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STABILITY OF EQUILIBRIUM STATES FOR A STOCHASTICALLY PERTURBED MOSQUITO POPULATION EQUATION

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Abstract. In the paper it is shown how the known results of stability theory can be simply applied to stability investigation of some nonlinear mathematical models with stochastic perturbations. The known discrete delay Mosquito population equation with an exponential nonlinearity is considered. It is assumed that this model is exposed to stochastic perturbations which are directly proportional to the deviation of a system state from an equilibrium point. The necessary and sufficient conditions for asymptotic mean square stability of two (zero and positive) equilibrium points of a linear approximation of the considered stochastic difference equations are obtained. These conditions at the same time are sufficient conditions for stability in probability of equilibrium points of the initial nonlinear equation. Numerical calculations and figures illustrate the obtained results. The proposed investigation procedure can be applied for arbitrary nonlinear equations with an order of nonlinearity higher than one.

Keywords. Mosquito population equation, equilibrium points, stochastic perturbations, stability in probability, asymptotic mean square stability

1 Introduction

Consider the known discrete delay Mosquito population equation [2, 3] in the form

$$\begin{aligned} x_{n+1} &= (ax_n + bx_{n-1})e^{-\nu x_n}, \\ x_0, x_{-1} &> 0, \qquad n = 0, 1, 2, \ldots . \end{aligned} \tag{1.1}$$

Usually (see, for instance, [2]) in (1.1) it is assumed that $a \in (0, 1)$, $b \in (0, \infty)$, $\nu = 1$. But we will suppose that $\nu > 0$ and the parameters a, b have arbitrary values.

Putting in the equation (1.1) $x_n = x^*$, we obtain that the equilibrium points of the equation (1.1) are defined by the algebraic equation

$$x^* = (ax^* + bx^*)e^{-\nu x^*}. \tag{1.2}$$

It is easy to see that the equation (1.2) has two solutions:

$$x_1^* = 0 \tag{1.3}$$

and

$$x_2^* = \frac{1}{\nu}\ln(a+b), \qquad a+b>0. \tag{1.4}$$

Below we will assume that the equation (1.1) is influenced by stochastic perturbations that are directly proportional to the deviation of the system state x_n from the equilibrium point x^*. Note that stochastic perturbations of such type were first proposed in [1] and successfully used later by other researchers for different mathematical models with continuous and discrete time (see, for example, [4, 5] and references therein).

By this assumption some sufficient conditions for stability in probability of the equilibrium points (1.3) and (1.4) are obtained. The obtained results are illustrated by numerical calculations and figures of stability regions and trajectories of the considered equations.

2 Stochastic perturbations and some auxiliary equations and definitions

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a basic probability space, $\mathfrak{F}_n \in \mathfrak{F}$, $n \in Z = \{0, 1, ...\}$, be a family of σ-algebras, $\mathbf{E}$ be an expectation, ξ_n, $n \in Z$, be a sequence of $\mathfrak{F}_n$-adapted random variables such that $\mathbf{E}\xi_n = 0$, $\mathbf{E}\xi_n^2 = 1$. Let a process x_n be a solution of the equation

$$\begin{gathered} x_{n+1} = (ax_n + bx_{n-1})e^{-\nu x_n} + \sigma(x_n - x^*)\xi_{n+1}, \\ n = 0, 1, 2, \dots , \end{gathered} \tag{2.1}$$

with an $\mathfrak{F}_0$-adapted initial function

$$x_i = \varphi_i, \qquad i \in Z_0 = \{-1, 0\}. \tag{2.2}$$

Here σ is an arbitrary constant and x^* is an equilibrium point ((1.3) or (1.4)) of the equation (1.1). Note that the equilibrium point x^* is a solution of the equation (2.1) too.

Putting in (2.1) $x_n = x^* + y_n$, via (1.2) we obtain

$$\begin{aligned} x^* + y_{n+1} =& (a(x^* + y_n) + b(x^* + y_{n-1}))e^{-\nu(x^*+y_n)} + \sigma y_n \xi_{n+1} \\ =& [(ay_n + by_{n-1})e^{-\nu x^*} + x^*]e^{-\nu y_n} + \sigma y_n \xi_{n+1} \end{aligned}$$

or

$$y_{n+1} = (ay_n + by_{n-1})e^{-\nu x^*}e^{-\nu y_n} + x^*(e^{-\nu y_n} - 1) + \sigma y_n \xi_{n+1}. \tag{2.3}$$

It is easy to see that stability of the zero solution of the equation (2.3) is equivalent to stability of the solution x^* of the equation (2.1).

Note that the equation (2.3) is a nonlinear equation with an order of nonlinearity higher than one. It is known [4, p.150] that in this case a sufficient condition for asymptotic mean square stability of the zero solution of the linear approximation

$$z_{n+1} = (ae^{-\nu x^*} - \nu x^*)z_n + bz_{n-1}e^{-\nu x^*} + \sigma z_n \xi_{n+1} \tag{2.4}$$

of the nonlinear equation (2.3) at the same time is a sufficient condition for stability in probability of the zero solution of the nonlinear equation (2.3).

It is easy to see that for the zero equilibrium point (1.3) the equations (2.3), (2.4) are respectively

$$y_{n+1} = (ay_n + by_{n-1})e^{-\nu y_n} + \sigma y_n \xi_{n+1} \tag{2.5}$$

and

$$z_{n+1} = az_n + bz_{n-1} + \sigma z_n \xi_{n+1}. \tag{2.6}$$

Similarly, for the equilibrium point (1.4) the equations (2.3), (2.4) are respectively

$$\begin{aligned} y_{n+1} = &\left(\frac{a}{a+b}y_n + \frac{b}{a+b}y_{n-1}\right)e^{-\nu y_n} \\ &+ \frac{1}{\nu}\ln(a+b)(e^{-\nu y_n} - 1) + \sigma y_n \xi_{n+1} \end{aligned} \tag{2.7}$$

and

$$z_{n+1} = \left(\frac{a}{a+b} - \ln(a+b)\right)z_n + \frac{b}{a+b}z_{n-1} + \sigma z_n \xi_{n+1}. \tag{2.8}$$

Definition 2.1. The zero solution of the equation (2.7) (or (2.5)) is called stable in probability if for any $\varepsilon > 0$ and $\varepsilon_1 > 0$ there exists a $\delta > 0$ such that the solution $y_i = y_i(\varphi)$ of the equation (2.7) (or (2.5)) satisfies the inequality $\mathbf{P}\{\sup_{i \in Z} |y_i| > \varepsilon\} < \varepsilon_1$ for any initial function φ_i such that $\mathbf{P}\{\sup_{i \in Z_0} |\varphi_i| < \delta\} = 1$.

Definition 2.2. The zero solution of the equation (2.8) (or (2.6)) is called mean square stable if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|z_i|^2 < \varepsilon$, $i \in Z$, for any initial function φ_i such that $\sup_{i \in Z_0} \mathbf{E}|\varphi_i|^2 < \delta$; asymptotically mean square stable if it is mean square stable and for each initial function φ_i such that $\sup_{i \in Z_0} \mathbf{E}|\varphi_i|^2 < \infty$ the solution z_i of the equation (2.8) (or (2.6)) satisfies the condition $\lim_{i \to \infty} \mathbf{E}|z_i|^2 = 0$.

3 Stability of the equilibrium points

Consider first the zero equilibrium (1.3). It is known [4, p.18] that the necessary and sufficient conditions for asymptotic mean square stability of the zero solution of the equation (2.6) are the inequalities

$$
\begin{gathered}
|b|<1, \qquad |a|<1-b, \\
\sigma^2<\frac{(1+b)[(1-b)^2-a^2]}{1-b}.
\end{gathered}
\tag{3.1}
$$

At the same time the inequalities (3.1) are [4, p.150] sufficient conditions for stability in probability of the zero solution of the equation (2.5) and therefore for stability in probability of the zero equilibrium point of the equation (2.1).

Similarly to (3.1) from (2.8) we obtain the necessary and sufficient conditions for asymptotic mean square stability of the zero solution of the equation (2.8)

$$
\begin{gathered}
\left|\frac{b}{a+b}\right|<1, \qquad \left|\frac{a}{a+b}-\ln(a+b)\right|<1-\frac{b}{a+b}, \\
\sigma^2<\frac{\left(1+\frac{b}{a+b}\right)\left(\left(1-\frac{b}{a+b}\right)^2-\left(\frac{a}{a+b}-\ln(a+b)\right)^2\right)}{1-\frac{b}{a+b}}.
\end{gathered}
\tag{3.2}
$$

Note that the conditions (3.2) can be represented in the form

$$
\begin{gathered}
a>0, \quad a+2b>0, \quad 1<a+b<\exp\left(\frac{2a}{a+b}\right), \\
\sigma^2<\left(1+2\frac{b}{a}\right)\left(\frac{2a}{a+b}-\ln(a+b)\right)\ln(a+b).
\end{gathered}
\tag{3.3}
$$

So, the conditions (3.3) are sufficient conditions for stability in probability of the zero solution of the equation (2.7) and therefore of the equilibrium point (1.4) of the equation (2.1).

Remark 3.1. Note that stability conditions (3.3) for the equilibrium point (1.4) do not depend on the parameter ν in spite of the equilibrium point (1.4) depends on ν. In reality it is not a big surprise because putting $x'_i=\nu x_i$ we transform the equation (1.1) to the equation $x'_{n+1}=(\alpha x'_n+\beta x'_{n-1})e^{-x'_n}$ that does not depend on ν. The equilibrium point (1.4) by this transformation equals $\ln(a+b)$, i.e., does not depend on ν too.

Remark 3.2. Via the conditions (3.3) we have $a+b>1$. So, by the conditions (3.3) from (1.4) it follows that $x_2^*>0$ and we have the zero equilibrium point x_1^* and the positive equilibrium point x_2^*.

4 Numerical simulation

To construct stability regions in the space of the parameters (a, b) let us transform the conditions (3.1) and (3.3) to more convenient for numerical calculations form.

From (3.1) it follows that the bound of a stability region is defined by the formulas

$$\begin{aligned} a &= \pm\sqrt{\frac{1-b}{1+b}(1-\sigma^2-b^2)}, \\ -\sqrt{1-\sigma^2} &< b < \sqrt{1-\sigma^2}. \end{aligned} \tag{4.1}$$

Via (3.3) we obtain that in the case $\sigma = 0$ the part of a stability region bound is defined as follows

$$b = \begin{cases} 1-a, & a \in (0,2), \\ -\frac{1}{2}a, & a \geq 2. \end{cases} \tag{4.2}$$

Putting in (3.3) $b = \mu a$, via $a > 0$ and $a + 2b > 0$ we obtain another part of a stability region bound in the parametrical form

$$a = \frac{1}{1+\mu}\exp\left(\frac{2}{1+\mu}\right), \quad b = \frac{\mu}{1+\mu}\exp\left(\frac{2}{1+\mu}\right), \quad \mu > -\frac{1}{2}. \tag{4.3}$$

Putting in (3.3) $b = \mu a$ in the case $\sigma^2 > 0$, we have

$$\sigma^2 < (1+2\mu)\left(\frac{2}{1+\mu} - \ln[a(1+\mu)]\right)\ln[a(1+\mu)]. \tag{4.4}$$

Solving this inequality with respect to a, we obtain the parametrical equation of another part of a stability region bound

$$\begin{aligned} a =& \frac{1}{1+\mu}\exp\left(\frac{1}{1+\mu} \pm \sqrt{\frac{1}{(1+\mu)^2} - \frac{\sigma^2}{1+2\mu}}\right), \\ b =& \mu a, \qquad \mu > \frac{1-\sigma^2-\sqrt{1-\sigma^2}}{\sigma^2}. \end{aligned} \tag{4.5}$$

In Fig.4.1 the bounds of stability regions for the zero equilibrium point (1.3) (triangle ABC) and the equilibrium point (1.4) (BCD) are shown in the deterministic case ($\sigma = 0$). These regions are constructed by virtue of the conditions (4.1) and (4.2), (4.3) respectively. Here the points A, B, C, D have the following coordinates: $A(-2,-1)$, $B(0,1)$, $C(2,-1)$, $D(2e^4, -e^4)$.

In Fig.4.2 together with the stability regions from Fig.4.1 in another scale the similar stability regions are shown for $\sigma = 0.85$: the bound 1 for the zero equilibrium point (1.3) and the bound 2 for the positive equilibrium point (1.4).

Below we will suppose that $\nu = 1$, $\sigma = 0.85$ and for simulation of solutions of the equations (2.5)-(2.8) we will simulate ξ_n as an uniformly distributed on $(-\sqrt{3}, \sqrt{3})$ random variable. So, $\mathbf{E}\xi_n = 0$, $\mathbf{E}\xi_n^2 = 1$.

First consider the equations (2.6), (2.1) in the point $K(0.2, -0.2)$ (see Fig.4.2), i.e. with $a = 0.2$, $b = -0.2$. This point belongs to stability region of the zero equilibrium point, i.e. $x^* = 0$. So, the zero solution of the equation (2.6) is asymptotically mean square stable and the zero solution of the equation (2.1) is stable in probability. In Fig.4.3 one can see 100 trajectories of a solution of the equation (2.6) with the initial function $z_{-1} = -4$, $z_0 = 2$. All trajectories go to zero. In Fig.4.4 300 trajectories of a solution of the equation (2.1) with the initial function $x_{-1} = -0.4$, $x_0 = 0.3$ are shown. All trajectories go to zero.

Consider now the equations (2.8), (2.1) in the point $L(2, 0.5)$ (see Fig.4.2), i.e. with $a = 2$, $b = 0.5$. This point belongs to stability region of the equilibrium point $x^* = 0.9163$. So, the zero solution of the equation (2.8) is asymptotically mean square stable and the solution x^* of the equation (2.1) is stable in probability. In Fig.4.5 one can see 100 trajectories of a solution of the equation (2.8) with the initial function $z_{-1} = x^* - 3.5$, $z_0 = x^* + 1$. All trajectories go to zero. In Fig.4.6 300 trajectories of a solution of the equation (2.1) with the initial function $x_{-1} = x^* + 0.4$, $x_0 = x^* - 0.3$ are shown. In spite of some trajectories go to $-\infty$ almost all trajectories go to the equilibrium point $x^* = 0.9163$.

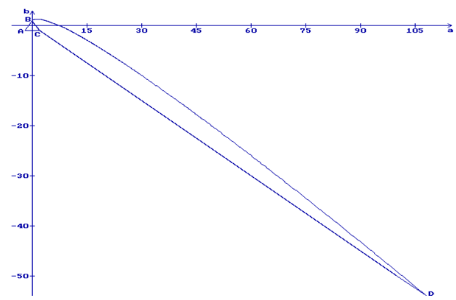

Figure 4.1: Regions of stability in probability of the equilibrium points (1.3) (triangle ABC) and (1.4) (BCD) for $\sigma = 0$.

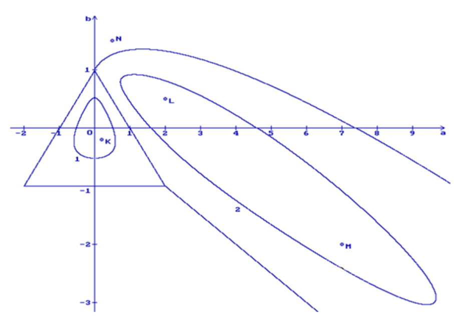

Figure 4.2: Regions of stability in probability of the equilibrium points (1.3) and (1.4) for $\sigma = 0$ and $\sigma = 0.85$.

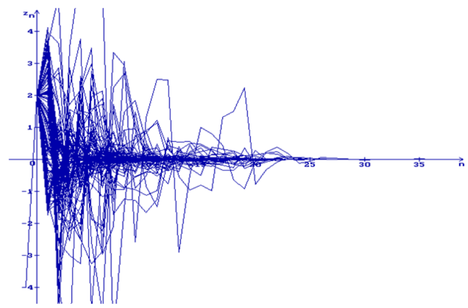

Figure 4.3: 100 trajectories of the equation (2.6) in the point K with $a = 0.2$, $b = -0.2$, $\sigma = 0.85$, $x^* = 0$, $x_{-1} = -4$, and $x_0 = 2$.

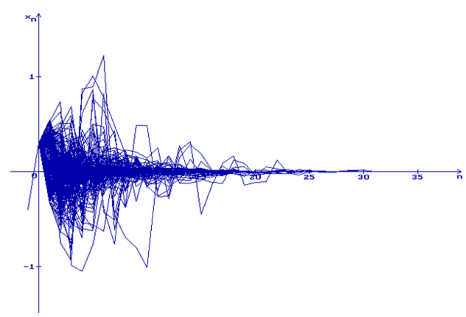

Figure 4.4: 300 trajectories of the equation (2.1) in the point K with $a = 0.2$, $b = -0.2$, $\sigma = 0.85$, $x^* = 0$, $x_{-1} = -0.4$, and $x_0 = 0.3$.

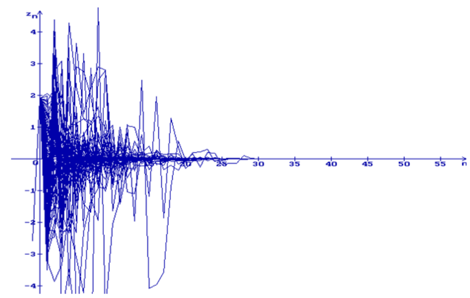

Figure 4.5: 100 trajectories of the equation (2.8) in the point L with $a = 2$, $b = 0.5$, $\sigma = 0.85$, $x^* = 0.9163$, $x_{-1} = x^* - 3.5$, and $x_0 = x^* + 1$.

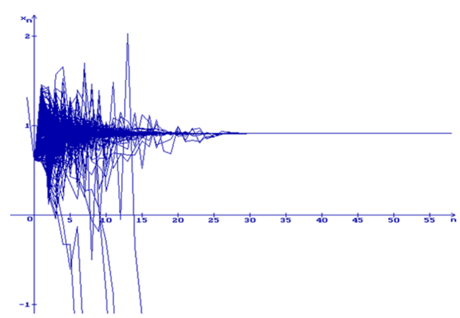

Figure 4.6: 300 trajectories of the equation (2.1) in the point L with $a = 2$, $b = 0.5$, $\sigma = 0.85$, $x^* = 0.9163$, $x_{-1} = X^* + 0.4$, and $x_0 = X^* + 0.3$.

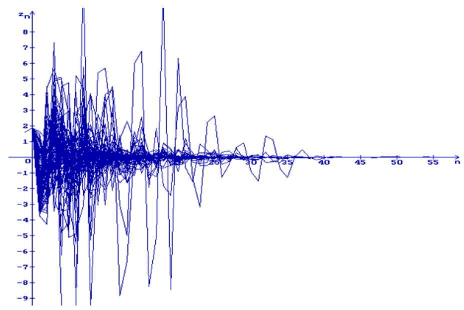

Figure 4.7: 100 trajectories of the equation (2.8) in the point M with $a = 7$, $b = -2$, $\sigma = 0.85$, $x^* = 1.6094$, $x_{-1} = x^*$, and $x_0 = x^* + 0.3$.

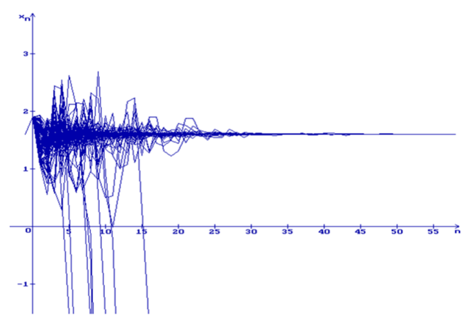

Figure 4.8: 100 trajectories of the equation (2.1) in the point M with $a = 7$, $b = -2$, $\sigma = 0.85$, $x^* = 1.6094$, $x_{-1} = X^*$, and $x_0 = X^* + 0.3$.

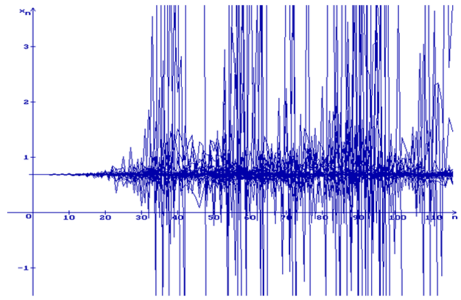

Figure 4.9: 100 trajectories of the equation (2.8) in the point N with $a = 0.5$, $b = 1.5$, $\sigma = 0.85$, $x^* = 0.6931$, $x_{-1} = 0$, and $x_0 = 0.001$.

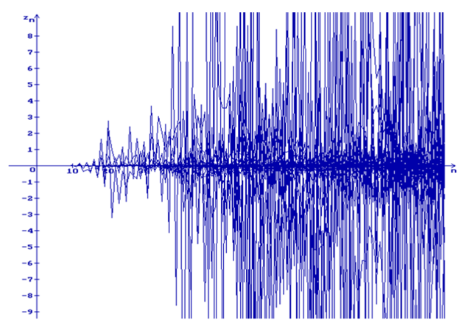

Figure 4.10: 100 trajectories of the equation (2.8) in the point N with $a = 0.5$, $b = 1.5$, $\sigma = 0.85$, $x^* = 0.6931$, $x_{-1} = X^* - 0.001$, and $x_0 = X^*$.

Similar pictures for the equations (2.8), (2.1) in the point $M(7,-2)$ (see Fig.4.2) one can see in Fig.4.7 and Fig.4.8 for $a=7$, $b=-2$ and $x^*=1.6094$, $x_{-1}=x^*$, $x_0=x^*+0.3$.

In conclusion consider a behavior of the solutions (2.8), (2.1) in the point $N(0.5,1.5)$ (see Fig.4.2), i.e. with $a=0.5$, $b=1.5$, which does not belong to stability region of the equilibrium point $x^*=0.6931$. In Fig.4.9 one can see 100 trajectories of a solution of the equation (2.8) with the initial function $z_{-1}=0$, $z_0=0.001$. The equilibrium point $x^*=0.6931$ is unstable, so, in spite of the initial function is almost zero, trajectories do not go to zero and fill whole space. Similar picture is shown in Fig.4.10: 100 trajectories of a solution of the equation (2.1) with the initial function $x_{-1}=x^*-0.001$, $x_0=x^*$ do not go to the equilibrium and fill whole space, in spite of the initial function almost equals to the equilibrium point $x^*=0.6931$.

References

[1] E. Beretta, V. Kolmanovskii and L. Shaikhet, Stability of epidemic model with time delays influenced by stochastic perturbations. Mathematics and Computers in Simulation (Special Issue “Delay Systems”), **45**(3-4) (1998), 269-277.

[2] X.Ding and R.Zhang, On the difference equation $x_{n+1}=(\alpha x_n+\beta x_{n-1})e^{-x_n}$, Advances in Difference Equations, (2008), doi: 10.1155/2008/876936.

[3] E.A.Grove, C.M.Kent, G.Ladas, S.Valicenti and R.Levins, Global stability in some population models, in Communications in Difference Equations (Poznan, 1998), Gordon and Breach, Amsterdam, The Netherlands, (2000), 149-176.

[4] L.Shaikhet, Lyapunov Functionals and Stability of Stochastic Difference Equations. Springer, London, Dordrecht, Heidelberg, New York, 2011.

[5] L.Shaikhet, Lyapunov Functionals and Stability of Stochastic Functional Differential Equations, Sprin-ger, Dordrecht, Heidelberg, New York, London, 2013.

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