About stability of a difference analogue of a nonlinear integro-differential equation of convolution type

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Abstract

A nonlinear integro-differential equation of convolution type with order of nonlinearity more than one and a stable trivial solution is considered. The integral in this equation has an exponential kernel and polynomial integrand. The difference analogue of the equation considered is constructed in the form of a difference equation with continuous time and it is shown that this difference analogue preserves the properties of stability of his original.
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1. Introduction

To use numerical investigation of functional differential equations it is very important to know whether the considered difference analogue of the original differential equation has the reliability to preserve some general properties of this equation, in particular, the property of stability. This problem is considered here by investigation of a difference analogue of the nonlinear integro-differential equation of convolution type.

Similar problems were considered already in [1] for deterministic nonlinear integro-differential equations and in [2] for linear stochastic integro-differential equations. In contrast to [1], here the difference analogue is constructed in the form of a difference equation with continuous time which is popular among researchers [3–10]. Besides that, here a more general form of nonlinearity is considered.

Stability conditions for asymptotic stability of the difference analogue are obtained via the general method of Lyapunov functional construction [8–11].

Below, the following definition and notation are used: $[t]$ is the integer part of an arbitrary real number t, $\Delta V(t) = V(t + \tau) - V(t)$ for an arbitrary $t \geq 0$ and the fixed positive number τ.

Lemma 1.1. *Arbitrary positive numbers $a, b, \alpha, \beta, \gamma$ satisfy the inequality*

$$a^\alpha b^\beta \leq \frac{\alpha}{\alpha + \beta} a^{\alpha+\beta} \gamma^\beta + \frac{\beta}{\alpha + \beta} b^{\alpha+\beta} \gamma^{-\alpha}.$$

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To prove Lemma 1.1 it is enough to show that the function

$$g(x) = \frac{\alpha}{\alpha + \beta} x^{\alpha+\beta} \gamma^\beta + \frac{\beta}{\alpha + \beta} b^{\alpha+\beta} \gamma^{-\alpha} - x^\alpha b^\beta, \quad x \geq 0,$$

reaches its minimum at the point $x_0 = b\gamma^{-1}$ and $g(x_0) = 0$.

2. Nonlinear integro-differential equation of convolution type

The nonlinear integro-differential equations of convolution type

$$\dot{x}(t) = \int_0^t K(t-s) f(x(s)) \mathrm{d}s$$

arise usually in problems related to evolutionary processes in ecology, in nuclear reactors, in control theory etc. [12, 13]. Here an equation of convolution type with exponential kernel $K(t) = -a\mathrm{e}^{-\lambda t}$, i.e.,

$$\dot{x}(t) = -a \int_0^t \mathrm{e}^{-\lambda(t-s)} f(x(s)) \mathrm{d}s, \tag{2.1}$$

is considered and it is supposed that $a > 0, \lambda > 0$,

$$f(x) = \sum_{i=1}^k \alpha_i x^{m_i}, \qquad m_i = \frac{2p_i + 1}{2q_i + 1}, \quad p_i \geq q_i \geq 0, \tag{2.2}$$

$\alpha_i > 0$, p_i and q_i are integers.

It is easy to check that the trivial solution of Eq. (2.1) is stable. Indeed, putting $x_1(t) = x(t)$, $x_2(t) = \dot{x}(t)$, one can transform Eq. (2.1) to the system of equations

$$\dot{x}_1(t) = x_2(t),$$
$$\dot{x}_2(t) = -af(x_1(t)) - \lambda x_2(t).$$

The function

$$V(t) = 2a \sum_{i=1}^k \frac{\alpha_i}{m_i + 1} x_1^{m_i+1}(t) + x_2^2(t)$$

is a Lyapunov function for this system since

$$m_i + 1 = \frac{2p_i + 1}{2q_i + 1} + 1 = \frac{2(p_i + q_i + 1)}{2q_i + 1}, \tag{2.3}$$

i.e., $V(t) > 0$ for $x_1^2(t) + x_2^2(t) > 0$, and $\dot{V}(t) = -2\lambda x_2^2(t) < 0$ unless $x_2(t) = 0$.

3. Difference analogue of the nonlinear integro-differential equation

Difference analogues arise, in particular, as the result of attempts to solve differential equations by numerical methods. To construct the difference analogue of Eq. (2.1) rewrite this equation in the equivalent form

$$\dot{x}(t) = -a \int_0^t \mathrm{e}^{-\lambda s} f(x(t-s)) \mathrm{d}s. \tag{3.1}$$

Let τ be a small enough positive number. Using representation (2.1) for $t \in [0, \tau)$ and (3.1) for $t \geq \tau$, we can construct a difference analogue in the form of the following difference equation with continuous time:

$$x(t) = x(0) - at^2 \mathrm{e}^{-\lambda t} f(x(0)), \quad t \in [0, \tau),$$
$$x(t + \tau) = x(t) - a\tau^2 F(t), \quad t \geq 0,$$
$$F(t) = \sum_{j=0}^{[\frac{t}{\tau}]} \mathrm{e}^{-\lambda \tau j} f(x(t - j\tau)). \tag{3.2}$$

If $t \in [0, \tau)$, then $F(t) = f(x(t))$. For $t \geq \tau$, transform $F(t)$ in the following way:

$$
\begin{aligned}
F(t) &= f(x(t)) + \sum_{j=1}^{[\frac{t}{\tau}]} e^{-\lambda \tau j} f(x(t - j\tau)) \\
&= f(x(t)) + \sum_{j=0}^{[\frac{t}{\tau}]-1} e^{-\lambda \tau (j+1)} f(x(t - (j+1)\tau)) \\
&= f(x(t)) + e^{-\lambda \tau} \sum_{j=0}^{[\frac{t-\tau}{\tau}]} e^{-\lambda \tau j} f(x(t - \tau - j\tau)) \\
&= f(x(t)) + e^{-\lambda \tau} F(t - \tau).
\end{aligned}
\tag{3.3}
$$

It follows from (3.2) that

$$
F(t) = -\frac{x(t + \tau) - x(t)}{a\tau^2}, \qquad F(t - \tau) = -\frac{x(t) - x(t - \tau)}{a\tau^2}.
$$

Substituting $F(t)$ and $F(t - \tau)$ from here into (3.3), we transform Eq. (3.2) to the form

$$
x(t + \tau) = x(t) - a\tau^2 f(x(t)) + e^{-\lambda \tau}(x(t) - x(t - \tau)), \quad t > \tau.
\tag{3.4}
$$

The process $x(t)$ is defined by Eq. (3.4) for $t > t_0 = 2\tau$ with the initial condition

$$
x(\theta) = \phi(\theta), \qquad \theta \in [t_0 - 2\tau, t_0] = [0, 2\tau],
\tag{3.5}
$$

where

$$
\phi(\theta) = \begin{cases} x(0) - a\theta^2 e^{-\lambda\theta} f(x(0)), & \theta \in [t_0 - 2\tau, t_0 - \tau) = [0, \tau), \\ \phi(\theta - \tau) - a\tau^2 f(\phi(\theta - \tau)), & \theta \in [t_0 - \tau, t_0) = [\tau, 2\tau]. \end{cases}
$$

Note that via (2.2) the order of nonlinearity of Eq. (3.4) is, generally speaking, more than one.

Definition 3.1. The solution of Eq. (3.4) with initial condition (3.5) is called asymptotically quasitrivial if $\lim_{j \to \infty} x(t + j\tau) = 0$ for each $t \in [t_0, t_0 + \tau)$.

Definition 3.2. The trivial solution of Eq. (3.4) is called stable if for any $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $|x(t)| < \epsilon$, for all $t \geq t_0$, if $\|\phi\| = \sup_{\theta \in [t_0 - 2\tau, t_0]} |\phi(\theta)| < \delta$.

Definition 3.3. The trivial solution of Eq. (3.4) is called asymptotically quasistable if it is stable and the solution of Eq. (3.4) is asymptotically quasitrivial for each initial condition (3.5).

Theorem 3.1. For a small enough $\tau > 0$ each bounded solution of Eq. (3.4) with initial condition (3.5) is asymptotically quasitrivial.

Proof. Using the general method of Lyapunov functional construction [10,11], we will construct a Lyapunov functional for Eq. (3.4) in the form $V(t) = V_1(t) + V_2(t)$, where

$$
V_1(t) = (x(t) - e^{-\lambda \tau} x(t - \tau))^2, \quad t \geq t_0,
\tag{3.6}
$$

is a Lyapunov functional for the auxiliary linear difference equation

$$
x(t + \tau) = x(t) + e^{-\lambda \tau}(x(t) - x(t - \tau)), \quad t > \tau.
\tag{3.7}
$$

Indeed, for Eq. (3.7) we have $\Delta V_1(t) = 0$.

Calculating $\Delta V_1(t)$ for Eq. (3.4) via (3.6), we obtain

$$
\begin{aligned}
\Delta V_1(t) &= (x(t + \tau) - e^{-\lambda \tau} x(t))^2 - (x(t) - e^{-\lambda \tau} x(t - \tau))^2 \\
&= (x(t) - e^{-\lambda \tau} x(t - \tau) - a\tau^2 f(x(t)))^2 - (x(t) - e^{-\lambda \tau} x(t - \tau))^2 \\
&= a^2 \tau^4 f^2(x(t)) - 2a\tau^2 f(x(t))x(t) + 2a\tau^2 e^{-\lambda \tau} f(x(t))x(t - \tau).
\end{aligned}
\tag{3.8}
$$

Using Lemma 1.1 for $\alpha = m_i$, $\beta = \gamma = 1$, from (3.8) we have

$$
\begin{aligned}
\Delta V_1(t) \leq{} & a^2\tau^4 f^2(x(t)) - 2a\tau^2 f(x(t))x(t) \\
& + 2a\tau^2 \mathrm{e}^{-\lambda\tau} \sum_{i=1}^{k} \alpha_i \left(\frac{m_i}{m_i+1} x^{m_i+1}(t) + \frac{1}{m_i+1} x^{m_i+1}(t-\tau) \right).
\end{aligned}
\tag{3.9}
$$

Put

$$
V_2(t) = 2a\tau^2 \mathrm{e}^{-\lambda\tau} \sum_{i=1}^{k} \frac{\alpha_i}{m_i+1} x^{m_i+1}(t-\tau), \quad t \geq t_0.
\tag{3.10}
$$

It follows from (2.3) that $V_2(t) \geq 0$. Estimating $\Delta V(t) = \Delta V_1(t) + \Delta V_2(t)$, via (3.9) and (3.10) we obtain

$$
\begin{aligned}
\Delta V(t) &\leq a^2\tau^4 f^2(x(t)) - 2a\tau^2(1 - \mathrm{e}^{-\lambda\tau}) f(x(t))x(t) \\
&\leq -\beta_1(\tau) f_1(x(t))(\beta_2(\tau) - f_2(x(t))), \quad t \geq t_0,
\end{aligned}
\tag{3.11}
$$

where

$$
\beta_1(\tau) = a^2\tau^4, \qquad \beta_2(\tau) = \frac{2(1 - \mathrm{e}^{-\lambda\tau})}{a\tau^2},
$$
$$
f_1(x) = \sum_{i=1}^{k} \alpha_i x^{m_i+1} > 0, \qquad f_2(x) = \sum_{i=1}^{k} \alpha_i x^{m_i-1} > 0, \quad x \neq 0.
\tag{3.12}
$$

Suppose that there exists $\tau_0 > 0$, such that the solution of Eq. (3.4) is uniformly bounded for $\tau \in [0, \tau_0]$, i.e., $|x(t)| \leq M$, $t \geq t_0$. Since $f_2(x)$ is a function that is non-decreasing for $x \geq 0$ and $\lim_{\tau \to 0} \beta_2(\tau) = \infty$, there exists a small enough $\tau > 0$ such that $f_2(x(t)) \leq f_2(M) < \beta_2(\tau)$. From here and (3.11) it follows that

$$
\Delta V(t) \leq -\gamma_1(\tau) f_1(x(t)), \quad t \geq t_0,
\tag{3.13}
$$

where $\gamma_1(\tau) = \beta_1(\tau)(\beta_2(\tau) - f_2(M)) > 0$. Rewrite (3.13) for $t + j\tau$, i.e.,

$$
\Delta V(t + j\tau) \leq -\gamma_1(\tau) f_1(x(t + j\tau)), \quad t \geq t_0, \; j = 0, 1, \ldots,
$$

and sum it from $j = 0$ to $j = i - 1$:

$$
V(t + i\tau) - V(t) \leq -\gamma_1(\tau) \sum_{j=0}^{i-1} f_1(x(t + j\tau)), \quad t \geq t_0.
\tag{3.14}
$$

From here it follows that

$$
\gamma_1(\tau) \sum_{j=0}^{\infty} f_1(x(t + j\tau)) \leq V(t) < \infty, \quad t \geq t_0.
$$

Therefore, $\lim_{j \to \infty} f_1(x(t + j\tau)) = 0$ for each $t \geq t_0$. Due to (2.3) and (3.12),

$$
0 \leq \alpha_1 x^{m_1+1}(t + j\tau) \leq f_1(x(t + j\tau)), \quad t \geq t_0.
$$

So, $\lim_{j \to \infty} |x(t + j\tau)| = 0$ for each $t \geq t_0$, i.e., the solution of Eq. (3.4) is asymptotically quasitrivial. The theorem is proven. $\square$

Theorem 3.2. *The trivial solution of Eq. (3.4) is stable.*

Proof. We will use here the functional $V(t)$, that was constructed in the proof of Theorem 3.1. Via (3.14) we have

$$
V(t + i\tau) \leq V(t), \quad i = 0, 1, \ldots, \; t \geq t_0.
$$

Putting $t = t_0 + j\tau + s$ with $j = [\frac{t - t_0}{\tau}]$ and $s \in [0, \tau)$, we obtain

$$
V(t_0 + (j+i)\tau + s) \leq V(t) = V(t_0 + j\tau + s) \leq V(t_0 + s).
\tag{3.15}
$$

From (3.6) we have

$$V_1(t_0 + s) = (x(t_0 + s) - e^{-\lambda\tau}\phi(t_0 + s - \tau))^2$$
$$\leq 2(|x(t_0 + s)|^2 + e^{-2\lambda\tau}\|\phi\|^2). \tag{3.16}$$

It follows from Eq. (3.4) and $t_0 + s - \tau = \tau + s \in [\tau, t_0)$ that

$$|x(t_0 + s)| \leq (1 + e^{-\lambda\tau})|\phi(\tau + s)| + a\tau^2|f(\phi(\tau + s))| + e^{-\lambda\tau}|\phi(s)|.$$

Due to (2.2),

$$|f(\phi(\theta))| \leq \sum_{i=1}^{k} \alpha_i |\phi(\theta)|^{m_i} \leq C_2\|\phi\|^{\nu}, \quad \theta \in [0, t_0],$$

where

$$C_2 = \sum_{i=1}^{k} \alpha_i, \qquad \nu = \begin{cases} 1 & \text{if } \|\phi\| \leq 1, \\ \max_{i=1,\dots,k} m_i & \text{if } \|\phi\| > 1. \end{cases}$$

Therefore,

$$|x(t_0 + s)| \leq C_3\|\phi\|^{\nu}, \qquad C_3 = 1 + 2e^{-\lambda\tau} + a\tau^2 C_2,$$

and using (3.16), we obtain

$$V_1(t_0 + s) \leq 2(C_3^2\|\phi\|^{2\nu} + e^{-2\lambda\tau}\|\phi\|^2). \tag{3.17}$$

It follows from (3.10) that

$$V_2(t_0 + s) = 2a\tau^2 e^{-\lambda\tau} \sum_{i=1}^{k} \frac{\alpha_i}{m_i + 1} \phi^{m_i+1}(t_0 + s - \tau) \leq C_1\|\phi\|^{\nu+1}, \tag{3.18}$$

where

$$C_1 = 2a\tau^2 e^{-\lambda\tau} \sum_{i=1}^{k} \frac{\alpha_i}{m_i + 1}.$$

From (3.15), (3.17) and (3.18), for the functional $V(t) = V_1(t) + V_2(t)$ there follows the inequality

$$V(t) \leq V(t_0 + s) \leq C_0\|\phi\|^{2\nu}, \quad t \geq t_0, \tag{3.19}$$
$$C_0 = C_1 + 2(C_3^2 + e^{-2\lambda\tau}).$$

Via (3.12), (3.14) and (3.19) we obtain

$$\gamma_1(\tau)\alpha_1|x(t)|^{m_1+1} \leq \gamma_1(\tau)f_1(x(t))$$
$$\leq \gamma_1(\tau) \sum_{j=0}^{i-1} f_1(x(t + j\tau)) \leq V(t) \leq C_0\|\phi\|^{2\nu}, \quad t \geq t_0.$$

So, for arbitrary $\epsilon > 0$ there exists a $\delta = \left(C_0^{-1}\gamma_1(\tau)\alpha_1\epsilon^{m_1+1}\right)^{\frac{1}{2\nu}} > 0$ such that $|x(t)| < \epsilon$, if $\|\phi\| < \delta$. The theorem is proven. $\quad\square$

Corollary 3.1. For a small enough $\tau > 0$ the trivial solution of Eq. (3.4) is asymptotically quasistable.

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