Dynamics of Continuous, Discrete and Impulsive Systems
Series **B:** *Applications & Algorithms* **19 (2012)** 709-729

STABILITY OF EQUILIBRIUM POINTS OF DIFFERENTIAL EQUATION WITH FRACTIONAL NONLINEARITY AND STOCHASTIC PERTURBATIONS

Leonid Shaikhet

Department of Higher Mathematics
Donetsk State University of Management
163a Chelyuskintsev Street, 83015 Donetsk, Ukraine
email:leonid.shaikhet@usa.net

Abstract. In this paper the stability of equilibrium points of the nonlinear differential equation with fractional nonlinearity is studied. It is supposed that this system is exposed to additive stochastic perturbations that are of the type of white noise and are directly proportional to the deviation of the system state from the equilibrium point. Sufficient conditions for stability in probability of equilibrium points of the considered nonlinear stochastic differential equation are obtained. Numerous color graphical illustrations of obtained stability regions and trajectories of solutions are plotted. The proposed method of stability investigation can be used for study of many other types of nonlinear systems with the order of nonlinearity higher than one.

Keywords. Differential equation, fractional nonlinearity, equilibrium point, stability, stochastic perturbations.

1 Introduction. Equilibrium points

The main contribution of this paper is a method for stability investigation of nonlinear systems with high order of nonlinearity and stochastic perturbations which is demonstrated on the nonlinear delay differential equation of the type of $\dot{x}(t) = -ax(t) + f(x(t-\tau))$. Systems of such type are enough popular in researches. See, for example, [5], [7], [11], the famous Nicholson blowflies equation [4], [9], [12] $\dot{x}(t) = -ax(t) + bx(t-\tau)e^{-\gamma x(t-\tau)}$, the Mackey-Glass model [8]

$$\dot{x}(t) = -ax(t) + \frac{bx(t-\tau)}{1+x^n(t-\tau)}.$$

On the other hand there is a very large interest in study of the behavior of solutions of nonlinear difference equations with the fractional nonlinearity of the type of

$$x_{n+1} = \frac{\mu + \sum_{j=0}^{k} a_j x_{n-j}}{\lambda + \sum_{j=0}^{k} b_j x_{n-j}}, \qquad n = 0, 1, ...,$$

(see [12] and a long list of the references therein).

Here similarly to [12] stability of equilibrium points of the nonlinear differential equation with fractional nonlinearity

$$\dot{x}(t) = -ax(t) + \frac{\mu + \sum_{j=0}^{k} a_j x(t-\tau_j)}{\lambda + \sum_{j=0}^{k} b_j x(t-\tau_j)}, \qquad t > 0, \tag{1}$$

and the initial condition

$$x(s) = \phi(s), \qquad s \in [-\tau, 0], \qquad \tau = \max\{\tau_1, .., \tau_k\}, \tag{2}$$

is investigated. Here μ, λ, a_j, b_j, $j = 0, ..., k$, $\tau_0 = 0$, $\tau_j > 0$, $j > 0$, are known constants.

Put

$$A = \sum_{j=0}^{k} a_j, \qquad B = \sum_{j=0}^{k} b_j, \tag{3}$$

and suppose that the equation (1) has some point of equilibrium $\hat{x}$ (not necessary a positive one) defined by the condition $\dot{x}(t) = 0$. So, via (1), (3) and the assumption

$$\lambda + B\hat{x} \neq 0 \tag{4}$$

the equilibrium point $\hat{x}$ is defined by the algebraic equation

$$a\hat{x} = \frac{\mu + A\hat{x}}{\lambda + B\hat{x}}. \tag{5}$$

If $aB \neq 0$ then by the condition (4) the equation (5) can be transformed to the form

$$aB\hat{x}^2 - (A - a\lambda)\hat{x} - \mu = 0. \tag{6}$$

Thus, if

$$(A - a\lambda)^2 + 4aB\mu > 0 \tag{7}$$

then the equation (1) has two points of equilibrium

$$\hat{x}_1 = \frac{A - a\lambda + \sqrt{(A - a\lambda)^2 + 4aB\mu}}{2aB} \tag{8}$$

and

$$\hat{x}_2 = \frac{A - a\lambda - \sqrt{(A - a\lambda)^2 + 4aB\mu}}{2aB}, \tag{9}$$

if

$$(A - a\lambda)^2 + 4aB\mu = 0 \tag{10}$$

then the equation (1) has only one point of equilibrium

$$\hat{x} = \frac{A - a\lambda}{2aB}. \tag{11}$$

At last if

$$(A - a\lambda)^2 + 4aB\mu < 0 \tag{12}$$

then the equation (1) has not equilibrium points.

Remark 1.1 Assume that $aB \neq 0$, $\mu = 0$. If $A \neq 0$ and $A \neq a\lambda$ then the equation (1) has two points of equilibrium:

$$\hat{x}_1 = \frac{A - a\lambda}{aB}, \qquad \hat{x}_2 = 0; \tag{13}$$

if $A = 0$ or $A = a\lambda$ then the equation (1) has only one point of equilibrium: $\hat{x} = 0$.

Remark 1.2 Assume that $aB = 0$. If $A \neq a\lambda$ then the equation (1) has only one equilibrium point

$$\hat{x} = -\frac{\mu}{A - a\lambda}.$$

Remark 1.3 Consider the case $\mu = B = 0$, $\lambda \neq 0$. If $A \neq a\lambda$ then the equation (1) has only one point of equilibrium: $\hat{x} = 0$, if $A = a\lambda$ then each solution $\hat{x} = const$ is an equilibrium point of the equation (1).

2 Stochastic perturbations, centering and linearization. Definitions and auxiliary statements

As it was first proposed in [2] and successfully used later in some other researches (see, for instance [1], [3], [12]), we will suppose that the equation (1) is exposed to stochastic perturbations of the type of white noise $\dot{w}(t)$ which are directly proportional to the deviation of the solution $x(t)$ of the equation (1) from the equilibrium point $\hat{x}$. Thus, (1) takes the form

$$\dot{x}(t) = -ax(t) + \frac{\mu + \sum_{j=0}^{k} a_j x(t - \tau_j)}{\lambda + \sum_{i=0}^{k} b_i x(t - \tau_i)} + \sigma(x(t) - \hat{x})\dot{w}(t). \tag{14}$$

Note that the equilibrium point $\hat{x}$ of the equation (1) is also the equilibrium point of the equation (14). Putting $x(t) = y(t) + \hat{x}$ and

$$\gamma_j = \frac{a_j - ab_j\hat{x}}{\lambda + B\hat{x}}, \qquad j = 0, ..., k, \tag{15}$$

we will center the equation (14) in the neighborhood of the point of the equilibrium $\hat{x}$. From (14), (15) it follows that $y(t)$ satisfies the equation

$$\dot{y}(t) = -ay(t) + \frac{\gamma_0 y(t) + \sum_{j=1}^{k} \gamma_j y(t-\tau_j)}{1 + \sum_{i=0}^{k} b_i(\lambda + B\hat{x})^{-1} y(t-\tau_i)} + \sigma y(t)\dot{w}(t), \tag{16}$$

It is clear that stability of the trivial solution of the equation (16) is equivalent to stability of the equilibrium point of the equation (14).

Together with the nonlinear equation (16) we will consider the linear part (in a neighborhood of the zero) of this equation

$$\dot{z}(t) = -(a - \gamma_0)z(t) + \sum_{j=1}^{k} \gamma_j z(t-\tau_j) + \sigma z(t)\dot{w}(t). \tag{17}$$

Two usual definitions for stability are used below [6].

Definition 2.1 The trivial solution of the equation (16) is called stable in probability if for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there exists $\delta > 0$ such that the solution $y(t) = y(t, \phi)$ satisfies the condition $\mathbf{P}\{\sup_{t\geq 0} |y(t,\phi)| > \varepsilon_1\} < \varepsilon_2$ for any initial function $\phi(s)$ such that $\mathbf{P}\{\sup_{s\in[-\tau,0]} |\phi(s)| \leq \delta\} = 1$.

Definition 2.2 The trivial solution of the equation (17) is called mean square stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that the solution $z(t) = z(t, \phi)$ satisfies the condition $\mathbf{E}|z(t,\phi)|^2 < \varepsilon$ for any initial function $\phi(s)$ such that $\sup_{s\in[-\tau,0]} \mathbf{E}|\phi(s)|^2 < \delta$. If besides $\lim_{t\to\infty} \mathbf{E}|z(t,\phi)|^2 = 0$ for any initial function $\phi(s)$ then the trivial solution of the equation (2.4) is called asymptotically mean square stable.

Below the following method for stability investigation is used. Since the order of nonlinearity of the equation (16) is higher than one then sufficient stability conditions for asymptotic mean square stability of the trivial solution of the constructed linear equation (17) (the linear part of (16)) at the same time are [14] sufficient conditions for stability in probability of the trivial solution of the nonlinear equation (16) and therefore for stability in probability of the equilibrium point of the equation (14). The sufficient conditions for asymptotic mean square stability of the trivial solution of the linear equation (17) were obtained via V.Kolmanovskii and L.Shaikhet general method of Lyapunov functionals construction (see [12], [13] and the references therein).

3 Stability of equilibrium points

The equation of the type of (17) is very well studied. The following lemmas consist some well known sufficient conditions for asymptotic mean square stability of the trivial solution of the equation (17).

Lemma 3.1 If

$$a > \gamma_0 + \sum_{j=1}^{k} |\gamma_j| + p, \qquad p = \frac{\sigma^2}{2}, \tag{18}$$

then the trivial solution of the equation (17), (15) is asymptotically mean square stable.

Lemma 3.2 If

$$\left(a - \sum_{j=0}^{k} \gamma_j\right)\left(1 - \sum_{j=1}^{k} |\gamma_j|\tau_j\right) > p, \qquad \sum_{j=1}^{k} |\gamma_j|\tau_j < 1, \tag{19}$$

then the trivial solution of the equation (17), (15) is asymptotically mean square stable.

The proofs of Lemmas 3.1 and 3.2 follows from [13] (the conditions (3.4) and (3.10)).

Remark 3.1 If in the equation (1) the delays are absent, i.e., $\tau_j = 0$, $j = 0, ..., k$, then the condition (19) is not worse than the condition (18) that does not depend on delays.

Suppose at first that the condition (10) holds. In this case the equation (14) has only one point of equilibrium $\hat{x}$ defined by (11) and via (15), (11)

$$\sum_{j=0}^{k} \gamma_j = \frac{A - aB\hat{x}}{\lambda + B\hat{x}} = \frac{A - \frac{1}{2}(A - a\lambda)}{\lambda + \frac{1}{2a}(A - a\lambda)} = a.$$

Thus, the stability condition (19) for the equilibrium point (11) does not hold. Moreover,

$$a = \sum_{j=0}^{k} \gamma_j \leq \gamma_0 + \sum_{j=1}^{k} |\gamma_j|.$$

Thus, the stability condition (18) for the equilibrium point (11) does not hold too.

Suppose now that the condition (7) holds. Then the equation (14) has two points of equilibrium $\hat{x}_1$ and $\hat{x}_2$ defined by (8) and (9) respectively. Put

$$S = \sqrt{(A - a\lambda)^2 + 4aB\mu}, \tag{20}$$

$$\gamma_j^{(l)} = \frac{a_j - ab_j\hat{x}_l}{\lambda + B\hat{x}_l}, \quad j = 0, ..., k, \tag{21}$$

Corollary 3.1 Assume that the condition (7) holds and $\gamma_0^{(l)} \geq 0$, $l = 1, 2$. Then for the fixed μ and λ the condition (18) cannot be true for the both points of equilibrium $\hat{x}_1$ and $\hat{x}_2$ together.

Proof: Via (18), (21), (8) for the equilibrium point $\hat{x}_1$ we obtain

$$1 > \frac{1}{a}\sum_{j=0}^{k}\left|\gamma_j^{(1)}\right| \geq \frac{1}{a}\left|\sum_{j=0}^{k}\gamma_j^{(1)}\right| = \frac{1}{a}\left|\frac{A - aB\hat{x}_1}{\lambda + B\hat{x}_1}\right|$$
$$= \left|\frac{A - \frac{1}{2}(A - a\lambda + S)}{a\lambda + \frac{1}{2}(A - a\lambda + S)}\right| = \left|\frac{A + a\lambda - S}{A + a\lambda + S}\right|.$$

Similarly for the equilibrium point $\hat{x}_2$ we have

$$1 > \frac{1}{a}\sum_{j=0}^{k}\left|\gamma_j^{(2)}\right| \geq \frac{1}{a}\left|\sum_{j=0}^{k}\gamma_j^{(2)}\right| = \frac{1}{a}\left|\frac{A - aB\hat{x}_2}{\lambda + B\hat{x}_2}\right|$$
$$= \left|\frac{A - \frac{1}{2}(A - a\lambda - S)}{a\lambda + \frac{1}{2}(A - a\lambda - S)}\right| = \left|\frac{A + a\lambda + S}{A + a\lambda - S}\right|.$$

Thus, we obtain two conflicting conditions. The proof is completed.

Corollary 3.2 Assume that the condition (7) holds and $a \neq 0$. If

$$\frac{2aS}{S + A + a\lambda}\left(1 - \sum_{j=1}^{k}|\gamma_j^{(1)}|\tau_j\right) > p, \qquad \sum_{j=1}^{k}|\gamma_j^{(1)}|\tau_j < 1, \tag{22}$$

then the equilibrium point $\hat{x} = \hat{x}_1$ (defined by (8)) of the equation (14) is stable in probability.

If

$$\frac{2aS}{S - A - a\lambda}\left(1 - \sum_{j=1}^{k}|\gamma_j^{(2)}|\tau_j\right) > p, \qquad \sum_{j=1}^{k}|\gamma_j^{(2)}|\tau_j < 1, \tag{23}$$

then the equilibrium point $\hat{x} = \hat{x}_2$ (defined by (9)) of the equation (14) is stable in probability.

Assume now that $a = 0$. If

$$\frac{A^2}{Q}\left(1 - \frac{|A|}{Q}\tau\right) > p, \qquad \tau = \sum_{j=1}^{k}|a_j|\tau_j < \frac{Q}{|A|}, \qquad Q = B\mu - A\lambda, \tag{24}$$

then the equilibrium point $\hat{x} = -\mu A^{-1}$ is stable in probability.

Proof: For $a \neq 0$ via (19) it is enough to note that for $\hat{x}_1$

$$a - \sum_{j=0}^{k}\gamma_j^{(1)} = a - \frac{A - aB\hat{x}_1}{\lambda + B\hat{x}_1} = a - \frac{A - \frac{1}{2}(A - a\lambda + S)}{\lambda + \frac{1}{2a}(A - a\lambda + S)} = \frac{2aS}{S + A + a\lambda}$$

and similarly for $\hat{x}_2$

$$a - \sum_{j=0}^{k} \gamma_j^{(2)} = a - \frac{A - aB\hat{x}_2}{\lambda + B\hat{x}_2} = a - \frac{A - \frac{1}{2}(A - a\lambda - S)}{\lambda + \frac{1}{2a}(A - a\lambda - S)} = \frac{2aS}{S - A - a\lambda}.$$

For $a = 0$ via Remark 1.2 and (15) we have

$$-\sum_{j=0}^{k} \gamma_j = -\frac{A}{\lambda - B\mu A^{-1}} = \frac{A^2}{Q}, \qquad \sum_{j=0}^{k} |\gamma_j|\tau_j = \frac{\tau}{|\lambda + B(-\mu)A^{-1}|} = \frac{|A|}{Q}\tau.$$

So, (19) implies (24). The proof is completed.

Remark 3.2 If $\tau_j = 0$, $j = 1, ..., k$, then the conditions (22), (23) take the forms

$$\frac{2aS}{S + A + a\lambda} > p, \qquad \frac{2aS}{S - A - a\lambda} > p, \tag{25}$$

respectively. Moreover, the inequalities (25) are necessary conditions for implementation of the conditions (22), (23) with arbitrary τ_j, $j = 1, ..., k$. If $a < 0$ or $p \geq 2a > 0$ then the conditions (22), (23) cannot be true for the same μ and λ. Really, if $a < 0$ then from (25) the contradiction follows: $0 < S < A + a\lambda < -S < 0$. If $p \geq 2a > 0$ then from (25) another contradiction follows: $0 \leq -(2a/p - 1)S < A + a\lambda < (2a/p - 1)S \leq 0$.

Corollary 3.3 Put

$$q = \begin{cases} \frac{p}{2a-p} & if \quad p < 2a, \\ \frac{p}{p-2a} & if \quad p > 2a, \\ +\infty & if \quad p = 2a, \end{cases} \tag{26}$$

assume that $a \neq 0$, $S > 0$, $\tau_j = 0$, $j = 1, ..., k$, and consider the following cases.

Case 1: $a > 0$, $B > 0$.

If

$$p < 2a, \qquad \mu > \begin{cases} \frac{q^2(A+a\lambda)^2 - (A-a\lambda)^2}{4aB} & for \quad \lambda \geq -\frac{A}{a}, \\ \frac{A}{B}\lambda & for \quad \lambda < -\frac{A}{a}, \end{cases} \tag{27}$$

or

$$p \geq 2a, \qquad \frac{A}{B}\lambda < \mu < \frac{q^2(A + a\lambda)^2 - (A - a\lambda)^2}{4aB}, \qquad \lambda < -\frac{A}{a}, \tag{28}$$

then the equilibrium point $\hat{x}_1$ is stable in probability.

If

$$p < 2a, \qquad \mu > \begin{cases} \frac{q^2(A+a\lambda)^2 - (A-a\lambda)^2}{4aB} & for \quad \lambda < -\frac{A}{a}, \\ \frac{A}{B}\lambda & for \quad \lambda \geq -\frac{A}{a}, \end{cases} \tag{29}$$

or

$$p \geq 2a, \qquad \frac{A}{B}\lambda < \mu < \frac{q^2(A+a\lambda)^2-(A-a\lambda)^2}{4aB}, \qquad \lambda \geq -\frac{A}{a}, \tag{30}$$

then the equilibrium point $\hat{x}_2$ is stable in probability.

Case 2: $a > 0$, $B < 0$.

If

$$p < 2a, \qquad \mu < \begin{cases} \frac{q^2(A+a\lambda)^2-(A-a\lambda)^2}{4aB} & for \quad \lambda \geq -\frac{A}{a}, \\ \frac{A}{B}\lambda & for \quad \lambda < -\frac{A}{a}, \end{cases} \tag{31}$$

or

$$p \geq 2a, \qquad \frac{A}{B}\lambda > \mu > \frac{q^2(A+a\lambda)^2-(A-a\lambda)^2}{4aB}, \qquad \lambda < -\frac{A}{a}, \tag{32}$$

then the equilibrium point $\hat{x}_1$ is stable in probability.

If

$$p < 2a, \qquad \mu < \begin{cases} \frac{q^2(A+a\lambda)^2-(A-a\lambda)^2}{4aB} & for \quad \lambda < -\frac{A}{a}, \\ \frac{A}{B}\lambda & for \quad \lambda \geq -\frac{A}{a}, \end{cases} \tag{33}$$

or

$$p \geq 2a, \qquad \frac{A}{B}\lambda > \mu > \frac{q^2(A+a\lambda)^2-(A-a\lambda)^2}{4aB}, \qquad \lambda \geq -\frac{A}{a}, \tag{34}$$

then the equilibrium point $\hat{x}_2$ is stable in probability.

Case 3: $a < 0$, $B > 0$.

If

$$\frac{A}{B}\lambda < \mu < \frac{q^2(A+a\lambda)^2-(A-a\lambda)^2}{4aB}, \qquad \lambda > -\frac{A}{a}, \tag{35}$$

then the equilibrium point $\hat{x}_1$ is stable in probability.

If

$$\frac{A}{B}\lambda < \mu < \frac{q^2(A+a\lambda)^2-(A-a\lambda)^2}{4aB}, \qquad \lambda < -\frac{A}{a}, \tag{36}$$

then the equilibrium point $\hat{x}_2$ is stable in probability.

Case 4: $a < 0$, $B < 0$.

If

$$\frac{A}{B}\lambda > \mu > \frac{q^2(A+a\lambda)^2-(A-a\lambda)^2}{4aB}, \qquad \lambda > -\frac{A}{a}, \tag{37}$$

then the equilibrium point $\hat{x}_1$ is stable in probability.

If

$$\frac{A}{B}\lambda > \mu > \frac{q^2(A+a\lambda)^2-(A-a\lambda)^2}{4aB}, \qquad \lambda < -\frac{A}{a}, \tag{38}$$

then the equilibrium point $\hat{x}_2$ is stable in probability.

Proof: It is enough to prove Case 1, the proofs of the others cases are similar.

Consider the equilibrium point $\hat{x}_1$. Assume first that $p < 2a$. If $A+a\lambda \geq 0$ then via (20) from the first line of (27) it follows that $S > q(A+a\lambda)$. Via (26) and $\tau_j = 0$, $j = 1, ..., k$, this inequality coincides with (22). If $A+a\lambda < 0$ then from the second line of (27) we have $B\mu > A\lambda$. So, via (20) we obtain $S > |A+a\lambda|$ and, therefore, $S > S + A + a\lambda > 0$. From this and $2a > p$ the condition (22) with $\tau_j = 0$, $j = 1, ..., k$ follows.

Let now $p > 2a$. Then via (28) $A + a\lambda < 0$ and $B\mu > A\lambda$. Thus, from (20), (28) it follows that $q|A + a\lambda| > S > |A + a\lambda|$. From this via (26) the condition (22) with $\tau_j = 0$, $j = 1, ..., k$ follows. Finally, if $p = 2a$ then (28) is equivalent to $B\mu > A\lambda$ and via (20) $S > |A + a\lambda|$ that implies (22) with $\tau_j = 0$, $j = 1, ..., k$.

Consider the equilibrium point $\hat{x}_2$. Assume first that $p < 2a$. If $A + a\lambda \geq 0$, then from the second line of (29) it follows that $B\mu > A\lambda$. Via (20) it means that $S > A + a\lambda$ and therefore $S > S - A - a\lambda$. From this and $p < 2a$ the condition (23) with $\tau_j = 0$, $j = 1, ..., k$ follows. If $A + a\lambda < 0$ then from the first line of (29) we obtain $S > q|A + a\lambda|$. From this and (26) the condition (23) with $\tau_j = 0$, $j = 1, ..., k$ follows.

Let now $p > 2a$. Then via (30) $A + a\lambda \geq 0$ and $B\mu > A\lambda$. Thus, from (20), (30) it follows that $q(A + a\lambda) > S > A + a\lambda \geq 0$. From this and (26) the condition (23) with $\tau_j = 0$, $j = 1, ..., k$ follows. At last if $p = 2a$ then (30) is equivalent to $B\mu > A\lambda$ and via (20) $S > A + a\lambda$ that implies (23) with $\tau_j = 0$, $j = 1, ..., k$. The proof is completed.

Corollary 3.4 Put $\tau = \sum_{j=1}^{k} |a_j|\tau_j$, $Q = B\mu - A\lambda$ and assume that $a = 0$, $AB \neq 0$, $\tau < Q|A|^{-1}$. If $B > 0$ and

$$\frac{A\lambda}{B} + \frac{A^2\left(1 - \sqrt{1 - 4p\tau|A^{-1}|}\right)}{2pB} < \mu < \frac{A\lambda}{B} + \frac{A^2\left(1 + \sqrt{1 - 4p\tau|A^{-1}|}\right)}{2pB}, \tag{39}$$

or if $B < 0$ and

$$\frac{A\lambda}{B} + \frac{A^2\left(1 + \sqrt{1 - 4p\tau|A^{-1}|}\right)}{2pB} < \mu < \frac{A\lambda}{B} + \frac{A^2\left(1 - \sqrt{1 - 4p\tau|A^{-1}|}\right)}{2pB}, \tag{40}$$

then the equilibrium point $\hat{x} = -\mu A^{-1}$ is stable in probability.

Proof: It is enough to note that the conditions (39), (40) are the solution of the inequality

$$pQ^2 - A^2Q + A^2|A|\tau < 0, \quad where \quad Q = B\mu - A\lambda > 0,$$

which is equivalent to (24).

4 Numerical analysis

Example 4.1 Consider the equation

$$\dot{x}(t) = -ax(t) + \frac{\mu + a_1 x(t-\tau_1) + a_2 x(t-\tau_2)}{\lambda + b_1 x(t-\tau_1) + b_2 x(t-\tau_2)} + \sigma(x(t) - \hat{x})\dot{w}(t), \qquad (41)$$

that is an equation of the type of (14) with $k = 2$, $a_0 = b_0 = 0$.

Case 1. Put $a = 1$, $a_1 = 1.5$, $a_2 = -0.5$, $b_1 = 1.2$, $b_2 = 1.8$, $\tau_1 = 0.4$, $\tau_2 = 0.3$, $\sigma = 1.2$. Thus, $a > 0$, $B = 1.2 + 1.8 = 3 > 0$, $p = 0.72 < 2a = 2$, $A = 1.5 - 0.5 = 1 > 0$.

In Figure 1 the regions of stability in probability for the equilibrium points $\hat{x}_1$ and $\hat{x}_2$ are shown in the space of the parameters (μ, λ): in the white region there are no equilibrium points; in the yellow region there are possible unstable equilibrium points; the red, cyan, magenta and grey regions are the regions for stability in probability of the equilibrium point $\hat{x} = \hat{x}_1$ given by the condition (18) (red and cyan) and the condition (22) (cyan, magenta and grey); the blue, green and grey regions are the regions for stability in probability of the equilibrium point $\hat{x} = \hat{x}_2$ given by the condition (18) (blue) and the condition (23) (blue, green and grey); in the grey region the both equilibrium points $\hat{x} = \hat{x}_1$ and $\hat{x} = \hat{x}_2$ are stable in probability. The curves 1 and 2 are the bounds of the equilibrium points $\hat{x}_1$ and $\hat{x}_2$ stability regions respectively given by the conditions (27) and (29) for the case $\tau_1 = \tau_2 = 0$. One can see that the stability regions obtained for positive delays are placed inside of the regions with the zero delays.

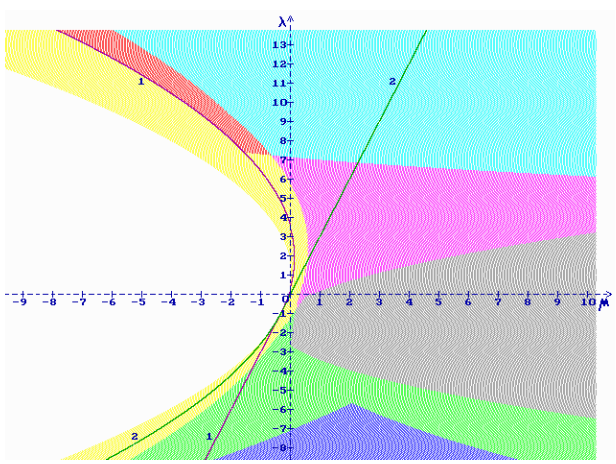

Figure 1: Stability regions for the equilibrium points $\hat{x}_1$ and $\hat{x}_2$ of the equation (41) by the values of the parameters: $a = 1$, $a_1 = 1.5$, $a_2 = -0.5$, $b_1 = 1.2$, $b_2 = 1.8$, $\tau_1 = 0.4$, $\tau_2 = 0.3$, $\sigma = 1.2$

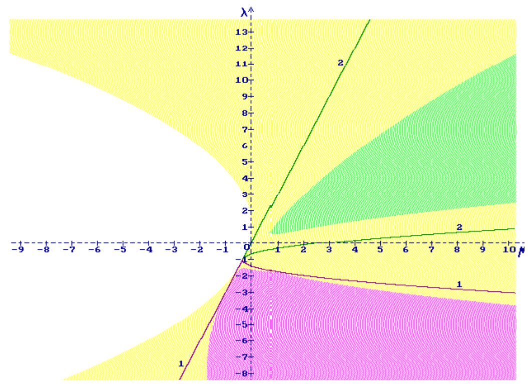

Figure 2: Stability regions for the equilibrium points $\hat{x}_1$ and $\hat{x}_2$ of the equation (41) by the values of the parameters: $a = 1$, $a_1 = 1.5$, $a_2 = -0.5$, $b_1 = 1.2$, $b_2 = 1.8$, $\tau_1 = 0.15$, $\tau_2 = 0.01$, $\sigma = 2.2$

Put now $\tau_1 = 0.15$, $\tau_2 = 0.01$, $\sigma = 2.2$. Then $p = 2.42 > 2a = 2$ and the condition (18) does not hold. The appropriate stability regions obtained with the same values of all other parameters by the conditions (28), (30) for the equilibrium points $\hat{x}_1$ (magenta) and $\hat{x}_2$ (green) are shown in Figure 2. Similarly to Figure 1 the stability regions obtained for positive delays are placed inside of the regions with the zero delays (the bounds 1 and 2).

Case 2. Put $a = 1$, $a_1 = -1.5$, $a_2 = -0.5$, $b_1 = -1.2$, $b_2 = -1.8$, $\tau_1 = 0.3$, $\tau_2 = 0.4$, $\sigma = 1.1$. Thus, $a > 0$, $B = -1.2 - 1.8 = -3 < 0$, $p = 0.605 < 2a = 2$, $A = -1.5 - 0.5 = -2 < 0$.

In Figure 3 the regions of stability in probability for the equilibrium points $\hat{x}_1$ and $\hat{x}_2$ are shown in the space of the parameters (μ, λ): in the white region there are no equilibrium points; in the yellow region there are possible unstable equilibrium points; the red, cyan, magenta and grey regions are the regions for stability in probability of the equilibrium point $\hat{x} = \hat{x}_1$ given by the condition (18) (red and cyan) and the condition (22) (cyan, magenta and grey); the blue, brown, green and grey regions are the regions for stability in probability of the equilibrium point $\hat{x} = \hat{x}_2$ given by the condition (18) (blue, brown) and the condition (23) (blue, brown, green and grey); in the grey region the both equilibrium points $\hat{x} = \hat{x}_1$ and $\hat{x} = \hat{x}_2$ are stable in probability. The curves 1 and 2 are the bounds of the equilibrium points $\hat{x}_1$ and $\hat{x}_2$ stability regions respectively given by the conditions (31) and (33) for the case $\tau_1 = \tau_2 = 0$. One can see that the stability regions obtained for positive delays are placed inside of the regions with the zero delays.

Put now $\tau_1 = 0.02$, $\tau_2 = 0.03$, $\sigma = 2.1$. Then $p = 2.205 > 2a = 2$ and the condition (18) does not hold. The appropriate stability regions obtained with the same values of all other parameters by the conditions (32), (34) for the equilibrium points $\hat{x}_1$ (magenta) and $\hat{x}_2$ (green) are shown in Figure 4.

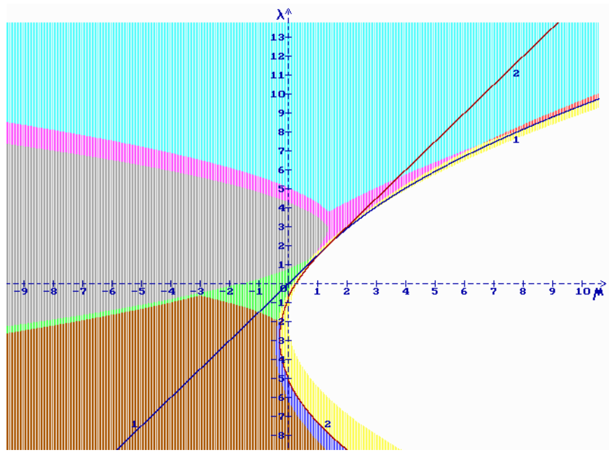

Figure 3: Stability regions for the equilibrium points $\hat{x}_1$ and $\hat{x}_2$ of the equation (41) by the values of the parameters: $a = 1$, $a_1 = -1.5$, $a_2 = -0.5$, $b_1 = -1.2$, $b_2 = -1.8$, $\tau_1 = 0.3$, $\tau_2 = 0.4$, $\sigma = 1.1$

Similarly to Figure 3 the stability regions obtained for positive delays are placed inside of the regions with the zero delays (the bounds 1 and 2).

Case 3. Put $a = -1.2$, $a_1 = 1.5$, $a_2 = 0.5$, $b_1 = 1.2$, $b_2 = 1.8$, $\tau_1 = 0.04$, $\tau_2 = 0.03$, $\sigma = 2$. Thus, $a < 0$, $B = 1.2 + 1.8 = 3 > 0$, $A = 1.5 + 0.5 = 2 > 0$, the condition (18) does not hold. In Figure 5 the regions of stability in probability for the equilibrium points $\hat{x}_1$ and $\hat{x}_2$ are shown in the space of the parameters (μ, λ): in the white region there are no equilibrium points; in the yellow region there are possible unstable equilibrium points; the magenta region is the region for stability in probability of the equilibrium point $\hat{x}_1$ given by the condition (22), the green region is the region for stability in probability of the equilibrium point $\hat{x}_2$ given by the condition (23). The curves 1 and 2 are the bounds of the equilibrium points $\hat{x}_1$ and $\hat{x}_2$ stability regions respectively given by the conditions (35) and (36) for the case $\tau_1 = \tau_2 = 0$. One can see that the stability regions obtained for positive delays are placed inside of the regions with the zero delays.

Case 4. Put $a = -1$, $a_1 = -1.5$, $a_2 = -0.5$, $b_1 = -1.2$, $b_2 = -1.8$, $\tau_1 = 0.04$, $\tau_2 = 0.05$, $\sigma = 1.7$. Thus, $a < 0$, $B = -1.2 - 1.8 = -3 < 0$, $A = -1.5 - 0.5 = -2 < 0$. The appropriate regions of stability in probability for the equilibrium points $\hat{x}_1$ and $\hat{x}_2$ obtained by the conditions (22), (23), (37), (38) are shown in Figure 6.

Example 4.2 Consider the equation

$$\dot{x}(t) = -ax(t) + \frac{\mu + a_0 x(t) + a_1 x(t - \tau_1)}{\lambda + b_0 x(t) + b_1 x(t - \tau_1)} + \sigma(x(t) - \hat{x})\dot{w}(t), \tag{42}$$

that is a particular case of the equation (14) with $k = 1$. The linear part of

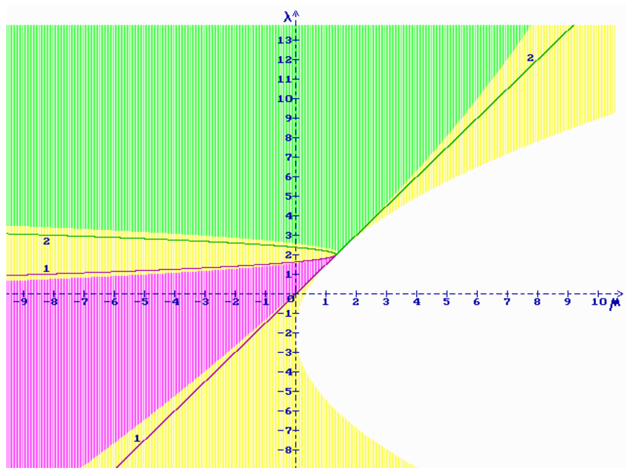

Figure 4: Stability regions for the equilibrium points $\hat{x}_1$ and $\hat{x}_2$ of the equation (41) by the values of the parameters: $a = 1$, $a_1 = -1.5$, $a_2 = -0.5$, $b_1 = -1.2$, $b_2 = -1.8$, $\tau_1 = 0.02$, $\tau_2 = 0.03$, $\sigma = 2.1$

the type of (21) for this equation has the form

$$\dot{z}(t) = \hat{\gamma}_0 z(t) + \gamma_1 z(t - \tau_1) + \sigma z(t)\dot{w}(t), \tag{43}$$

where

$$\hat{\gamma}_0 = \gamma_0 - a, \qquad \gamma_j = \frac{a_j - ab_j\hat{x}}{\lambda + B\hat{x}}, \quad j = 0, 1, \qquad B = b_0 + b_1.$$

The necessary and sufficient condition for asymptotic mean square stability of the trivial solution of the equation (43) is ([12], p.8, Lemma 1.5)

$$\hat{\gamma}_0 + \gamma_1 < 0, \qquad G^{-1} > p, \tag{44}$$

where

$$G = \begin{cases} \frac{\gamma_1 q^{-1}\sin(q\tau) - 1}{\hat{\gamma}_0 + \gamma_1 \cos(q\tau)}, & \gamma_1 + |\hat{\gamma}_0| < 0, \quad q = \sqrt{\gamma_1^2 - \hat{\gamma}_0^2}, \\ \frac{1 + |\hat{\gamma}_0|\tau}{2|\hat{\gamma}_0|}, & \gamma_1 = \hat{\gamma}_0 < 0, \\ \frac{\gamma_1 q^{-1}\sinh(q\tau) - 1}{\hat{\gamma}_0 + \gamma_1 \cosh(q\tau)}, & \hat{\gamma}_0 + |\gamma_1| < 0, \quad q = \sqrt{\hat{\gamma}_0^2 - \gamma_1^2}. \end{cases} \tag{45}$$

Note that if in the equation (43) $\tau_1 = 0$ then the sufficient condition (19) for asymptotic mean square stability of the trivial solution of the equation (43) takes the form $a > \gamma_0 + \gamma_1 + p$ and coincides with the necessary and sufficient condition (44) for asymptotic mean square stability of the trivial solution of the equation (43). Let us show that for small enough delay the sufficient stability conditions (18), (19) together are enough close to the necessary and sufficient stability condition (44).

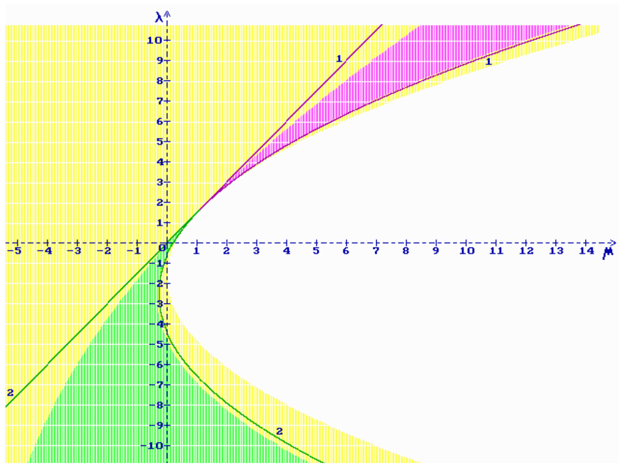

Figure 5: Stability regions for the equilibrium points $\hat{x}_1$ and $\hat{x}_2$ of the equation (41) by the values of the parameters: $a = -1.2$, $a_1 = 1.5$, $a_2 = 0.5$, $b_1 = 1.2$, $b_2 = 1.8$, $\tau_1 = 0.04$, $\tau_2 = 0.03$, $\sigma = 2$

In Figure 7 the stability regions for the equilibrium point $\hat{x}_1$ given by the condition (18) (green and magenta), by the condition (22) (magenta and cyan) and by the condition (44) (grey, green, magenta and cyan) are shown for the following values of the parameters

$$a = 1, \quad a_0 = -0.4, \quad b_0 = 0.2, \quad a_1 = b_1 = 1.5, \quad \tau_1 = 0.4, \quad \sigma = 1.3. \quad (46)$$

One can see that both stability conditions (18) and (22) complement each other and both these conditions together give the region of stability (green, magenta and cyan) that is close enough to the exact stability region obtained by the necessary and sufficient stability condition (44).

In Figure 8 the similar picture for the same values of the parameters (45) is shown for the equilibrium point $\hat{x}_2$: both stability conditions (18) (magenta and green (small region placed between magenta and the bound 2)) and (22) (magenta and cyan) complement each other and both these conditions together give the region of stability (green, magenta and cyan) that is close enough to the exact stability region (green, magenta, cyan and grey) obtained by the necessary and sufficient stability condition (44).

For numerical simulation of solutions of the equations (42) and (43) the algorithm for numerical simulation of the Wiener process trajectories is used that is described in [10]. 25 trajectories of the Wiener process obtained via this algorithm are shown in Figure 9.

Consider the point A with $\mu = 4$, $\lambda = -2$. This point belongs to the stability regions for the both equilibrium points: $\hat{x}_1 = 2.696$ (Figure 7) and $\hat{x}_2 = -0.873$ (Figure 8). In the point $A(4, -2)$ the trivial solution of the equation (43) is asymptotically mean square stable. Thus, in the point A all

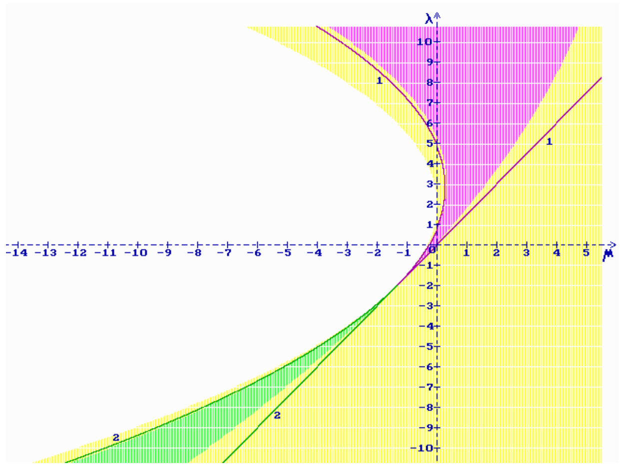

Figure 6: Stability regions for the equilibrium points $\hat{x}_1$ and $\hat{x}_2$ of the equation (41) by the values of the parameters: $a = -1$, $a_1 = -1.5$, $a_2 = -0.5$, $b_1 = -1.2$, $b_2 = -1.8$, $\tau_1 = 0.04$, $\tau_2 = 0.05$, $\sigma = 1.7$

trajectories of the equation (43) solutions with different given initial functions and the values of the parameters (45) converge to zero if $t \to \infty$. 200 such trajectories are shown in Figure 10 by the initial functions ($s \leq 0$)

$$x(s) = \hat{x}_1 + \frac{j}{33}\cos\left(\frac{10}{7}s\right) - 8.5, \qquad j = 0, 2, 4, ..., 198,$$
$$x(s) = \frac{25}{28}\hat{x}_1 - \frac{j}{33}\cos\left(\frac{10}{7}s\right) + 3, \qquad j = 1, 3, 5, ..., 199.$$

In Figure 11 trajectories of solutions of the nonlinear equation (42) are shown in the point A for the values of the parameters (45). In the point A the equilibrium point $\hat{x}_1 = 2.696$ of the equation (42) is stable in probability. Thus, in the point A 50 trajectories of solutions of the equation (42) with the initial functions

$$x(s) = \hat{x}_1 - \frac{2j}{33}\cos\left(\frac{10}{7}s\right) + 1.5, \qquad s \leq 0, \qquad j = 1, 2, ..., 50,$$

that belong to some neighborhood of the equilibrium point $\hat{x}_1$, converge to $\hat{x}_1$ if $t \to \infty$ (magenta trajectories) but other 50 trajectories of solution with the one initial function

$$x(s) = \hat{x}_1 + \frac{6}{11}\cos\left(\frac{10}{7}s\right) - 2.2, \qquad s \leq 0,$$

that is placed out of the neighborhood of $\hat{x}_1$, fill by itself the whole space (green trajectories). Only some of these trajectories, that come to neighborhood of $\hat{x}_1$ converge to $\hat{x}_1$ if $t \to \infty$.

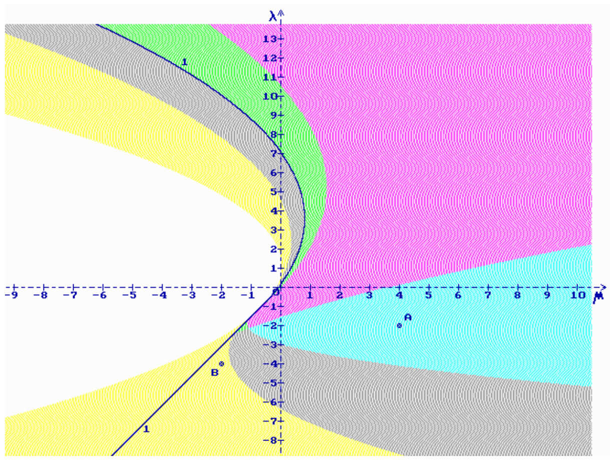

Figure 7: Stability regions for the equilibrium point $\hat{x}_1 = 2.696$ of the equation (42) by the values of the parameters: $a = 1$, $a_0 = -0.4$, $b_0 = 0.2$, $a_1 = 1.5$, $b_1 = 1.5$, $\tau_1 = 0.4$, $\sigma = 1.3$

Figure 12 is similar to Figure 11, but it shows 100 trajectories for the equilibrium point $\hat{x}_2 = -0.873$: 50 trajectories (magenta) with the initial functions

$$x(s) = \hat{x}_2 - \frac{j}{15}\cos\left(\frac{5}{3}s\right) + 2.1, \qquad s \le 0, \qquad j = 1, 2, ..., 50,$$

that belong to a small enough neighborhood of the equilibrium point $\hat{x}_2$ converge to this equilibrium and 50 trajectories (green) with one initial function

$$x(s) = \hat{x}_2 + \frac{4}{11}\cos(2s) + 1.8, \qquad s \le 0,$$

that is placed out of this neighborhood of $\hat{x}_2$ fill by itself the whole space.

Consider now the point B with $\mu = -2$, $\lambda = -4$ (Figure 7). This point does not belong to stability region for the equilibrium point $\hat{x}_1 = 2.536$, thus, in the point $B(-2, -4)$ the equilibrium point $\hat{x}_1$ is unstable. In Figure 13 five hundred trajectories of the solution of the equation (42) are shown with the initial function

$$x(s) = \hat{x}_1 + 0.015\sin\left(\frac{10}{3}s\right), \qquad s \le 0,$$

that is placed close enough to the equilibrium point $\hat{x}_1$. One can see that the trajectories do not converge to $\hat{x}_1$ and fill by itself the whole space.

In Figure 14 the similar picture is shown for the unstable equilibrium point $\hat{x}_2 = -2$ in the point C with $\mu = 4$, $\lambda = 2.5$ (Figure 8) with the initial

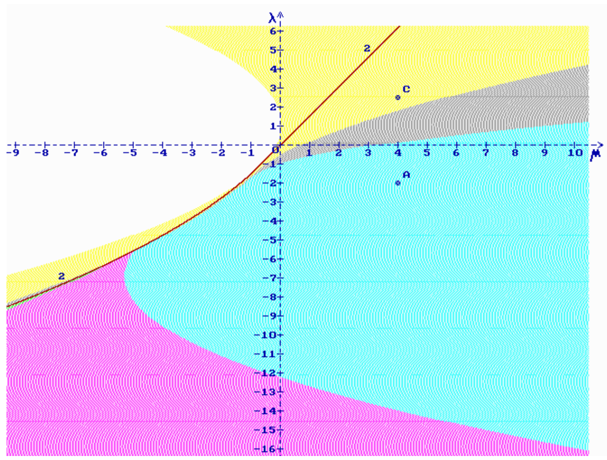

Figure 8: Stability regions for the equilibrium point $\hat{x}_2 = -0.873$ of the equation (42) by the values of the parameters: $a = 1$, $a_0 = -0.4$, $b_0 = 0.2$, $a_1 = 1.5$, $b_1 = 1.5$, $\tau_1 = 0.4$, $\sigma = 1.3$

function

$$x(s) = \hat{x}_2 - 0.025 \cos\left(\frac{5}{3}s\right), \qquad s \le 0.$$

5 Conclusions

In this paper we have obtained sufficient conditions for stability in probability of the equilibrium points of a differential equation with fractional nonlinearity and stochastic perturbations. The obtained results are illustrated by figures with stability regions and trajectories of stable and unstable solutions.

The detailed description of the proposed method including the numerical simulations of stochastic perturbations and solutions of the considered equations will help to researchers in stability theory and different applications in investigation of systems with various other types of nonlinearities of high orders.

6 Acknowledgements

The author thanks anonymous referees for very useful suggestions which improved the article content and presentation of the results.

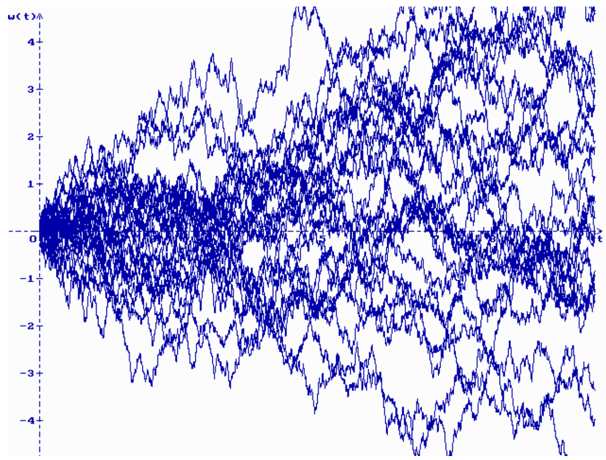

Figure 9: 25 trajectories of the Wiener process

7 References

[1] M. Bandyopadhyay, J. Chattopadhyay, Ratio dependent predator-prey model: effect of environmental fluctuation and stability, *Nonlinearity*, **18**, (2005) 913-936.

[2] E. Beretta, V. Kolmanovskii and L. Shaikhet, Stability of epidemic model with time delays influenced by stochastic perturbations, *Mathematics and Computers in Simulation*, special issue "Delay Systems", **45**(3-4), (1998) 269-277.

[3] M. Carletti, On the stability properties of a stochastic model for phage-bacteria interaction in open marine environment, *Mathematical Biosciences*, **175**, (2002) 117-131.

[4] W.S.C. Gurney, S.P. Blythe and R.M. Nisbet, Nicholson's blowflies revisited, *Nature*, **287**, (1980) 17-21.

[5] I. Gyori and S. Trofimchuk, Global attractivity in $\dot{x}(t) = -\delta x(t) + pf(x(t-\tau))$, *Dynamic Systems and Applications*, **8**, (1999) 197-210.

[6] V.B. Kolmanovskii, V.R. Nosov, Stability of functional differential equations. Academic Press, London, 1986.

[7] E. Liz, G. Rost, On global attractor of delay differential equations with unimodal feedback, *Discrete and Continuous Dynamical Systems*, **24**(4), (2009) 1215-1224.

[8] M.C. Mackey and L. Glass, Oscillation and chaos in physiological control system, *Science*, **197**, (1997) 287-289.

[9] A.J. Nicholson, An outline of the dynamics of animal populations, *Australian Journal of Zoology*, **2**, (1954) 9-65.

[10] S.I. Resnick, Adventures in stochastic processes. Birkhauser, Boston, Mass, USA, 1992.

[11] G. Rost and J. Wu, Domain decomposition method for the global dynamics of delay differential equations with unimodal feedback, *Proc. of the Royal Society, London Ser. A, Math. Phys. Eng. Sci.*, **463**, 2007, pp. 2655-2669.

[12] L. Shaikhet, Lyapunov functionals and stability of stochastic difference equations, Springer, London, Dordrecht, Heidelberg, New York, 2011.

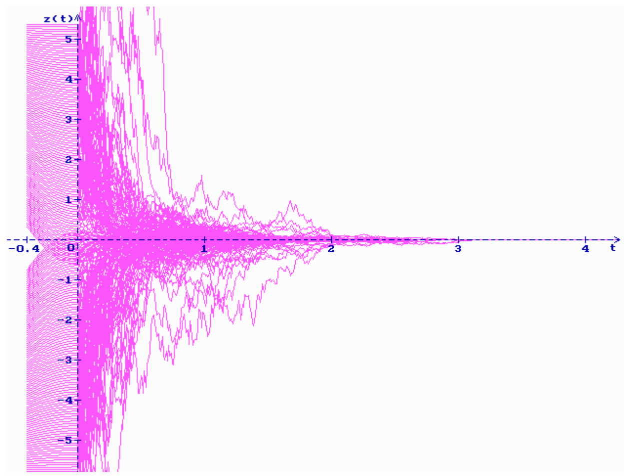

Figure 10: 200 trajectories of solutions of the equation (43) in the point A(4,-2) with different initial functions

[13] L. Shaikhet, Some new aspects of Lyapunov type theorems for stochastic differential equations of neutral type, *SIAM Journal on Control and Optimization*, **48**(7), (2010) 4481-4499.

[14] L. Shaikhet, Stability in probability of nonlinear stochastic hereditary systems, *Dynamic Systems and Applications*, **4**(2), (1995) 199-204.

Received February 2012; revised September 2012.

http://monotone.uwaterloo.ca/∼journal/

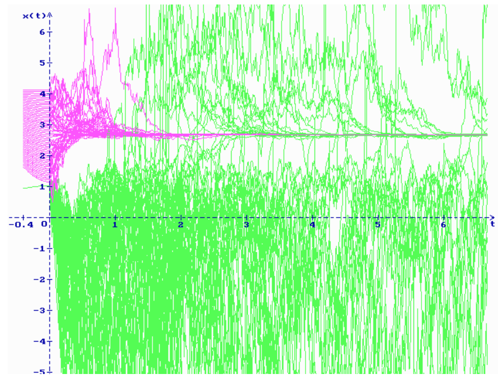

Figure 11: 100 trajectories of solutions of the equation (42) in the point A(4,-2) for the stable equilibrium point $\hat{x}_1 = 2.696$

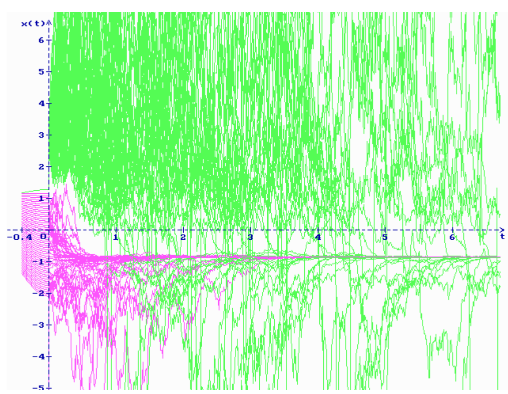

Figure 12: 100 trajectories of solutions of the equation (42) in the point A(4,-2) for the stable equilibrium point $\hat{x}_2 = -0.873$

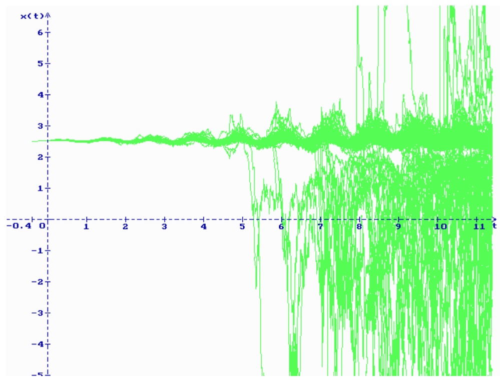

Figure 13: 500 trajectories of the solution of the equation (42) in the point B(-2,-4) for the unstable equilibrium point $\hat{x}_1 = 2.536$

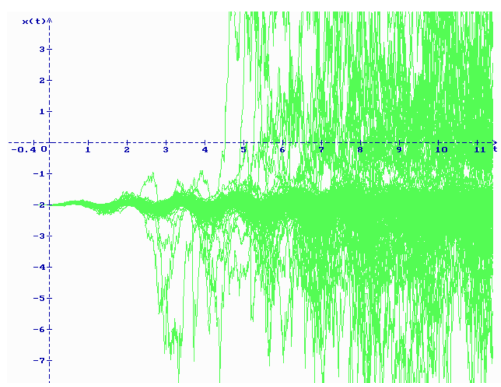

Figure 14: 500 trajectories of the solution of the equation (42) in the point C(4,2.5) for the unstable equilibrium point $\hat{x}_2 = -2$