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# Stability of equilibrium states for a stochastically perturbed exponential type system of difference equations

Leonid Shaikhet

*Department of Higher Mathematics, Donetsk State University of Management, Chelyuskintsev 163-a, 83015 Donetsk, Ukraine*

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ABSTRACT

In the paper it is shown how the known results of stability theory can be simply applied to stability investigation of equilibrium points of some systems of nonlinear difference equations with stochastic perturbations. A system of two difference equations with exponential nonlinearity is considered and it is shown that instead of the zero equilibrium this system can have also a positive equilibrium. Sufficient conditions for stability in probability of the both equilibriums of the initial nonlinear system with stochastic perturbations are obtained. Numerical simulations and figures illustrate a convergence of the positive solutions of the considered system to one of two (zero or positive) equilibriums in deterministic and stochastic cases. The proposed investigation procedure can be applied for arbitrary nonlinear equations with an order of nonlinearity higher than one.



## 1. Nonlinear systems and equilibrium states

Nonlinear differential equations with exponential nonlinearity often enough are used in different applied mathematical models. Consider, for instance, the economic delay differential neoclassical growth model [1,2]

$$\dot{x}(t) = ax^{\gamma}(t-\tau)e^{-bx(t-\tau)} - cx(t),$$

where $a, b, c, \gamma$ are positive parameters, $\tau$ is the delay in the production process. If, in particular, $\gamma = 1$, then this model describes the population dynamics of well-known Nicholson's blowflies, which is one of the most important mathematical models in ecology [3,4]. In connection with numerical simulation of special interest is the investigation of discrete analogues of this model [5,6]. Difference equations with exponential nonlinearity are used also immediately in different applied models [7], in particular, in the known discrete delay Mosquito population equation [8,9]

$$x_{n+1} = (ax_n + bx_{n-1})e^{-\nu x_n}.$$

To construct more complicated models some authors use also systems of difference equations with exponential nonlinearity [10–13]. Other models described by difference equations containing exponential terms one can find in [14] and in the references cited therein.

One of the most popular research directions for nonlinear system is asymptotic behavior and stability of its positive equilibriums [1,2,4,6,8,9,14–16]. Here stability of both the zero and positive equilibriums of the system of two nonlinear difference equations

$$\begin{aligned} x_1(n+1) &= ax_1(n) + bx_2(n-1)e^{-\mu x_1(n)}, \\ x_2(n+1) &= cx_2(n) + dx_1(n-1)e^{-\nu x_2(n)}, \\ n &= 0, 1, 2, \ldots, \end{aligned} \tag{1.1}$$

*E-mail address:* leonid.shaikhet@usa.net.

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with positive initial functions

$$x_1(j) = \varphi_1(j), \qquad x_2(j) = \varphi_2(j), \quad j = -1, 0, \tag{1.2}$$

and positive parameters $a, b, c, d, \mu, \nu$ is studied. Asymptotic behavior of the positive solutions of the system (1.1), (1.2) (in the case $\mu = \nu = 1$) is investigated in [14].

Putting in the system (1.1) $x_k(n) = x_k^*,\ k = 1, 2$, we obtain that the equilibrium points of this system are defined by the system of two algebraic equations

$$\begin{aligned} (1-a)x_1^* &= bx_2^* e^{-\mu x_1^*}, \\ (1-c)x_2^* &= dx_1^* e^{-\nu x_2^*}. \end{aligned} \tag{1.3}$$

It is easy to see that for arbitrary positive values of the parameters the system (1.3) has the zero solution $E_0 = (0, 0)$.

Let us suppose that $x_1^* \neq 0, x_2^* \neq 0$. Then from (1.3) we obtain

$$\eta = \frac{bd}{(1-a)(1-c)} = e^{\mu x_1^*} e^{\nu x_2^*}. \tag{1.4}$$

From (1.4) it follows that for positive $x_1^*$ and $x_2^*$ the condition $\eta > 1$ holds.

**Lemma 1.1.** *If the conditions*

$$bd > (1-a)(1-c), \quad a < 1,\ c < 1, \tag{1.5}$$

*hold, then the positive equilibrium point $E_+ = (x_1^*, x_2^*)$ of the system (1.1) there exists and satisfies the conditions*

$$\mu x_1^* + \nu x_2^* = \ln \eta, \tag{1.6}$$

$$x_1^* = \frac{\ln \eta}{\mu + \nu b^{-1}(1-a)e^{\mu x_1^*}}, \tag{1.7}$$

$$x_2^* = \frac{\ln \eta}{\nu + \mu d^{-1}(1-c)e^{\nu x_2^*}}, \tag{1.8}$$

*where $\eta$ is defined in (1.4).*

**Proof.** From (1.5), (1.4) it follows that $\eta > 1$. The condition (1.6) follows immediately from (1.4). Substituting $x_2^*$ from (1.6) into the first equation (1.3) we obtain (1.7). Substituting $x_1^*$ from (1.6) into the second equation (1.3) we obtain (1.8). The proof is completed.

**Lemma 1.2.** *If the conditions (1.5) hold, then the positive equilibrium point $E_+ = (x_1^*, x_2^*)$ satisfies the inequalities*

$$\frac{\ln \eta}{\mu + \nu d(1-c)^{-1}} < x_1^* < \frac{\ln \eta}{\mu + \nu b^{-1}(1-a)}, \tag{1.9}$$

$$\frac{\ln \eta}{\nu + \mu b(1-a)^{-1}} < x_2^* < \frac{\ln \eta}{\nu + \mu d^{-1}(1-c)}. \tag{1.10}$$

$$\frac{1-a}{b} < \frac{x_2^*}{x_1^*} < \frac{d}{1-c}. \tag{1.11}$$

**Proof.** The right bounds of the inequalities (1.9), (1.10) follow immediately from the conditions (1.7), (1.8) respectively. To get the left bound of the inequality (1.9) note that via (1.6) and the right bound of (1.10) we have

$$\begin{aligned} x_1^* &= \frac{1}{\mu}(\ln \eta - \nu x_2^*) \\ &> \frac{1}{\mu}\left(\ln \eta - \frac{\nu \ln \eta}{\nu + \mu d^{-1}(1-c)}\right) \\ &= \frac{\ln \eta}{\mu}\left(\frac{\mu d^{-1}(1-c)}{\nu + \mu d^{-1}(1-c)}\right) \\ &= \frac{\ln \eta}{\mu + \nu d(1-c)^{-1}}. \end{aligned}$$

Similarly the left bound of the inequality (1.10) can be obtained. The right bound of the inequality (1.11) simply follows from (1.9), (1.10)

$$\frac{x_2^*}{x_1^*} < \frac{\ln\eta[\mu + \nu d(1-c)^{-1}]}{[\nu + \mu d^{-1}(1-c)]\ln\eta} = \frac{d}{1-c}.$$

Similarly the left bound of the inequality (1.11) can be obtained. The proof is completed.

**Remark 1.1.** The inequalities (1.9), (1.10) specify and generalize the corresponding inequalities obtained in [14] for the case $\mu = \nu = 1$

$$\frac{\ln\eta}{1+d(1-c)^{-1}} < x_1^* < \ln\eta,$$
$$\frac{\ln\eta}{1+b(1-a)^{-1}} < x_2^* < \ln\eta.$$

## 2. Stochastic perturbations and some auxiliary equations, definitions, statements

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a basic probability space, $\mathfrak{F}_n \in \mathfrak{F}$, $n \in Z = \{0, 1, \ldots\}$, be a family of $\sigma$-algebras, $\mathbf{E}$ be an expectation, $\xi_k(n)$, $k = 1, 2$, $n \in Z$, are mutually independent sequences of $\mathfrak{F}_n$-adapted random variables such that

$$\mathbf{E}\xi_k(n) = 0, \qquad \mathbf{E}\xi_k^2(n) = 1, \qquad \mathbf{E}\xi_1(n)\xi_2(n) = 0. \tag{2.1}$$

Let us suppose that the system (1.1) is influenced by stochastic perturbations that are directly proportional to the deviation of the system state $(x_1(n), x_2(n))$ from the equilibrium point $(x_1^*, x_2^*)$. It means that if the deviation of the system state from the equilibrium point increases, then the stochastic perturbations also increase. The stochastic perturbations of such type were first proposed in [17] and successfully used later by many other researchers for different mathematical models with continuous and discrete time (see [4,6,9] and references cited therein).

By this assumption the system (1.1), (1.2) takes the form

$$\begin{aligned} x_1(n+1) &= ax_1(n) + bx_2(n-1)e^{-\mu x_1(n)} + \sigma_1(x_1(n) - x_1^*)\xi_1(n+1), \\ x_2(n+1) &= cx_2(n) + dx_1(n-1)e^{-\nu x_2(n)} + \sigma_2(x_2(n) - x_2^*)\xi_2(n+1), \end{aligned} \tag{2.2}$$

with an $\mathfrak{F}_0$-adapted initial function

$$x_k(l) = \varphi_k(l), \quad l \in Z_0 = \{-1, 0\}. \tag{2.3}$$

Here $\sigma_1$ and $\sigma_2$ are arbitrary constants and $(x_1^*, x_2^*)$ is an equilibrium point $E_+$ of the system (1.1) defined by (1.7), (1.8). Note that the equilibrium point $(x_1^*, x_2^*)$ is a solution of Eq. (2.2) too.

Putting in (2.2) $x_k(n) = y_k(n) + x_k^*$, $k = 1, 2$, and using (1.3), we obtain

$$\begin{aligned} y_1(n+1) &= ay_1(n) + bx_2^*e^{-\mu x_1^*}(e^{-\mu y_1(n)} - 1) + by_2(n-1)e^{-\mu(y_1(n)+x_1^*)} + \sigma_1 y_1(n)\xi_1(n+1), \\ y_2(n+1) &= cy_2(n) + dx_1^*e^{-\nu x_2^*}(e^{-\nu y_2(n)} - 1) + dy_1(n-1)e^{-\nu(y_2(n)+x_2^*)} + \sigma_2 y_2(n)\xi_2(n+1). \end{aligned} \tag{2.4}$$

It is easy to see that stability of the zero solution of the system (2.4) is equivalent to stability of the solution $(x_1^*, x_2^*)$ of the system (2.2).

Note that Eqs. (2.4) are nonlinear equations with an order of nonlinearity higher than one. Together with the nonlinear system (2.4) we will consider the linear approximation

$$\begin{aligned} z_1(n+1) &= (a - \mu bx_2^*e^{-\mu x_1^*})z_1(n) + be^{-\mu x_1^*}z_2(n-1) + \sigma_1 z_1(n)\xi_1(n+1), \\ z_2(n+1) &= (c - \nu dx_1^*e^{-\nu x_2^*})z_2(n) + de^{-\nu x_2^*}z_1(n-1) + \sigma_2 z_2(n)\xi_2(n+1), \end{aligned} \tag{2.5}$$

of the system (2.4).

It is easy to see that for the zero equilibrium $E_0$ the systems (2.4), (2.5) respectively are

$$\begin{aligned} y_1(n+1) &= ay_1(n) + by_2(n-1)e^{-\mu y_1(n)} + \sigma_1 y_1(n)\xi_1(n+1), \\ y_2(n+1) &= cy_2(n) + dy_1(n-1)e^{-\nu y_2(n)} + \sigma_2 y_2(n)\xi_2(n+1), \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} z_1(n+1) &= az_1(n) + bz_2(n-1) + \sigma_1 z_1(n)\xi_1(n+1), \\ z_2(n+1) &= cz_2(n) + dz_1(n-1) + \sigma_2 z_2(n)\xi_2(n+1). \end{aligned} \tag{2.7}$$

Put now $y(n) = (y_1(n), y_2(n))'$, $z(n) = (z_1(n), z_2(n))'$, $\varphi(n) = (\varphi_1(n), \varphi_2(n))'$.

**Definition 2.1.** The zero solution of the system (2.4) (or (2.6)) is called stable in probability if for any $\varepsilon > 0$ and $\varepsilon_1 > 0$ there exists a $\delta > 0$ such that the solution $y(n) = y(n, \varphi)$ of the system (2.4) (or (2.6)) satisfies the inequality $\mathbf{P}\{\sup_{n\in Z} |y(n)| > \varepsilon\} < \varepsilon_1$ for any initial function (2.3) such that $\mathbf{P}\{\|\varphi\|_0 < \delta\} = 1$, where $\|\varphi\|_0 = \max_{l\in Z_0} |\varphi(l)|$.

**Definition 2.2.** The zero solution of the system (2.5) (or (2.7)) is called mean square stable if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|z(n)|^2 < \varepsilon$, $n \in Z$, for any initial function (2.3) such that $\|\varphi\|^2 = \max_{l\in Z_0} \mathbf{E}|\varphi(l)|^2 < \delta$; asymptotically mean square stable if it is mean square stable and for each initial function (2.3) such that $\|\varphi\|^2 < \infty$ the solution $z(n)$ of the system (2.5) (or (2.7)) satisfies the condition $\lim_{n\to\infty} \mathbf{E}|z(n)|^2 = 0$.

Let $\mathbf{E}_i = \mathbf{E}\{./\mathfrak{F}_i\}$ be the conditional expectation with respect to the $\sigma$-algebra $\mathfrak{F}_i$. Put $U_\varepsilon = \{x : |x| \le \varepsilon\}$ and $\Delta V_i = V_{i+1} - V_i$.

**Theorem 2.1** ([6]). *For the system (2.4) (or (2.6)) assume there exists a functional $V_i = V(i, y(-1), \dots, y(i))$ satisfying the conditions*

$$V(i, y(-1), \dots, y(i)) \ge c_0|y(i)|^2, \tag{2.8}$$

$$V(0, \varphi(-1), \dots, \varphi(0)) \le c_1\|\varphi\|^2, \tag{2.9}$$

$$\mathbf{E}_i\Delta V_i \le 0, \qquad x_j \in U_\varepsilon, \qquad -h \le j \le i, \quad i \in Z, \tag{2.10}$$

*where $\varepsilon > 0$, $c_0 > 0$, $c_1 > 0$. Then the trivial solution of Eq. (2.4) (or (2.6)) is stable in probability.*

**Theorem 2.2** ([6]). *For the system (2.5) (or (2.7)) there exists a nonnegative functional $V_i = V(i, z(-1), \dots, z(i))$ satisfying the conditions (2.9) and*

$$\mathbf{E}\Delta V_i \le -c_2\mathbf{E}|z(i)|^2, \quad i \in Z, \tag{2.11}$$

*where $c_2 > 0$. Then the zero solution of the system (2.5) (or (2.7)) is asymptotically mean square stable.*

**Remark 2.1.** Let us suppose that for some nonlinear stochastic difference equations with an order of nonlinearity higher than one there exists a functional $V_i$, which satisfies the conditions (2.8), (2.9), (2.11) for the linear part of the considered nonlinear equation. As it is shown in [6, p. 150] this functional satisfies also the condition (2.10) for the initial nonlinear equation. Thus, to get sufficient conditions for stability in probability of the zero solution of the nonlinear system (2.4) (or (2.6)) it is enough by virtue of some functional, which satisfies the conditions (2.8), (2.9), (2.11), to get sufficient conditions for asymptotic mean square stability of the zero solution of the linear system (2.5) (or (2.7)).

## 3. Stability condition

Consider the system of two linear stochastic difference equations

$$\begin{aligned} z_1(n+1) &= a_{11}z_1(n) + b_{12}z_2(n-1) + \sigma_1 z_1(n)\xi_1(n+1), \\ z_2(n+1) &= a_{22}z_2(n) + b_{21}z_1(n-1) + \sigma_2 z_2(n)\xi_2(n+1), \end{aligned} \tag{3.1}$$

where $\sigma_1, \sigma_1$ are given constants and $\xi_k(n)$, $k = 1, 2$, $n \in Z$, are mutually independent sequences of $\mathfrak{F}_n$-adapted random variables satisfying the conditions (2.1).

It is easy to see that the systems (2.4) and (2.6) are particular cases of the system (3.1).

To get asymptotic mean square stability condition for the zero solution of the system (3.1) put

$$\begin{aligned} &A_0 = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix}, \\ &A = \begin{pmatrix} 0 & E \\ A_1 & A_0 \end{pmatrix}, \qquad D = \begin{pmatrix} D_{11} & D_{12} \\ D'_{12} & D_{22} \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix}. \end{aligned} \tag{3.2}$$

Here $A_0, A_1$ are matrices of dimension $2 \times 2$, $A, D, U$ are matrices of dimension $4 \times 4$, where

$$\begin{aligned} &E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}, \\ &D_{11} = \begin{pmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{pmatrix}, \qquad D_{12} = \begin{pmatrix} d_{13} & d_{14} \\ d_{23} & d_{24} \end{pmatrix}, \qquad D_{22} = \begin{pmatrix} d_{33} & d_{34} \\ d_{34} & d_{44} \end{pmatrix}, \end{aligned} \tag{3.3}$$

are matrices of dimension $2 \times 2$.

Note that $P$, $D$ and $U$ are symmetric matrices. For two symmetric matrices $P$ and $Q$ we will write $P > Q$ if the matrix $P - Q$ is a positive definite matrix.

**Theorem 3.1.** *Let for some positive definite matrix $P$ the matrix equation*

$$A'DA - D = -U \tag{3.4}$$

*have a positive semidefinite solution $D$ of the form (3.2), (3.3) such that*

$$P > \begin{pmatrix} d_{33}\sigma_1^2 & 0 \\ 0 & d_{44}\sigma_2^2 \end{pmatrix}. \tag{3.5}$$

*Then the zero solution of Eq. (3.1) is asymptotically mean square stable.*

**Proof.** Using the matrices $A_0$ and $A_1$ defined in (3.2) and

$$\Theta(\xi(n)) = \begin{pmatrix} \sigma_1\xi_1(n) & 0 \\ 0 & \sigma_2\xi_2(n) \end{pmatrix}, \qquad z(n) = \begin{pmatrix} z_1(n) \\ z_2(n) \end{pmatrix}, \tag{3.6}$$

represent the system (3.1) in the form

$$z(n+1) = A_0 z(n) + A_1 z(n-1) + \Theta(\xi(n+1))z(n). \tag{3.7}$$

It is easy to see that by virtue of the matrix $A$ defined in (3.2) and

$$B(\xi(n)) = \begin{pmatrix} 0 & 0 \\ 0 & \Theta(\xi(n)) \end{pmatrix}, \qquad w(n) = \begin{pmatrix} z(n-1) \\ z(n) \end{pmatrix}, \tag{3.8}$$

Eq. (3.7) can be written as follows

$$w(n+1) = [A + B(\xi(n+1))]w(n). \tag{3.9}$$

Consider the Lyapunov functional

$$V(n) = w'(n)Dw(n),$$

with the matrix $D$ defined in (3.2), (3.3). Calculating the expectation of

$$\Delta V(n) = V(n+1) - V(n)$$

via (3.9), (3.4) and the properties (2.1) of the sequences $\xi_k(n)$ we obtain

$$\begin{aligned} \mathbf{E}\Delta V(n) &= \mathbf{E}(w'(n+1)Dw(n+1) - w'(n)Dw(n)) \\ &= \mathbf{E}w'(n)\big([A + B(\xi(n+1))]'D[A + B(\xi(n+1))] - D\big)w(n) \\ &= \mathbf{E}w'(n)\big[A'DA - D + B'(\xi(n+1))DB(\xi(n+1))\big]w(n) \\ &= \mathbf{E}w'(n)\big[-U + B'(\xi(n+1))DB(\xi(n+1))\big]w(n). \end{aligned} \tag{3.10}$$

Note that via (3.2), (3.8) we have

$$\begin{aligned} B'(\xi(n))DB(\xi(n)) &= \begin{pmatrix} 0 & 0 \\ 0 & \Theta(\xi(n)) \end{pmatrix} \begin{pmatrix} D_{11} & D_{12} \\ D'_{12} & D_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \Theta(\xi(n)) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & \Theta(\xi(n))D_{22}\Theta(\xi(n)) \end{pmatrix} \end{aligned} \tag{3.11}$$

and via (3.3), (3.6) using the properties (2.1) of $\xi_k(n)$ we obtain

$$\begin{aligned} \mathbf{E}\Theta(\xi(n))D_{22}\Theta(\xi(n)) &= \mathbf{E}\begin{pmatrix} \sigma_1\xi_1(n) & 0 \\ 0 & \sigma_2\xi_2(n) \end{pmatrix} \begin{pmatrix} d_{33} & d_{34} \\ d_{34} & 4d_{44} \end{pmatrix} \begin{pmatrix} \sigma_1\xi_1(n) & 0 \\ 0 & \sigma_2\xi_2(n) \end{pmatrix} \\ &= \mathbf{E}\begin{pmatrix} d_{33}\sigma_1^2\xi_1^2(n) & d_{34}\sigma_1\sigma_2\xi_1(n)\xi_2(n) \\ d_{34}\sigma_1\sigma_2\xi_1(n)\xi_2(n) & d_{44}\sigma_2^2\xi_2^2(n) \end{pmatrix} \\ &= \begin{pmatrix} d_{33}\sigma_1^2 & 0 \\ 0 & d_{44}\sigma_2^2 \end{pmatrix}. \end{aligned} \tag{3.12}$$

From (3.10), (3.11), (3.12) via (3.2), (3.8) it follows that

$$\begin{aligned}
\mathbf{E}\Delta V(n) &= \mathbf{E}\mathrm{Tr}((w(n)w'(n))[-U + B'(\xi(n+1))DB(\xi(n+1))]) \\
&= \mathbf{E}\mathrm{Tr}\left(\begin{pmatrix} z(n-1)z'(n-1) & z(n-1)z'(n) \\ z(n)z'(n-1) & z(n)z(n)\end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 0 & -P + \Theta(\xi(n+1))D_{22}\Theta(\xi(n+1))\end{pmatrix}\right) \\
&= \mathbf{E}\mathrm{Tr}\begin{pmatrix} 0 & (z(n-1)z'(n))[-P + \Theta(\xi(n+1))D_{22}\Theta(\xi(n+1))] \\ 0 & (z(n)z'(n))[-P + \Theta(\xi(n+1))D_{22}\Theta(\xi(n+1))]\end{pmatrix} \\
&= \mathrm{Tr}(\mathbf{E}(z(n)z'(n))[-P + \mathbf{E}\Theta(\xi(n+1))D_{22}\Theta(\xi(n+1))]) \\
&= \mathrm{Tr}\left(\mathbf{E}(z(n)z'(n))\left[-P + \begin{pmatrix} d_{33}\sigma_1^2 & 0 \\ 0 & d_{44}\sigma_2^2\end{pmatrix}\right]\right) \\
&= \mathbf{E}z'(n)\left[-P + \begin{pmatrix} d_{33}\sigma_1^2 & 0 \\ 0 & d_{44}\sigma_2^2\end{pmatrix}\right]z(n).
\end{aligned}$$

From this and (3.5) it follows that $\mathbf{E}\Delta V(n) \le -c\mathbf{E}|z(n)|^2$ and via Theorem 2.2 the zero solution of Eq. (3.1) is asymptotically mean square stable. The proof is completed.

**Corollary 3.1.** *If the condition of Theorem 3.1 holds for the system (2.7) (or (2.5)), then via Remark 2.1 the zero solution of the system (2.6) (or (2.4)) is stable in probability and therefore the zero (or positive) equilibrium of the system (1.1) with stochastic perturbations is stable in probability.*

**Remark 3.1.** Via (3.2), (3.3) the matrix equation (3.4) is equivalent to the following system of 10 algebraic equations

$$\begin{aligned}
&b_{21}^2 d_{44} = d_{11}, \qquad b_{12}^2 d_{33} = d_{22}, \qquad b_{12}b_{21}d_{34} = d_{12}, \\
&b_{21}d_{14} + a_{11}b_{21}d_{34} = d_{13}, \qquad b_{21}d_{24} + a_{22}b_{21}d_{44} = d_{14}, \\
&b_{12}d_{13} + a_{11}b_{12}d_{33} = d_{23}, \qquad b_{12}d_{23} + a_{22}b_{12}d_{34} = d_{24}, \\
&d_{11} + 2a_{11}d_{13} + (a_{11}^2 - 1)d_{33} = -p_{11}, \\
&d_{22} + 2a_{22}d_{24} + (a_{22}^2 - 1)d_{44} = -p_{22}, \\
&d_{12} + a_{22}d_{14} + a_{11}d_{23} + (a_{11}a_{22} - 1)d_{34} = -p_{12}.
\end{aligned} \tag{3.13}$$

Solving this system with respect to $d_{ij}$, $1 \le i \le j \le 4$, (see Appendix) one can get the condition (3.5) in an explicit form. Choosing different positive definite matrices $P$ one can get different stability conditions.

**Remark 3.2.** Note that the proposed investigation procedure can be applied for arbitrary nonlinear difference equations with an order of nonlinearity higher than one. For instance, by the condition (1.5) the system

$$\begin{aligned}
x_1(n+1) &= [ax_1(n) + bx_2(n-1)]e^{-\mu x_1(n)}, \\
x_2(n+1) &= [cx_2(n) + dx_1(n-1)]e^{-\nu x_2(n)},
\end{aligned} \tag{3.14}$$

which a bit differs from (1.1) instead of the zero equilibrium has a positive equilibrium too. Similarly to Lemmas 1.1 and 1.2 the following statement can be proved.

**Lemma 3.1.** *If the conditions (1.5) hold, then the positive equilibrium point $E_+ = (x_1^*, x_2^*)$ of the system (3.14) there exists and satisfies the conditions*

$$\begin{aligned}
e^{\frac{\nu}{b}x_1^*(e^{\mu x_1^*} - a)} &= c + \frac{bd}{e^{\mu x_1^*} - a}, \\
e^{\frac{\mu}{d}x_2^*(e^{\nu x_2^*} - c)} &= a + \frac{bd}{e^{\nu x_2^*} - c}, \\
x_2^* &= \frac{x_1^*}{b}(e^{\mu x_1^*} - a),
\end{aligned}$$

*and also (1.11). Further investigation of stability of the system (3.14) with stochastic perturbations is similar to the investigation of the system (1.1).*

## 4. Numerical simulations

Below the obtained results are illustrated by numerical simulations of the system (1.1) solution for both zero and positive equilibriums.

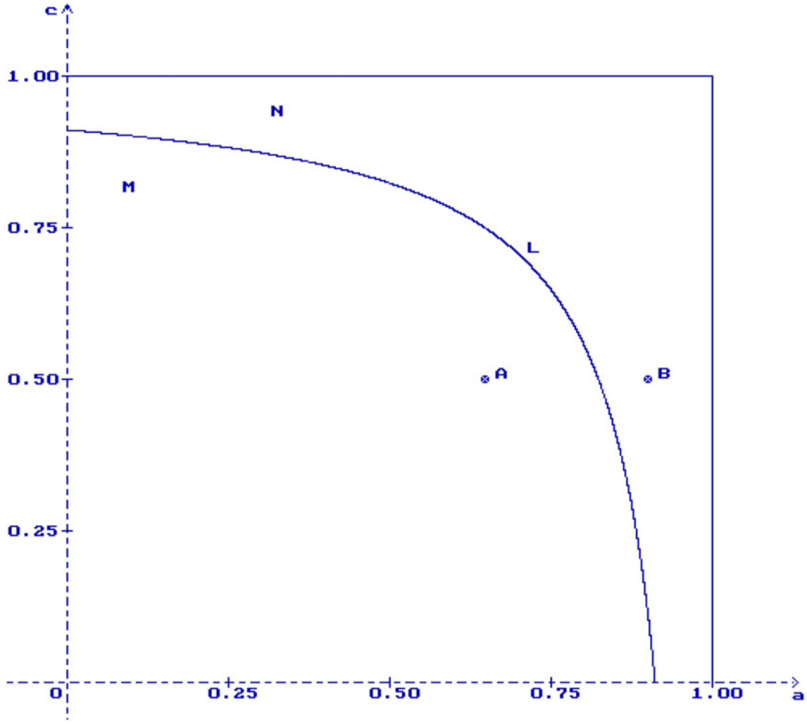


**Fig. 4.1.** The curve $L$ defined by the equation $bd = (1-a)(1-c)$ and the points $A(0.65, 0.5)$ and $B(0.9, 0.5)$.

Consider the system (1.1) with the following values of the parameters:

$$a = 0.65, \qquad b = 0.25, \qquad c = 0.5, \qquad d = 0.35, \qquad \mu = \nu = 0.5. \tag{4.1}$$

Via (1.4) in this case $\eta = 0.5 < 1$ and therefore (Lemma 1.1) the zero equilibrium there exists only.

In Fig. 4.1 in the space of the parameters $(a, c)$ two regions are shown: the region $M$, where the system (1.1) can have the zero equilibrium only, and the region $N$, where the system (1.1) besides the zero equilibrium has the positive equilibrium too. Via Lemma 1.1 both these regions are separated by the curve $L$ defined by the equation

$$bd = (1-a)(1-c)$$

for the values of the parameters $b, d$ given in (4.1). So, in the point $A(0.65, 0.5)$ the system (1.1) has the zero equilibrium $E_0$ only, in the point $B(0.9, 0.5)$ the system (1.1) has both equilibriums $E_0$ and $E_+$.

Solving the system (3.13) for the matrix $P$ with the elements

$$p_{11} = p_{22} = 1, \qquad p_{12} = 0, \tag{4.2}$$

we obtain the positive definite matrix

$$D = \begin{pmatrix} 0.239 & 0.089 & 0.385 & 0.437 \\ 0.089 & 0.188 & 0.586 & 0.274 \\ 0.385 & 0.586 & 3.012 & 1.020 \\ 0.437 & 0.274 & 1.020 & 1.950 \end{pmatrix}. \tag{4.3}$$

So, from the conditions (3.5) and (4.2), (4.3) for the point $A(0.65, 0.5)$ we obtain the following statements: if

$$\sigma_1 < 1/3.012 = 0.332 \quad \text{and} \quad \sigma_2 < 1/1.950 = 0.513, \tag{4.4}$$

then

- the zero solution of the system (2.7) is asymptotically mean square stable;
- the zero solution of the system (2.6) (the zero equilibrium by stochastic perturbations) is stable in probability.

In Fig. 4.2 the solutions of the systems (2.7) ($z_1$—blue, $z_2$—lilac) and (2.6) ($y_1$—green, $y_2$—red) are shown for $\sigma_1 = \sigma_2 = 0$ and the initial conditions

$$\begin{aligned} z_1(-1) = y_1(-1) = 1.10, \qquad z_1(0) = y_1(0) = 0.30, \\ z_2(-1) = y_2(-1) = 0.40, \qquad z_2(0) = y_2(0) = 1.00. \end{aligned} \tag{4.5}$$

The both solutions converge to zero.

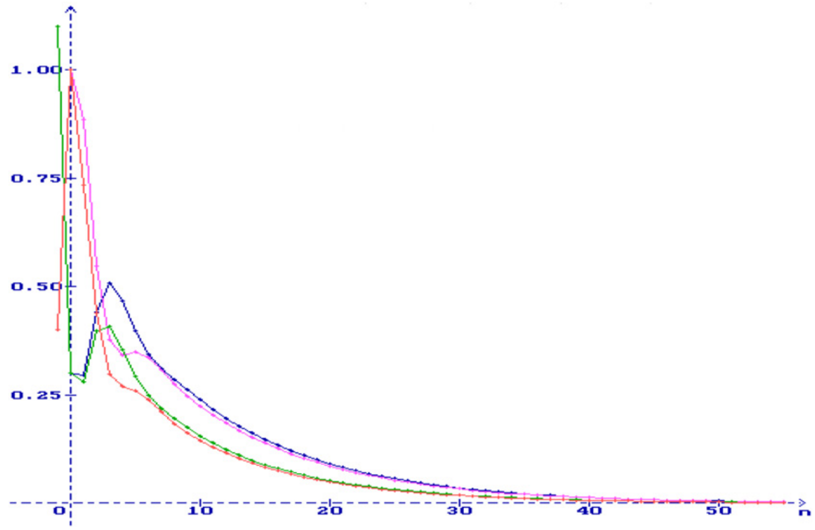


**Fig. 4.2.** The solutions of the systems (2.7) ($z_1$—blue, $z_2$—lilac) and (2.6) ($y_1$—green, $y_2$—red) in the case $\sigma_1 = \sigma_2 = 0$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Put now $a = 0.9$. Using the values of all other parameters given in (4.1) via (1.4) we have $\eta = 1.75 > 1$. So, in the point $B(0.9, 0.5)$ the system (1.1) has a positive equilibrium and from (1.6)–(1.8) we get $x_1^* = 0.712$, $x_2^* = 0.407$. Note also that for these $x_1^*, x_2^*$ the conditions (1.9)–(1.11) hold:

$$0.658 < x_1^* = 0.712 < 0.895,$$
$$0.320 < x_2^* = 0.407 < 0.461,$$
$$0.4 < \frac{x_2^*}{x_1^*} = 0.572 < 0.7.$$

Solving the system (3.13) for the matrix $P$ with the elements (4.2) we obtain the positive definite matrix

$$D = \begin{pmatrix} 0.158 & 0.121 & 0.696 & 0.346 \\ 0.121 & 0.286 & 1.535 & 0.437 \\ 0.696 & 1.535 & 9.338 & 2.418 \\ 0.346 & 0.437 & 2.418 & 1.943 \end{pmatrix}. \tag{4.6}$$

So, from the conditions (3.5) and (4.2), (4.6) for the point $B(0.9, 0.5)$ we obtain the following statements: if

$$\sigma_1 < 1/9.338 = 0.107 \quad \text{and} \quad \sigma_2 < 1/1.943 = 0.515, \tag{4.7}$$

then

- the zero solution of the system (2.5) is asymptotically mean square stable;
- the zero solution of the system (2.4) (and therefore the equilibrium point $(x_1^*, x_2^*) = (0.712, 0.407)$ of the system (2.2)) is stable in probability.

In Fig. 4.3 the solutions of the systems (2.5) ($z_1$—blue, $z_2$—lilac) and (2.2) ($x_1$—green, $x_2$—red) are shown for $\sigma_1 = \sigma_2 = 0$ and the initial conditions

$$\begin{aligned} z_1(-1) &= 0.700, & z_1(0) &= 0.300, \\ z_2(-1) &= 0.400, & z_2(0) &= 0.750, \\ x_1(-1) &= 0.001, & x_1(0) &= 0.000, \\ x_2(-1) &= 0.000, & x_2(0) &= 0.001. \end{aligned} \tag{4.8}$$

Here one can see the following:

- the zero solution of the system (2.5) is asymptotically mean square stable, so, the solution of the system (2.5) with nonzero initial functions goes to the zero;
- the zero solution of the system (2.2) is unstable, so, the solution of the system (2.2) with almost zero initial functions (see (4.8)) converges to the asymptotically stable positive equilibrium $(x_1^*, x_2^*) = (0.712, 0.407)$.

The last statement means in particular that by the considered values of the parameters the zero equilibrium of the system (1.1) (since the system (2.2) with $\sigma_1 = \sigma_2 = 0$ coincides with (1.1)) is unstable. Really, solving the system (3.13) for the

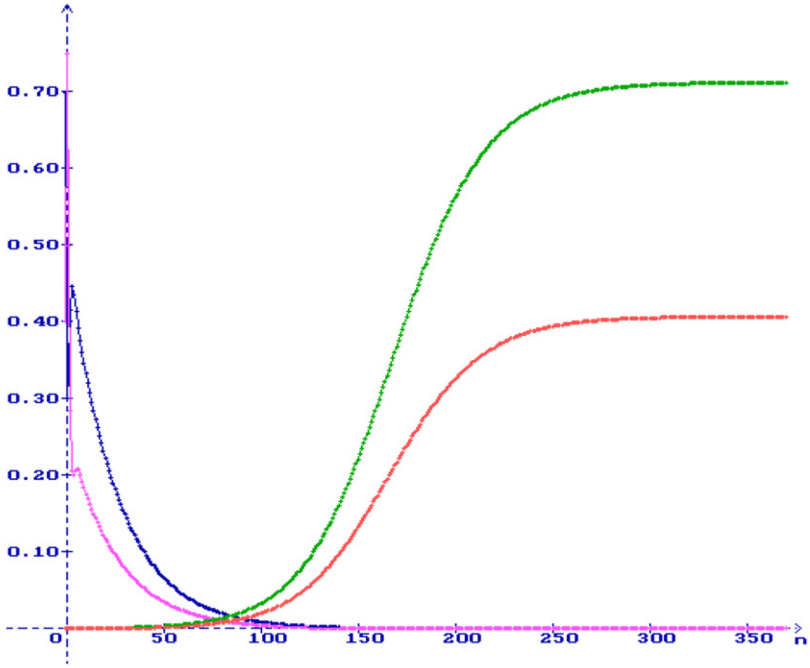


**Fig. 4.3.** The solutions of the systems (2.5) ($z_1$—blue, $z_2$—lilac) and (2.2) ($x_1$—green, $x_2$—red) in the case $\sigma_1 = \sigma_2 = 0$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

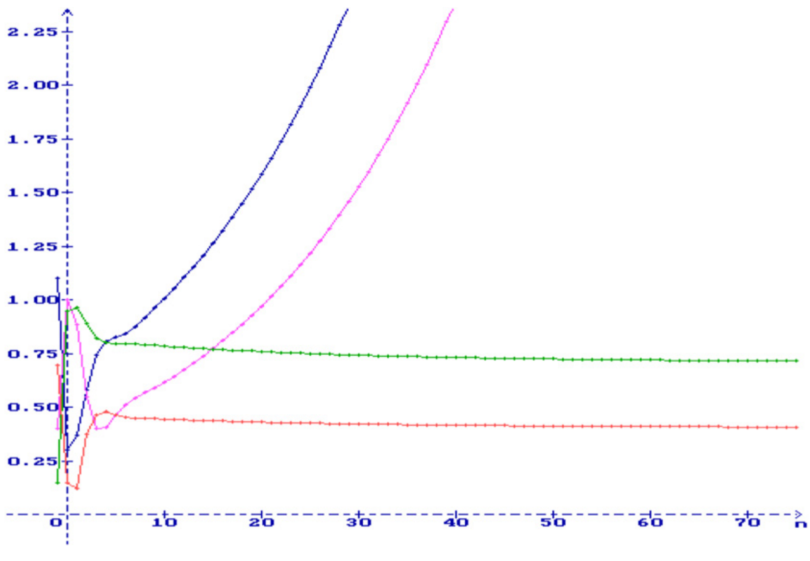


**Fig. 4.4.** The solutions of the systems (2.7) ($z_1$—blue, $z_2$—lilac) and (2.6) ($y_1$—green, $y_2$—red) in the case $\sigma_1 = \sigma_2 = 0$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

matrix $P$ with the elements (4.2) we obtain that all diagonal elements of the matrix $D$ are negative:

$$d_{11} = -0.017, \qquad d_{22} = -0.334, \qquad d_{33} = -5.347, \qquad d_{44} = -0.139,$$

i.e., the matrix $D$ is not a positive definite. In Fig. 4.4 the solutions of the systems (2.7) ($z_1$—blue, $z_2$—lilac) and (2.6) ($y_1$—green, $y_2$—red) are shown for $\sigma_1 = \sigma_2 = 0$ and the initial conditions (4.5). The zero solution of the system (2.7) is unstable, therefore, the solution converges to the infinity. The solution of the system (2.6) (by $\sigma_1 = \sigma_2 = 0$ it coincides with (1.1)) converges to the positive equilibrium point $E_+(0.712, 0.407)$.

Consider now numerical simulation of the system (1.1) with stochastic perturbations that are directly proportional to the deviation of the system state from the equilibrium point. More exactly consider the system (2.6) that is the system (1.1) with stochastic perturbations around the zero equilibrium $E_0(0, 0)$ and the system (2.2) that is the system (1.1) with stochastic perturbations around the positive equilibrium $E_+(0.712, 0.407)$.

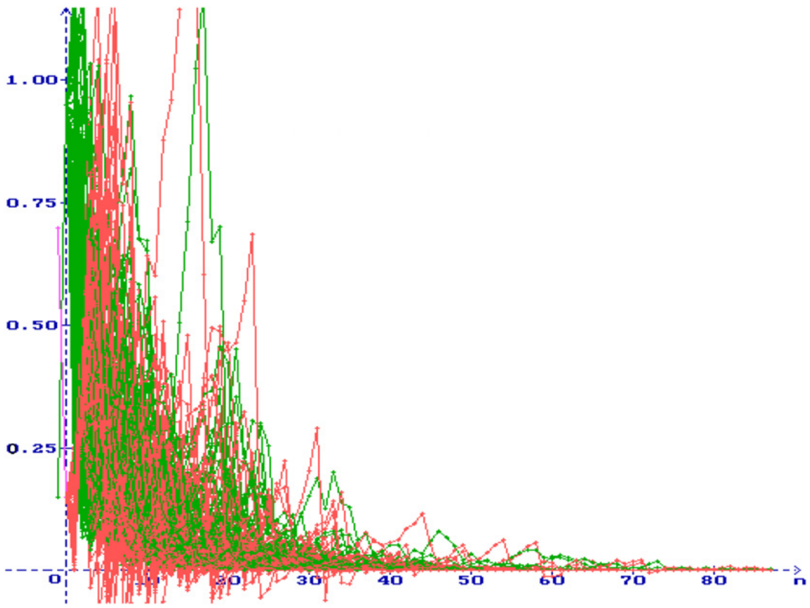


**Fig. 4.5.** 100 trajectories of the solution of the system (2.6) ($y_1$—green, $y_2$—red) in the point $A(0.65, 0.5)$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

For numerical simulation of the system (2.6) solution we will use the values of the parameters given in (4.1) and put $\sigma_1 = 0.33, \sigma_2 = 0.51$. Via (4.4) for these values of $\sigma_1, \sigma_2$ the zero equilibrium $E_0(0, 0)$ in the point $A(0.65, 0.5)$ (see Fig. 4.1) is stable in probability.

For simulation of stochastic perturbations we will use sequences of random variables $\xi(n)$ uniformly distributed on the interval $(-\sqrt{3}, \sqrt{3})$. So, $\mathbf{E}\xi(n) = 0, \mathbf{E}\xi^2(n) = 1$.

In Fig. 4.5 100 trajectories of the solution of the system (2.6) with the initial conditions (4.5) are shown. According to stability in probability of the zero equilibrium of the system (2.6) all trajectories converge to zero ($y_1$—green, $y_2$—red).

Put now $a = 0.9$ and consider behavior of the zero equilibrium $E_0(0, 0)$ in the point $B(0.9, 0.5)$. In Fig. 4.6 50 trajectories of the system (2.6) solution ($y_1$—green, $y_2$—red) are shown for the initial conditions (4.5) and small enough $\sigma_1, \sigma_2$: $\sigma_1 = 0.1$, $\sigma_2 = 0.2$. In the point $B(0.9, 0.5)$ the zero equilibrium of the system (2.6) is unstable, so, the trajectories fill whole space.

In Fig. 4.7 100 trajectories of the system (2.2) solution with the initial conditions

$$x_1(-1) = 0.9, \qquad x_1(0) = 0.3,$$
$$x_2(-1) = 0.1, \qquad x_2(0) = 0.9,$$

and $\sigma_1 = 0.1, \sigma_2 = 0.51$ are shown. Via (4.7) by these values of the parameters $\sigma_1, \sigma_2$ the positive equilibrium of the system (2.2) in the point $B$ is stable in probability, so, all trajectories converge to the positive equilibrium $E_+(0.712, 0.407)$.

In Fig. 4.8 for comparison with Fig. 4.7 a similar picture is shown with the initial conditions which are close to $E_+(0.712, 0.407)$ ($x_1(-1) = x_1(0) = 0.7, x_2(-1) = x_2(0) = 0.4$) and doubled noise levels. One can see that by these noise levels the equilibrium $E_+$ is unstable.

## 5. Conclusions

In this paper, a system of nonlinear difference equations with exponential nonlinearity is considered. It is supposed that this system is exposed to stochastic perturbations that are directly proportional to the deviation of the system state from one of two (the zero or positive) equilibrium points. The special procedure is proposed which allows to get sufficient conditions for stability in probability of the system equilibriums. The obtained results are illustrated by numerical simulations of solutions of the considered system.

The proposed investigation procedure can be applied for arbitrary nonlinear difference equations with an order of nonlinearity higher than one.

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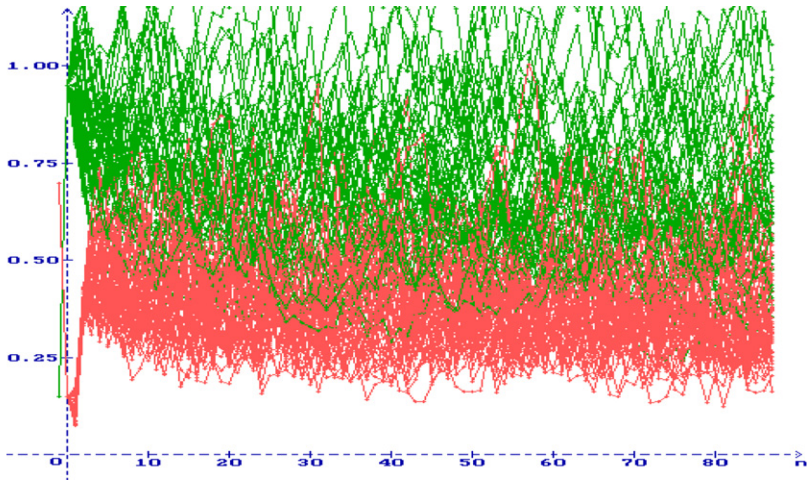


**Fig. 4.6.** 50 trajectories of the solution of the system (2.6) ($y_1$—green, $y_2$—red) in the point $B(0.9, 0.5)$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

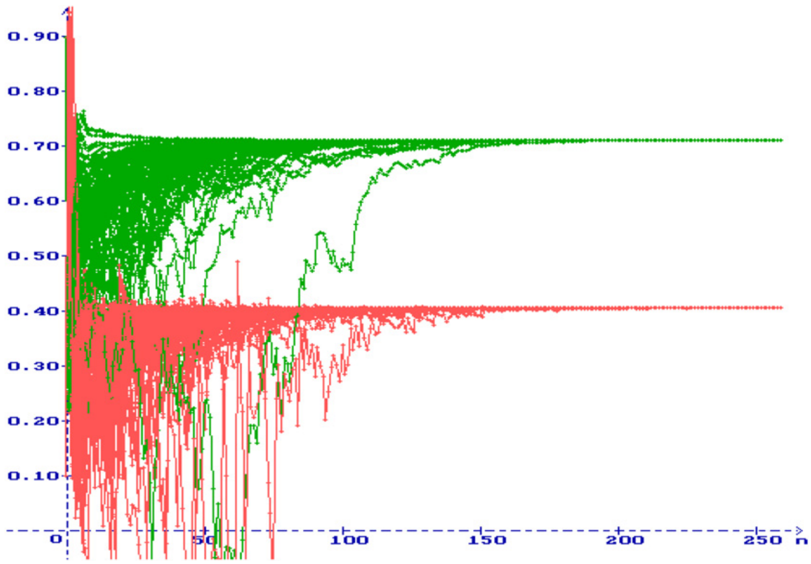


**Fig. 4.7.** 100 trajectories of the solution of the system (2.2) ($x_1$—green, $x_2$—red) in the point $B(0.9, 0.5)$ with $\sigma_1 = 0.1$, $\sigma_2 = 0.51$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

## Appendix. Solution of the system (3.13)

Put

$$\alpha_1 = \frac{a_{11}b_{12}^2}{1-b_{12}^2b_{21}^2}, \qquad \alpha_2 = \frac{\gamma_2 b_{12}}{1-b_{12}^2b_{21}^2}, \qquad \alpha_3 = \frac{a_{22}b_{12}^2b_{21}^2}{1-b_{12}^2b_{21}^2},$$
$$\beta_0 = \frac{p_{22}}{2\alpha_2 a_{22}}, \qquad \beta_1 = \frac{b_{12}^2}{2\alpha_2 a_{22}} + \frac{\alpha_1}{\alpha_2}, \qquad \beta_2 = \frac{1-a_{22}^2}{2\alpha_2 a_{22}} - \frac{\alpha_3}{\alpha_2},$$
$$\gamma_1 = a_{11} + \alpha_2 b_{21}, \qquad \gamma_2 = a_{22} + a_{11}b_{12}b_{21}, \qquad \gamma_3 = \alpha_3 + a_{22}$$
$$Q_{00} = \alpha_2\gamma_2 b_{21} + a_{11}a_{22} + b_{12}b_{21}(1+a_{11}^2) - 1,$$
$$Q_{10} = 2\beta_0\gamma_1 a_{11}b_{21} - p_{11},$$
$$Q_{11} = a_{11}(a_{11} + 2\alpha_1 b_{21}^2) - 2\beta_1\gamma_1 a_{11}b_{21} - 1,$$
$$Q_{12} = 2\beta_2\gamma_1 a_{11}b_{21} + b_{21}^2(2a_{11}\gamma_3 + 1),$$
$$Q_{20} = \beta_0 Q_{00} - p_{12},$$

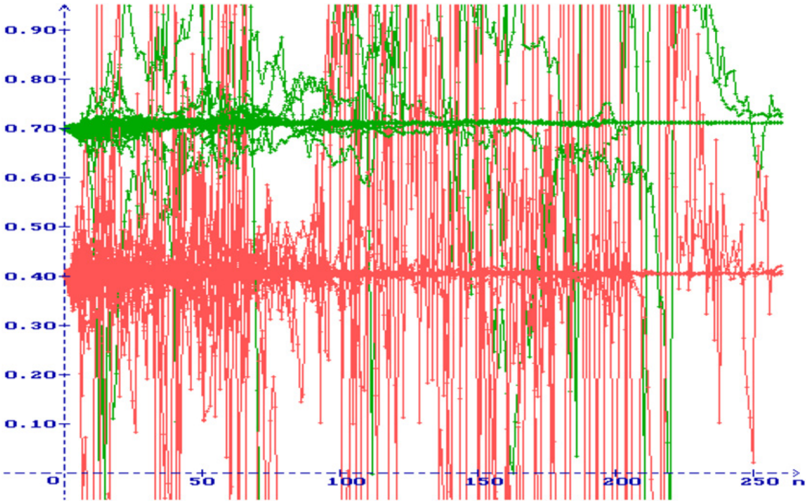


**Fig. 4.8.** 100 trajectories of the solution of the system (2.2) solution ($x_1$—green, $x_2$—red) in the point $B(0.9, 0.5)$ with doubled noise levels. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$Q_{21} = \alpha_1\gamma_2 b_{21} + a_{11}^2 b_{12} - \beta_1 Q_{00},$$
$$Q_{22} = \beta_2 Q_{00} + \gamma_2\gamma_3 b_{21},$$
$$\rho = Q_{11}Q_{22} - Q_{12}Q_{21}.$$

Then the solution of the system (3.13) can be obtained by the following way:

$$d_{33} = \frac{1}{\rho}(Q_{10}Q_{22} - Q_{12}Q_{20}), \qquad d_{44} = \frac{1}{\rho}(Q_{11}Q_{20} - Q_{10}Q_{21}),$$
$$d_{34} = -\beta_0 - \beta_1 d_{33} + \beta_2 d_{44},$$
$$d_{24} = \frac{1}{2a_{22}}((1 - a_{22}^2)d_{44} - b_{12}^2 d_{33} - p_{22}), \qquad d_{23} = \frac{1}{b_{12}}(d_{24} - a_{22}b_{12}d_{34}),$$
$$d_{14} = b_{21}d_{24} + a_{22}b_{21}d_{44}, \qquad d_{13} = b_{21}d_{14} + a_{11}b_{21}d_{34},$$
$$d_{11} = b_{21}^2 d_{44}, \qquad d_{12} = b_{12}b_{21}d_{34}, \qquad d_{22} = b_{12}^2 d_{33}.$$

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