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Some Peculiarities of the General Method of Lyapunov Functionals Construction

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Abstract—The general method of Lyapunov functionals construction for stability investigation of stochastic hereditary systems which was proposed and developed before is considered. Some features of this method for difference systems which allow one to use the method more effectively are discussed.

Keywords—Method of Lyapunov functionals construction, Stability of stochastic difference equations.

Let $\{\Omega, \mathbf{P}, \sigma\}$ be a probability space, h be a given nonnegative number, i be a discrete time, $i \in Z_0 \cup Z$, $Z_0 = \{-h, \dots, 0\}$, $Z = \{0, 1, \dots\}$, $f_i \in \sigma$, $i \in Z$, be a sequence of σ-algebras, $\mathbf{E}$ be the mathematical expectation, $\xi_0, \xi_1, \dots$ be a sequence of mutually independent random variables, $\xi_i \in R^m$, ξ_i be f_{i+1}-adapted and independent on f_i, $\mathbf{E}\xi_i = 0$, $\mathbf{E}\xi_i\xi_i' = I$, $i \in Z$, I be identity matrix, process $x_i \in R^n$ be a solution of the equation

$$x_{i+1} = F(i, x_{-h}, \dots, x_i) + \sum_{j=0}^{i} G(i, j, x_{-h}, \dots, x_j)\xi_j, \qquad i \in Z, \tag{1}$$

with initial function

$$x_i = \varphi_i, \qquad i \in Z_0.$$

Here $F : Z * S \Leftrightarrow \mathbf{R}^n$, $G : Z * Z * S \Leftrightarrow \mathbf{R}^{n*m}$, S is a space of sequences with elements from $\mathbf{R}^n$. It is assumed that $F(i, \dots)$ is independent on x_j for $j > i$, $G(i, j, \dots)$ is independent on x_k for $k > j$, $F(i, 0, \dots, 0) = 0$, $G(i, j, 0, \dots, 0) = 0$.

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DEFINITION. *The zero solution of equation (1) is called mean square stable if for any* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that* $\mathbf{E}|x_i|^2 < \epsilon$, $i \in Z$, *if* $\|\varphi\|^2 = \sup_{i \in Z_0} \mathbf{E}|\varphi_i|^2 < \delta$. *If, besides,* $\lim_{i\to\infty} \mathbf{E}|x_i|^2 = 0$ *for all initial functions* φ, *then the zero solution of equation (1) is called asymptotically mean square stable.*

THEOREM. *(See [1].) Let there exist a nonnegative functional* $V_i = V(i, x_{-h}, \dots, x_i)$, $i \in Z$, *which satisfies the conditions*

$$\mathbf{E}V(0, x_{-h}, \dots, x_0) \le c_1 \|\varphi\|^2,$$
$$\mathbf{E}\Delta V_i \le -c_2 \mathbf{E}|x_i|^2, \qquad i \in Z,$$

where $\Delta V_i = V_{i+1} - V_i$, $c_1 > 0$, $c_2 > 0$. *Then the zero solution of equation (1) is asymptotically mean square stable.*

From Theorem 1 it follows that the stability investigation of stochastic equations can be reduced to construction of appropriate Lyapunov functionals. Following the general method of Lyapunov functionals construction, which was proposed and developed in [1–24], it is necessary to construct Lyapunov functional V_i in the form $V_i = V_{1i} + V_{2i}$, where the main component V_{1i} must be chosen by a special way. This choice is not unique. Hence, for each choice of V_{1i} we can construct other Lyapunov functionals and therefore to get other stability conditions. Besides choosing different ways of estimation of $\mathbf{E}\Delta V_{1i}$ we can construct different Lyapunov functionals and as a result can again obtain different stability conditions.

Let us demonstrate these peculiarities of the general method of Lyapunov functionals construction for the equation

$$x_{i+1} = a x_i + b \sum_{j=1}^{k} (k+1-j) x_{i-j} + \sigma \sum_{j=0}^{m} (m+1-j) x_{i-j} \xi_i. \tag{2}$$

Here it is supposed that $k \ge 0$, $m \ge 0$.

Following [1], we will construct Lyapunov functional for equation (2) in the form $V_i = V_{1i} + V_{2i}$, where $V_{1i} = x_i^2$. Calculating $\mathbf{E}\Delta V_{1i}$ by virtue of (2), we get

$$\begin{aligned}
\mathbf{E}\Delta V_{1i} &= \mathbf{E}\left(x_{i+1}^2 - x_i^2\right) = \left(a^2 - 1\right)\mathbf{E}x_i^2 + 2ab \sum_{j=1}^{k} (k+1-j)\mathbf{E}x_i x_{i-j} \\
&\quad + b^2 \mathbf{E}\left(\sum_{j=1}^{k} (k+1-j) x_{i-j}\right)^2 + \sigma^2 \mathbf{E}\left(\sum_{j=0}^{m} (m+1-j) x_{i-j}\xi_i\right)^2 \\
&\le (a^2-1)\mathbf{E}x_i^2 + |ab| \sum_{j=1}^{k} (k+1-j)\left(\mathbf{E}x_i^2 + \mathbf{E}x_{i-j}^2\right) \\
&\quad + b^2 \sum_{l=1}^{k} (k+1-l) \sum_{j=1}^{k} (k+1-j)\mathbf{E}x_{i-j}^2 \\
&\quad + \sigma^2 \sum_{l=0}^{m} (m+1-l) \sum_{j=0}^{m} (m+1-j)\mathbf{E}x_{i-j}^2 \\
&= \left(a^2 + |ab| \frac{k(k+1)}{2} + \sigma^2 \frac{(m+1)^2(m+2)}{2} - 1\right)\mathbf{E}x_i^2 \\
&\quad + \left(|ab| + b^2 \lambda_{k1}\right) \sum_{j=1}^{k} (k+1-j)\mathbf{E}x_{i-j}^2 + \sigma^2 \lambda_{m0} \sum_{j=1}^{m} (m+1-j)\mathbf{E}x_{i-j}^2,
\end{aligned}$$

where

$$\lambda_{kl} = \frac{1}{2}(k+1-l)(k+2-l). \tag{3}$$

Choosing the functional V_{2i} in the form

$$V_{2i} = \sum_{l=i-k}^{i-1} A_{i-l} x_l^2 + \sum_{l=i-m}^{i-1} B_{i-l} x_l^2,$$

where

$$A_l = \left(|ab| + b^2 \lambda_{k1}\right) \lambda_{kl}, \qquad B_l = \sigma^2 \lambda_{m0} \lambda_{ml}, \tag{4}$$

we get

$$\begin{aligned}
\Delta V_{2i} &= \sum_{l=i+1-k}^{i} A_{i+1-l} x_l^2 + \sum_{l=i+1-m}^{i} B_{i+1-l} x_l^2 - V_{2i} \\
&= (A_1 + B_1) x_i^2 + \sum_{l=i+1-k}^{i-1} (A_{i+1-l} - A_{i-l}) x_l^2 - A_k x_{i-k}^2 \\
&\quad + \sum_{l=i+1-m}^{i-1} (B_{i+1-l} - B_{i-l}) x_l^2 - B_m x_{i-m}^2 \\
&= (A_1 + B_1) x_i^2 - \left(|ab| + b^2 \lambda_{k1}\right) \cdot \sum_{j=1}^{k} (k+1-j) x_{i-j}^2 - \sigma^2 \lambda_{m0} \sum_{j=1}^{m} (m+1-j) x_{i-j}^2.
\end{aligned}$$

From (3),(4), it follows that

$$A_1 = |ab| \frac{k(k+1)}{2} + b^2 \frac{k^2(k+1)^2}{4}, \qquad B_1 = \sigma^2 \frac{m(m+1)^2(m+2)}{4}.$$

Therefore, for $V_i = V_{1i} + V_{2i}$ we obtain

$$\mathbf{E}\Delta V_i \le \left[\left(|a| + |b| \frac{k(k+1)}{2}\right)^2 + p - 1\right] \mathbf{E} x_i^2,$$

where

$$p = \frac{\sigma^2}{4} (m+1)^2 (m+2)^2.$$

From here and Theorem 1, it follows that the inequality

$$\left(|a| + |b| \frac{k(k+1)}{2}\right)^2 + p < 1 \tag{5}$$

is a sufficient condition of asymptotic mean square stability of the zero solution of equation (2).

Let us show that choosing another way of estimation of $\mathbf{E}\Delta V_{1i}$ and supposing some additional conditions on a and b, we can get stability conditions which differ from (5).

So let us use the functional V_{1i} in the form $V_{1i} = x_i^2$ again. Calculating $\mathbf{E}\Delta V_{1i}$ and using (3), we get as previously

$$\begin{aligned}
\mathbf{E}\Delta V_{1i} &= \left(a^2 - 1\right) \mathbf{E} x_i^2 + 2ab \sum_{j=1}^{k} (k+1-j) \mathbf{E} x_i x_{i-j} \\
&\quad + b^2 \mathbf{E} \left(\sum_{j=1}^{k} (k+1-j) x_{i-j}\right)^2 + \sigma^2 \mathbf{E} \left(\sum_{j=0}^{m} (m+1-j) x_{i-j} \xi_i\right)^2 \\
&\le \left(a^2 + \sigma^2 \frac{(m+1)^2(m+2)}{2} - 1\right) \mathbf{E} x_i^2 + 2ab \sum_{j=1}^{k} (k+1-j) \mathbf{E} x_i x_{i-j} \\
&\quad + b^2 \lambda_{k1} \sum_{j=1}^{k} (k+1-j) \mathbf{E} x_{i-j}^2 + \sigma^2 \lambda_{m0} \sum_{j=1}^{m} (m+1-j) \mathbf{E} x_{i-j}^2.
\end{aligned}$$

Suppose now that $ab \le 0$ and put

$$V_{2i} = |ab| \sum_{j=1}^{k+1} \left(\sum_{l=1}^{j} x_{i-l} \right)^2 + \sum_{l=i-k}^{i-1} C_{i-l} x_l^2 + \sum_{l=i-m}^{i-1} B_{i-l} x_l^2,$$

where B_l are defined by (4), (3), and $C_l = b^2 \lambda_{k1} \lambda_{kl}$.

Calculating $\mathbf{E}\Delta V_{2i}$ as previously, we get

$$\mathbf{E}\Delta V_{2i} = |ab| \mathbf{E}\gamma_i + (C_1 + B_1)\mathbf{E}x_i^2$$
$$-b^2 \lambda_{k1} \sum_{j=1}^{k} (k+1-j)\mathbf{E}x_{i-j}^2 - \sigma^2 \lambda_{m0} \sum_{j=1}^{m} (m+1-j)\mathbf{E}x_{i-j}^2,$$

where

$$\gamma_i = \sum_{j=1}^{k+1} \left[\left(\sum_{l=1}^{j} x_{i+1-l} \right)^2 - \left(\sum_{l=1}^{j} x_{i-l} \right)^2 \right].$$

Note that

$$\begin{aligned}
\gamma_i &= \sum_{j=1}^{k+1} \left[\left(\sum_{l=0}^{j-1} x_{i-l} \right)^2 - \left(\sum_{l=1}^{j} x_{i-l} \right)^2 \right] \\
&= \sum_{j=1}^{k+1} \left[\left(x_i + \sum_{l=1}^{j-1} x_{i-l} \right)^2 - \left(x_{i-j} + \sum_{l=1}^{j-1} x_{i-l} \right)^2 \right] \\
&= \sum_{j=1}^{k+1} \left[x_i^2 + 2x_i \sum_{l=1}^{j-1} x_{i-l} - \left(x_{i-j}^2 + 2x_{i-j} \sum_{l=1}^{j-1} x_{i-l} \right) \right] \\
&= (k+1)x_i^2 + 2x_i \sum_{l=1}^{k} (k+1-l) x_{i-l} - \rho_i,
\end{aligned}$$

where

$$\rho_i = \sum_{j=1}^{k+1} \left(x_{i-j}^2 + 2x_{i-j} \sum_{l=1}^{j-1} x_{i-l} \right).$$

It is easy to see that

$$\begin{aligned}
\rho_i &= \sum_{j=1}^{k+1} x_{i-j}^2 + \sum_{j=1}^{k+1} x_{i-j} \sum_{l=1}^{j-1} x_{i-l} + \sum_{l=1}^{k} x_{i-l} \sum_{j=l+1}^{k+1} x_{i-j} \\
&= \sum_{j=1}^{k+1} x_{i-j} \sum_{l=1}^{j} x_{i-l} + \sum_{j=1}^{k+1} x_{i-j} \sum_{l=j+1}^{k+1} x_{i-l} \\
&= \sum_{j=1}^{k+1} x_{i-j} \sum_{l=1}^{k+1} x_{i-l} = \left(\sum_{j=1}^{k+1} x_{i-j} \right)^2 \ge 0.
\end{aligned}$$

It means that

$$\gamma_i \le (k+1)x_i^2 + 2x_i \sum_{l=1}^{k} (k+1-l) x_{i-l}.$$

Note that

$$\sigma^2 \frac{(m+1)^2(m+2)}{2} + B_1 = p.$$

Therefore, for $V_i = V_{1i} + V_{2i}$, we have

$$\mathbf{E}\Delta V_i \leq \left(a^2 + |ab|(k+1) + b^2\,\frac{k^2(k+1)^2}{4} + p - 1\right)\mathbf{E}x_i^2.$$

Thus, the inequalities

$$a^2 + |ab|(k+1) + b^2\,\frac{k^2(k+1)^2}{4} + p < 1, \qquad ab \leq 0, \tag{6}$$

are a sufficient condition of asymptotic mean square stability of the zero solution of equation (2).

Rewriting condition (5) in the form

$$a^2 + |ab|k(k+1) + b^2\,\frac{k^2(k+1)^2}{4} + p < 1$$

and collating conditions (5) and (6) by $ab \leq 0$, it is easy to see that by $k = 1$ condition (6) coincides with (5), but by $k > 1$ condition (6) is better than (5).

REMARK. As it follows from [1] constructing other Lyapunov functionals we can obtain sufficient conditions of asymptotic mean square stability of the zero solution of equation (2) in other forms, for example,

$$p < \left(1 - a - b\,\frac{k(k+1)}{2}\right)\left(1 + a + b\,\frac{k(k+1)}{2} - |b|\,\frac{k(k+1)(k+2)}{3}\right), \qquad \left|a + b\,\frac{k(k+1)}{2}\right| < 1,$$

or

$$\frac{|b|\,(k(k-1)/2)\,[2|a| + (1-bk)|b|\,(k(k+3)/2)] + (1-bk)p}{(1+bk)\,[(1-bk)^2 - a^2]} < 1, \qquad |b|k < 1, \quad |a| < 1 - bk.$$

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