigation of mean square stability of linear systems. For example, the sufficient condition of stability in probability of steady state solution of the well known Volterra population equation is obtained.

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1. INTRODUCTION

Many processes in automatic regulation, physics, mechanics, biology, economy, etc. can be modelled by functional differential equations (also called hereditary equations). The study of the hereditary systems have the large literature [1-8]. In this paper we consider the nonlinear stochastic integro-differential equation

$$dx(t) = \left(\int_0^\infty dK_0(s)x(t-s) + g_0(t,x_t))\right) dt$$

$$+\sum_{i=1}^N \left(\int_0^\infty dK_i(s)x(t-s) + g_i(t,x_t)\right) d\xi_i(t),\ x_0 = \varphi \in H_0, \tag{1.1}$$

where H_0 is defined below.

Let $\{\Omega, f, P\}$ be a probability space, $\{f_t, t \geq 0\}$ be the family of σ-algebras, $f_t \subset f$, H_0 be the space of f_0-adapted functions $\varphi(s) \subset R^n$, $s \leq 0$, $\|\varphi\| = \sup_{s\leq 0} |\varphi(s)|$, $x_t = x(t+\theta)$, $\theta \leq 0$, $\xi_1(t), \dots, \xi_N(t)$ are independent f_t-adapted scalar Wiener processes, K_i, $i = 0, 1, \dots, N$, $n \times n$ matrices such that

$$\int_0^\infty |dK_i(s)| < \infty,\ \int_0^\infty s|dK_i(s)| < \infty. \tag{1.2}$$

We assume furthermore that functions $g_i(t,\varphi)$, $i = 0, 1 \dots, N$, satisfy

$$|g_i(t,\varphi)| \leq \int_0^\infty dR_i(s)|\varphi(-s)|^{\alpha_i},\ \|\varphi\| \leq \delta,\ \alpha_i > 1, \tag{1.3}$$

δ is sufficiently small,

$$dR_i(s) \geq 0,\ \int_0^\infty dR_i(s) < \infty,\ \int_0^\infty s dR_i(s) < \infty. \tag{1.4}$$

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The trivial solution of Equation (1.1) will be called stable in probability if for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$ there exists $\delta > 0$ such that solution $x(t) = x(t,\varphi)$ of Equation (1.1) satisfies

$$P\{\sup_{t\geq 0} |x(t,\varphi)| > \epsilon_1\} < \epsilon_2$$

for any initial function $\varphi \in H_0$ satisfying $P\{\|\varphi\| \leq \delta\} = 1$.

Generating operator L of Equation (1.1) is defined by the formula

$$LV(t,\varphi) = \overline{\lim_{\Delta\to 0}} \frac{1}{\Delta}[V(t+\Delta, y_{t+\Delta}) - V(t,\varphi)].$$

Here $y(s)$ is the solution of Equation (1.1) for $s \geq t$ with initial function $y_t = y(t+\theta) = \varphi(\theta)$, $\theta \leq 0$.

Now we will describe a class of functionals for which this operator can be calculated. We reduce the arbitrary functional $V(t,\varphi)$, $t \geq 0$, $\varphi \in H_0$, to the form $V(t,\varphi) = V(t,\varphi(0),\varphi(s))$, $s < 0$, and define

$$V_\varphi(t,x) = V(t,\varphi) = V(t,x_t) = V(t,x,x(t+s)), s < 0, \varphi = x_t, x = \varphi(0) = x(t).$$

Let D be the class of functionals $V(t,\varphi)$ for which function $V_\varphi(t,x)$ has two continuous derivatives with respect to x and one bounded derivative with respect to t for almost all $t \geq 0$. For the functionals from D generating operator L of Equation (1.1) is defined and is given by

$$LV(t,x_t) = L_0V(t,x_t) + g_0'(t,x_t)\frac{\partial V_\varphi(t,x)}{\partial x}$$

$$+\sum_{i=1}^{N} g_i'(t,x_i)\frac{\partial^2 V_\varphi(t,x)}{\partial x^2}\int_0^\infty dK_i(s)x(t-s)$$

$$+\frac{1}{2}\sum_{i=1}^{N} g_i'(t,x_i)\frac{\partial^2 V_\varphi(t,x)}{\partial x^2} g_i(t,x_t). \tag{1.5}$$

Here

$$L_0V(t,x_t) = \frac{\partial V_\varphi(t,x)}{\partial t} + \left(\int_0^\infty dK_0(s)x(t-s)\right)'\frac{\partial V_\varphi(t,x)}{\partial x}$$

$$+\frac{1}{2}\sum_{i=1}^{N}\left(\int_0^\infty dK_i(s)x(t-s)\right)'\frac{\partial^2 V_\varphi(t,x)}{\partial x^2}\int_0^\infty dK_i(s)x(t-s). \tag{1.6}$$

Together with nonlinear Equation (1.1) we shall consider its "linear part", i.e., linear equation

$$dx(t) = \int_0^\infty dK_0(s)x(t-s)dt + \sum_{i=1}^{N}\int_0^\infty dK_i(s)x(t-s)d\xi_i(t). \tag{1.7}$$

Generating operator L_0 of Equation (1.7) has a form similar to that in (1.6).

2. STABILITY IN PROBABILITY

Theorem 2.1: *Let there exist the functional* $V(t,\varphi) \in D$ *such that*

$$k_1|\varphi(0)|^2 \leq V(t,\varphi) \leq k_2|\varphi(s)|^2, \ k_i > 0 \tag{2.1}$$

and $LV(t,\varphi) \leq 0$ *for any function* $\varphi \in H_0$ *such that* $P\{\|\varphi\| \leq \delta\} = 1$, $\delta > 0$ *be sufficiently small. Then the trivial solution of Equation* (1.1) *is stable in probability.*

The proof of this theorem can be found in [2, 3].

Theorem 2.2: *Let there exist the functional* $V_0(t,\varphi) \in D$ *for which the inequalities of the type* (2.1) *hold and be such that*

$$L_0V_0(t,\varphi) \leq -c_0|\varphi(0)|^2,$$

$$\left|\frac{\partial V_\varphi^0(t,x)}{\partial x}\right| \leq c_1|x| + \int_0^\infty dP(\tau)\int_{-\tau}^0 |\varphi(s)|ds, \ \left|\frac{\partial^2 V_\varphi^0(t,x)}{\partial x^2}\right| \leq c_2, \tag{2.2}$$

$$c_i > 0, \ i = 0,1,2, \ \int_0^\infty \tau^k dP(\tau) < \infty, \ k = 1,2.$$

Then the trivial solution of Equation (1.1) *is stable in probability.*

Proof: We will construct now the functional V which satisfies the conditions of Theorem 2.1. The functional V will be constructed in the form $V = V_0 + V_1$, where V_0 is the functional which satisfies the conditions (2.2). Using (1.5) we obtain

$$\begin{aligned} LV = LV_0 + LV_1 = L_0V_0 + LV_1 + g_0'(t,x_t)\frac{\partial V_\varphi^0(t,x)}{\partial x} \\ + \sum_{i=1}^N g_i'(t,x_t)\frac{\partial^2 V_\varphi^0(t,x)}{\partial x^2}\int_0^\infty dK_i(s)x(t-s) \\ + \frac{1}{2}\sum_{i=1}^N g_i'(t,x_t)\frac{\partial^2 V_\varphi^0(t,x)}{\partial x^2} g_i(t,x_t). \end{aligned} \tag{2.3}$$

By virtue of inequalities (1.3), (2.2) we estimate the terms from (2.3). We get

$$\begin{aligned} \left|g_0'(t,x_t)\frac{\partial V_\varphi^0(t,x)}{\partial x}\right| &\leq |g_0(t,x_t)|\left(c_1|x(t)| + \int_0^\infty dP(\tau)\int_{t-\tau}^t |x(s)|ds\right) \\ &\leq c_1|g_0(t,x_t)||x(t)| + \int_0^\infty dP(\tau)\int_{t-\tau}^t |g_0(t,x_t)||x(s)|ds) \\ &\leq \left(c_1 dR_0(0) + \frac{c_1}{2}\int_{+0}^\infty dR_0(s) + \frac{1}{2}dR_0(0)\int_0^\infty \tau dP(\tau)\right)\delta^{\alpha_0-1}|x(t)|^2 \end{aligned}$$

$$+\frac{1}{2}\delta^{\alpha_0-1}\left(c_1+\int_0^\infty \tau dP(\tau)\right)\int_{+0}^\infty dR_0(s)|x(t-s)|^2$$

$$+\frac{1}{2}\delta^{\alpha_0-1}\left(\int_0^\infty dR_0(\theta)\int_0^\infty dP(\tau)\int_{t-\tau}^t |x(s)|^2 ds\right). \tag{2.4}$$

Similarly we can obtain

$$\sum_{i=1}^N \left| g_i'(t,x_t)\frac{\partial^2 V_\varphi^0(t,x)}{\partial x^2}\int_0^\infty dK_i(s)x(t-s)\right|$$

$$\leq c_2\sum_{i=1}^N \int_0^\infty dR_i(s)|x(t-s)|^{\alpha_i}\int_0^\infty dK_i(\tau)|x(t-\tau)|$$

$$\leq \frac{c_2}{2}\sum_{i=1}^N \delta^{\alpha_i-1}dR_0(0)\int_0^\infty |dK_i(\tau)||x(t)|^2$$

$$+\frac{c_2}{2}\sum_{i=1}^N \delta^{\alpha_i-1}\int_0^\infty dR_i(s)\int_0^\infty |dK_i(\theta)||x(t-\theta)|^2$$

$$+\frac{c_2}{2}\sum_{i=1}^N \delta^{\alpha_i-1}\int_0^\infty dK_i(\theta)\int_{+0}^\infty |dR_i(s)||x(t-s)|^2 \tag{2.5}$$

and

$$\sum_{i=1}^N \left| g_i'(t,x_t)\frac{\partial^2 V_\varphi^0(t,x)}{\partial x^2}g_i(t,x_t)\right| \leq c_2\sum_{i=1}^N |g_i(t,x_t)|^2$$

$$\leq c_2\sum_{i=1}^N \delta^{2(\alpha_i-1)}\int_0^\infty dR_i(s)dR_i(0)|x(t)|^2$$

$$+c_2\sum_{i=1}^N \delta^{2(\alpha_i-1)}\int_0^\infty dR_i(s)\int_{+0}^\infty dR_i(\tau)|x(t-\tau)|^2. \tag{2.6}$$

We define the functional V_1 as follows

$$V_1(t,x_t)=\frac{1}{2}\delta^{\alpha_0-1}\left(c_1+\int_0^\infty \tau dP(\tau)\right)\int_{+0}^\infty dR_0(\theta)\int_{t-\theta}^t |x(s)|^2 ds$$

$$+\frac{1}{2}\delta^{\alpha_0-1}\left(\int_0^\infty dR_0(\theta)\int_0^\infty dP(\tau)\int_{t-\tau}^t (s+\tau-t)|x(s)|^2 ds\right)$$

$$+\frac{c_2}{2}\sum_{i=1}^N \delta^{\alpha_i=1}\left(\int_0^\infty dR_I(s)\int_0^\infty |dK_i(\theta)|\int_{t-\theta}^t |x(s)|^2 ds\right)$$

$$+\frac{c_2}{2}\sum_{i=1}^N \delta^{\alpha_i-1}\left(\int_0^\infty |dK_i(\theta)|+(\delta^{\alpha_i-1})\int_0^\infty dR_i(\theta)\right)\int_{+0}^\infty dR_i(\tau)\int_{t-\tau}^t |x(s)|^2 ds. \tag{2.7}$$

It is easy to see that $0 \leq V_1(t,\varphi) \leq c\|\varphi\|^2$, $c > 0$. Therefore the functional V satisfies the conditions (2.1). Now we will show that $LV \leq 0$. Note that $V_\varphi^1(t,x) = V_1(t,x_t) = V_\varphi^1(t)$. It follows that $LV_1 = \frac{d}{dt}V_\varphi^1(t)$. As a result of (2.3)-(2.7) we get

$$LV \leq -|x(t)|^2 \left[c_0 - \delta^{\alpha_0 - 1}\left(c_1 + \int_0^\infty \tau dP(\tau)\right)\int_0^\infty dR_0(s) \right.$$

$$\left. -c_2 \sum_{i=1}^N \delta^{\alpha_i - 1}\int_0^\infty dR_i(s)\int_0^\infty |dK_i(\theta)| - \frac{c_2}{2}\sum_{i=1}^N \left(\delta^{\alpha_i - 1}\int_0^\infty dR_i(\tau)\right)^2 \right].$$

For sufficiently small δ the term in square brackets is positive. Therefore $LV \leq 0$ and the functional $V = V_0 + V_1$ satisfies all the conditions of Theorem 2.1. It follows that the trivial solution of Equation (1.1) is stable in probability. The proof is therefore complete.

Remark 2.1: The asymptotical mean square stability of trivial solution of Equation (1.7) follows (see [2, 3]) from existence of the functional V_0, which satisfies the conditions of Theorem 2.2. Therefore in order to obtain sufficient conditions of stability in probability of trivial solution of nonlinear Equation (1.1) it is sufficient to obtain by virtue of some Lyapunov functional sufficient conditions of asymptotical mean square stability of trivial solution of the "linear part" of Equation (1.1), i.e., of Equation (1.7). For example, it is easy to show that functional

$$V_0(t,x_t) = |x(t)|^2 + \nu\left|x(t) + \int_{+0}^\infty dK_0(\tau)\int_{t-\tau}^t x(s)ds\right|^2$$

$$+\nu\int_{+0}^\infty \left|\left(\int_0^\infty dK_0(\theta)\right)' dK_0(\tau)\right| \int_{t-\tau}^t (s+\tau-t)|x(s)|^2 ds$$

$$+\int_{+0}^\infty |dK_0(\tau)|\int_{t-\tau}^t |x(s)|^2 ds + (\nu+1)\sum_{i=1}^N \int_0^\infty |dK_i(\theta)|\int_0^\infty |dK_i(\tau)|\int_{t-\tau}^t |x(s)|^2 ds,$$

$\nu \geq 0$, satisfies the conditions of Theorem 2.2 if the matrix

$$Q = \int_0^\infty dK_0(s) + \inf_{\nu\geq 0}\frac{1}{\nu+1}\left[\int_{+0}^\infty |dK_0(s)| - \int_{+0}^\infty dK_0(s)\right.$$

$$\left. +\nu\int_{+0}^\infty \tau\left|\left(\int_0^\infty dK_0(s)\right)' dK_0(\tau)\right|\right] + \frac{1}{2}\sum_{i=1}^N\left(\int_0^\infty |dK_i(s)|\right)^2 \tag{2.8}$$

is negative definite; i.e., $x'Qx \leq -c|x|^2$, $c > 0$. Here I is the identity matrix.

Theorem 2.3: *Let matrix* (2.8) *be negative definite. Then the trivial solution of Equation* (1.1) *is stable in probability.*

3. EXAMPLE

Consider the well known Volterra population equation [8]

$$\dot{x}(t) = rx(t)\left[1 - \frac{1}{K}\int_0^\infty x(t-s)dH(s)\right], \tag{3.1}$$

$$r > 0,\ K > 0,\ dH(s) \geq 0,\ \int_0^\infty dH(s) = 1,\ \int_0^\infty s dH(s) < \infty.$$

We assume that the parameter r is susceptible to stochastic perturbations of the "white noise" type $\dot{\xi}(t)$. Then the Equation (3.1) is transformed into the stochastic integro-differential equation

$$\dot{x}(t) = r(1 + \sigma\dot{\xi}(t))x(t)[1 - \frac{1}{K}\int_0^\infty x(t-s)dH(s)]. \tag{3.2}$$

Let us linearize Equation (3.2) in the neighborhood of the steady state point $x(t) = K$. We get

$$\dot{x}(t) = -r\int_0^\infty x(t-s)dH(s) - r\sigma\int_0^\infty x(t-s)dH(s)\dot{\xi}(t). \tag{3.3}$$

We get from Remark 2.1 and Theorem 2.3 that inequality

$$\min\left[2\int_{+0}^\infty dH(s), r\int_0^\infty s dH(s)\right] + \frac{r\sigma^2}{2} \leq 1$$

is sufficient for asymptotical mean square stability of trivial solution of Equation (3.3) and for stability in probability of the steady state solution $x(t) = K$ of Equation (3.2).

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