

—Brief Paper—

# SOME NEW ASPECT OF LYAPUNOV TYPE THEOREMS FOR STOCHASTIC DIFFERENCE EQUATIONS WITH CONTINUOUS TIME

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## ABSTRACT

Some new Lyapunov type theorems for stochastic difference equations with continuous time are proven. It is shown that these theorems simplify an application of Lyapunov functionals construction method.

**KeyWords:** Lyapunov type theorems, stochastic difference equations, stability, method of Lyapunov functionals construction.

Stability investigation of hereditary systems [1-3] is connected often with construction of some appropriate Lyapunov functionals. One general method of Lyapunov functionals construction was proposed and developed in [4-11] for both stochastic differential equations with after-effect and stochastic difference equations with discrete time. After some modification of the basic Lyapunov-type stability theorem, this method was also used for stochastic difference equations with continuous time [12-14], which are popular enough in researches [15-20]. Here some new aspect of Lyapunov type theorems is shown, which allows to simplify an application of the general method of Lyapunov functionals construction for stochastic difference equations with continuous time. The theorems obtained here can similarly be applied for stochastic differential equations and stochastic difference equations with discrete time.

## I. DEFINITIONS AND BASIC LYAPUNOV TYPE THEOREM

Let  $\{\Omega, \mathcal{F}, \mathbf{P}\}$  be a probability space,  $\{\mathcal{F}_t, t \geq t_0\}$  be a nondecreasing family of sub- $\sigma$ -algebras of  $\mathcal{F}$ , i.e.  $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$  for  $t_1 < t_2$ ,  $\mathbf{E}$  be the expectation with respect to

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the measure  $\mathbf{P}$ ,  $\mathbf{E}_t = \mathbf{E}(\cdot/\mathcal{F}_t)$  be the conditional expectation with respect to  $\sigma$ -algebra  $\mathcal{F}_t$ .

Consider the stochastic difference equation

$$\begin{aligned} x(t+h_0) = & a_1(t, x(t), x(t-h_1), x(t-h_2), \dots) \\ & + a_2(t, x(t), x(t-h_1), x(t-h_2), \dots) \xi(t+h_0), \\ & t > t_0 - h_0, \end{aligned} \quad (1)$$

with the initial condition

$$x(\theta) = \phi(\theta), \quad \theta \in \Theta = [t_0 - h, t_0], \quad h = h_0 + \max_{j \geq 1} h_j \quad (2)$$

Here  $x \in \mathbf{R}^n$ ,  $h_0, h_1, \dots$  are positive constants, the functionals  $a_1 \in \mathbf{R}^n$  and  $a_2 \in \mathbf{R}^{n \times m}$  satisfy the condition

$$|a_i(t, x_0, x_1, x_2, \dots)|^2 \leq \sum_{j=0}^{\infty} a_{ij} |x_j|^2, \quad A = \sum_{i=1}^2 \sum_{j=0}^{\infty} a_{ij} < \infty, \quad (3)$$

$\phi(\theta)$ ,  $\theta \in \Theta$ , is a  $\mathcal{F}_{t_0}$ -measurable function, the perturbation  $\xi(t) \in \mathbf{R}^m$  is a  $\mathcal{F}_t$ -measurable stationary stochastic process such that

$$\mathbf{E}_t \xi(s) = 0, \quad \mathbf{E}_t \xi(s) \xi'(s) = I, \quad s - t \geq h_0. \quad (4)$$

A solution of problem (1), (2) is an  $\mathcal{F}_t$ -measurable process  $x(t) = x(t; t_0, \phi)$ , which is equal to the initial function  $\phi(t)$  from (2) for  $t \leq t_0$  and with probability 1 defined by (1) for  $t > t_0$ .

**Definition 1.1.** The solution of Eq. (1) with initial condition (2) is called uniformly mean square bounded if there exists a positive number  $C$  such that for all  $t \geq t_0$

$$\mathbf{E} |x(t; t_0, \phi)|^2 \leq C. \quad (5)$$

**Definition 1.2.** The trivial solution of Eqs. (1), (2) is called mean square stable if for any  $\epsilon > 0$  and  $t_0$  there exists a  $\delta = \delta(\epsilon, t_0) > 0$  such that  $\mathbf{E}|x(t; t_0, \phi)|^2 < \epsilon$  for all  $t \geq t_0$  if  $\|\phi\|^2 = \sup_{\theta \in \Theta} \mathbf{E}|\phi(\theta)|^2 < \delta$ .

**Definition 1.3.** The solution of Eq. (1) with initial condition (2) is called asymptotically mean square trivial if

$$\lim_{t \rightarrow \infty} \mathbf{E} |x(t; t_0, \phi)|^2 = 0. \quad (6)$$

**Definition 1.4.** The solution of Eq. (1) with initial condition (2) is called asymptotically mean square quasitrivial if for each  $t \in [t_0, t_0 + h_0)$

$$\lim_{j \rightarrow \infty} \mathbf{E} |x(t + jh_0; t_0, \phi)|^2 = 0. \quad (7)$$

**Definition 1.5.** The trivial solution of Eqs. (1), (2) is called asymptotically mean square stable if it is mean square stable and for each initial function  $\phi$  the solution of Eq. (1) is asymptotically mean square trivial.

**Definition 1.6.** The trivial solution of Eqs. (1), (2) is called asymptotically mean square quasistable if it is mean square stable and for each initial function  $\phi$  the solution of Eq. (1) is asymptotically mean square quasitrivial.

**Definition 1.7.** The solution of Eq. (1) with initial condition (2) is called uniformly mean square summable if

$$\sup_{t \in [t_0, t_0 + h_0)} \sum_{j=0}^{\infty} \mathbf{E} |x(t + jh_0; t_0, \phi)|^2 < \infty. \quad (8)$$

**Definition 1.8.** The solution of Eq. (1) with initial condition (2) is called mean square integrable if

$$\int_{t_0}^{\infty} \mathbf{E} |x(t; t_0, \phi)|^2 dt < \infty. \quad (9)$$

**Remark 1.1.** If the solution of Eqs. (1), (2) is asymptotically mean square trivial then it is also asymptotically mean square quasitrivial, but the inverse statement is not true.

**Remark 1.2.** It follows from condition (8) that

$$\sup_{t \geq t_0} \sum_{j=0}^{\infty} \mathbf{E} |x(t + jh_0; t_0, \phi)|^2 < \infty.$$

Really, arbitrary  $t \geq t_0$  can be represented in the form  $t = s + kh_0$  with an integer  $k \geq 0$  and  $s \in [t_0, t_0 + h_0)$ . So,

$$\begin{aligned} & \sum_{j=0}^{\infty} \mathbf{E} |x(t + jh_0; t_0, \phi)|^2 \\ &= \sum_{j=0}^{\infty} \mathbf{E} |x(s + kh_0 + jh_0; t_0, \phi)|^2 \\ &= \sum_{j=k}^{\infty} \mathbf{E} |x(s + jh_0; t_0, \phi)|^2 \\ &\leq \sup_{s \in [t_0, t_0 + h_0)} \sum_{j=0}^{\infty} \mathbf{E} |x(s + jh_0; t_0, \phi)|^2 < \infty. \end{aligned}$$

**Remark 1.3.** If the solution of Eqs. (1), (2) is uniformly mean square summable then it is uniformly mean square bounded and asymptotically mean square quasitrivial.

Everywhere below it is supposed that the functional  $V(t) = V(t, x(t), x(t - h_1), x(t - h_2), \dots)$  equals zero if and only if  $x(t) = x(t - h_1) = x(t - h_2) = \dots = 0$ . Let also  $[t]$  be the integer part of a number  $t$  and  $\Delta V(t) = V(t + h_0) - V(t)$ .

**Theorem 1.1.** [14] Let there exists a nonnegative functional  $V(t) = V(t, x(t), x(t - h_1), x(t - h_2), \dots)$ , such that

$$\mathbf{E}V(t) \leq c_1 \sup_{s \leq t} \mathbf{E} |x(s)|^2, \quad t \in [t_0, t_0 + h_0), \quad (10)$$

$$\mathbf{E}\Delta V(t) \leq -c_2 \mathbf{E} |x(t)|^2, \quad t \geq t_0, \quad (11)$$

where  $c_1, c_2$  are positive numbers. Then the trivial solution of Eqs. (1), (2) is asymptotically mean square quasistable.

**Remark 1.4.** [14] If the conditions of Theorem 1.1 hold then the solution of Eq. (1) for each initial function (2) is uniformly mean square summable and mean square integrable.

**Corollary 1.1.** Let there exists a nonnegative functional  $V(t) = V(t, x(t), x(t - h_1), x(t - h_2), \dots)$ , which satisfies conditions (10) and  $\mathbf{E}\Delta V(t) = -c \mathbf{E}|x(t)|^2, t \geq t_0$ . Then the inequality  $c > 0$  is the necessary and sufficient condition for asymptotic mean square quasistability of the trivial solution of Eq. (1).

**Proof.** A sufficiency follows from Theorem 1.1. To prove a necessity it is enough to note that if  $c \leq 0$  then

$$\sum_{j=0}^i \mathbf{E}\Delta V(t + jh_0) = \mathbf{E}V(t + ih_0) - \mathbf{E}V(t) \geq 0$$

or  $\mathbf{E}V(t + ih_0) \geq \mathbf{E}V(t) > 0$ . It means that the trivial solution of Eq. (1) cannot be asymptotically mean square quasistable. ■

## II. FORMAL PROCEDURE OF LYAPUNOV FUNCTIONALS CONSTRUCTION

It follows from Theorem 1.1 and Remark 1.4 that an investigation of asymptotic behaviour of the solution of Eq. (1) can be reduced to construction of appropriate Lyapunov functionals.

Below some formal procedure of Lyapunov functionals construction for Eqs. (1), (2) is proposed. This procedure consists of four steps.

Step 1. Represent the functionals  $a_1$  and  $a_2$  at the right-hand side of Eq. (1) in the form

$$\begin{aligned} a_1(t, x(t), x(t-h_1), x(t-h_2), \dots) &= F_1(t) + F_2(t) + \Delta F_3(t), \\ a_2(t, x(t), x(t-h_1), x(t-h_2), \dots) &= G_1(t) + G_2(t), \end{aligned} \quad (12)$$

where

$$\begin{aligned} F_1(t) &= F_1(t, x(t), x(t-h_1), \dots, x(t-h_k)), \\ G_1(t) &= G_1(t, x(t), x(t-h_1), \dots, x(t-h_k)), \quad k \geq 0, \\ F_j(t) &= F_j(t, x(t), x(t-h_1), x(t-h_2), \dots), \quad j = 2, 3, \\ G_2(t) &= G_2(t, x(t), x(t-h_1), x(t-h_2), \dots), \\ F_1(t, 0, \dots, 0) &\equiv F_2(t, 0, 0, \dots) \equiv F_3(t, 0, 0, \dots) \\ &\equiv G_1(t, 0, \dots, 0) \equiv G_2(t, 0, 0, \dots) \equiv 0. \end{aligned}$$

Step 2. Suppose that for the auxiliary equation

$$\begin{aligned} y(t+h_0) &= F_1(t, y(t), y(t-h_1), \dots, y(t-h_k)) \\ &+ G_1(t, y(t), y(t-h_1), \dots, y(t-h_k)) \xi(t+h_0), \\ &t > t_0 - h_0, \end{aligned}$$

there exists a Lyapunov functional  $v(t) = v(t, y(t), y(t-h_1), \dots, y(t-h_k))$ , which satisfies the conditions of Theorem 1.1.

Step 3. Consider Lyapunov functional  $V(t)$  for Eq. (1) in the form  $V(t) = V_1(t) + V_2(t)$ , where the main component is  $V_1(t) = v(t, x(t) - F_3(t), x(t-h_1), \dots, x(t-h_k))$ . Calculate  $\mathbf{E}\Delta V_1(t)$  and, in a reasonable way, estimate it.

Step 4. In order to satisfy the conditions of Theorem 1.1, the additional component  $V_2(t)$  of the functional  $V(t)$  is chosen by some standard way.

Construction of Lyapunov functionals via this procedure is demonstrated in [4-14] for different types of hereditary systems.

Note that some standard way for construction of additional functional  $V_2$  allows to simplify the fourth step of the procedure and do not use the functional  $V_2$  at all. Below corresponding auxiliary Lyapunov type theorems are considered.

## III. AUXILIARY LYAPUNOV TYPE THEOREMS

The following theorems in some cases allow to construct Lyapunov functionals with conditions that are weaker than (11).

**Theorem 3.1.** Let there exists a nonnegative functional  $V_1(t) = V_1(t, x(t), x(t-h_1), x(t-h_2), \dots)$ , which satisfies condition (10) and the conditions

$$\mathbf{E}\Delta V_1(t) \leq a \mathbf{E} |x(t)|^2 + \sum_{j=1}^{N(t)} A(t, t-jh_0) \mathbf{E} |x(t-jh_0)|^2,$$

$$N(t) = \left\lceil \frac{t+h}{h_0} \right\rceil, \quad A(t, s) \geq 0, \quad s \leq t, \quad t \geq t_0, \quad (13)$$

$$a + b < 0, \quad b = \sup_{t \geq t_0} \sum_{j=1}^{\infty} A(t + jh_0, t). \quad (14)$$

Then the trivial solution of Eqs. (1), (2) is asymptotically mean square quasistable.

**Proof.** According to the procedure of Lyapunov functionals construction described above, let us construct the functional  $V(t)$  in the form  $V(t) = V_1(t) + V_2(t)$ , where  $V_1(t)$  satisfies conditions (13), (14) and

$$V_2(t) = \sum_{m=1}^{N(t)} \mathbf{E} |x(t-mh_0)|^2 \sum_{j=m}^{\infty} A(t + (j-m)h_0, t-mh_0).$$

Note that  $N(t+h_0) = N(t) + 1$ . So, calculating  $\Delta V_2(t)$ , we obtain

$$\begin{aligned} \Delta V_2(t) &= \sum_{m=1}^{N(t)+1} \mathbf{E} |x(t+h_0-mh_0)|^2 \\ &\quad \cdot \sum_{j=m}^{\infty} A(t+h_0+(j-m)h_0, t+h_0-mh_0) - V_2(t) \\ &= \mathbf{E} |x(t)|^2 \sum_{j=1}^{\infty} A(t+jh_0, t) \\ &\quad + \sum_{m=2}^{N(t)+1} \mathbf{E} |x(t+h_0-mh_0)|^2 \\ &\quad \cdot \sum_{j=m}^{\infty} A(t+h_0+(j-m)h_0, t+h_0-mh_0) - V_2(t) \\ &= \mathbf{E} |x(t)|^2 \sum_{j=1}^{\infty} A(t+jh_0, t) \\ &\quad + \sum_{k=1}^{N(t)} \mathbf{E} |x(t-kh_0)|^2 \sum_{j=k+1}^{\infty} A(t+(j-k)h_0, t-kh_0) \\ &\quad - \sum_{m=1}^{N(t)} \mathbf{E} |x(t-mh_0)|^2 \sum_{j=m}^{\infty} A(t+(j-m)h_0, t-mh_0) \\ &= \mathbf{E} |x(t)|^2 \sum_{j=1}^{\infty} A(t+jh_0, t) \\ &\quad - \sum_{m=1}^{N(t)} A(t, t-mh_0) \mathbf{E} |x(t-mh_0)|^2 \end{aligned}$$

From here and (13), (14), for the functional  $V(t) = V_1(t) + V_2(t)$  we get  $\mathbf{E}\Delta V(t) \leq (a + b) \mathbf{E}|x(t)|^2$ . Together with (14) this inequality implies (11). So, there exists the functional  $V(t)$ , which satisfies the conditions of Theorem 1, *i.e.*, the trivial solution of Eqs. (1), (2) is asymptotically mean square quasistable. The theorem is proven. ■

**Theorem 3.2.** Let there exists a nonnegative functional  $V(t) = V(t, x(t), x(t - h_1), x(t - h_2), \dots)$ , which satisfies conditions (10) and

$$\mathbf{E}\Delta V(t) \leq a \mathbf{E}|x(t)|^2 + b \mathbf{E}|x(t - kh_0)|^2, \quad t \geq t_0, \quad (15)$$

where  $k$  is a positive integer. If the solution of Eqs. (1), (2) is uniformly mean square bounded but is not uniformly mean square summable then

$$a + b \geq 0. \quad (16)$$

**Proof.** Rewrite (15) for  $t + jh_0$  with  $t \geq t_0, j = 0, 1, \dots, i.e.,$

$$\mathbf{E}\Delta V(t + jh_0) \leq a \mathbf{E}|x(t + jh_0)|^2 + b \mathbf{E}|x(t + (j - k)h_0)|^2. \quad (17)$$

Summing (17) from  $j = 0$  to  $j = i + k$ , we obtain

$$\begin{aligned} & \mathbf{E}V(t + (i + k + 1)h_0) - \mathbf{E}V(t) \\ & \leq a \sum_{j=0}^{i+k} \mathbf{E}|x(t + jh_0)|^2 + b \sum_{j=0}^{i+k} \mathbf{E}|x(t + (j - k)h_0)|^2 \\ & = a \sum_{j=0}^{i+k} \mathbf{E}|x(t + jh_0)|^2 + b \sum_{j=0}^i \mathbf{E}|x(t + jh_0)|^2 \\ & \quad + b \sum_{j=-k}^{-1} \mathbf{E}|x(t + jh_0)|^2 \\ & = (a + b) \sum_{j=0}^i \mathbf{E}|x(t + jh_0)|^2 + a \sum_{j=i+1}^{i+k} \mathbf{E}|x(t + jh_0)|^2 \\ & \quad + b \sum_{j=-k}^{-1} \mathbf{E}|x(t + jh_0)|^2. \end{aligned}$$

From here and  $V(t) \geq 0$  it follows

$$\begin{aligned} & -(a + b) \sum_{j=0}^i \mathbf{E}|x(t + jh_0)|^2 \\ & \leq \mathbf{E}V(t) + a \sum_{j=i+1}^{i+k} \mathbf{E}|x(t + jh_0)|^2 \\ & \quad + b \sum_{j=-k}^{-1} \mathbf{E}|x(t + jh_0)|^2, \quad t \geq t_0. \quad (18) \end{aligned}$$

Consider  $t \in [t_0, t_0 + h_0]$ . Since the solution of Eqs. (1), (2) is uniformly mean square bounded, *i.e.*,  $\mathbf{E}|x(t)|^2 \leq C$ , then using (10), (2), we have

$$-(a + b) \sum_{j=0}^i \mathbf{E}|x(t + jh_0)|^2 \leq c_1 C + k(|a|C + |b| \|\phi\|^2) < \infty.$$

Let us suppose that (16) does not hold, *i.e.*,  $a + b < 0$ . Then condition (8) holds, *i.e.*, the solution of Eqs. (1), (2) is uniformly mean square summable, and we obtain the contradiction with the condition of Theorem 3.2. Therefore, (16) holds. The theorem is proven. ■

**Corollary 3.1.** Let there exists a nonnegative functional  $V(t) = V(t, x(t), x(t - h_1), x(t - h_2), \dots)$ , which satisfies conditions (10), (15) and

$$a + b < 0. \quad (19)$$

Then the solution of Eqs. (1), (2) is either mean square unbounded or uniformly mean square summable.

**Corollary 3.2.** Let there exists a nonnegative functional  $V(t) = V(t, x(t), x(t - h_1), x(t - h_2), \dots)$ , which satisfies conditions (10) and (15). If the solution of Eqs. (1), (2) is uniformly mean square bounded but is not mean square integrable then condition (16) holds. If condition (19) holds then the solution of Eqs. (1), (2) is either mean square unbounded or mean square integrable.

Really, integrating (15) from  $s = t_0$  to  $s = T$ , we obtain

$$\begin{aligned} \int_{t_0}^T \mathbf{E}\Delta V(s) ds & = \int_T^{T+h_0} \mathbf{E}V(s) ds - \int_{t_0}^{t_0+h_0} \mathbf{E}V(s) ds \\ & \leq a \int_{t_0}^T \mathbf{E}|x(s)|^2 ds + b \int_{t_0-kh_0}^{T-kh_0} \mathbf{E}|x(s)|^2 ds \\ & = (a + b) \int_{t_0}^T \mathbf{E}|x(s)|^2 ds + b \int_{t_0-kh_0}^{t_0} \mathbf{E}|x(s)|^2 ds \\ & \quad - b \int_{T-kh_0}^T \mathbf{E}|x(s)|^2 ds. \end{aligned}$$

Using  $V(t) \geq 0$ , (10), (2), we have

$$\begin{aligned} & -(a + b) \int_{t_0}^T \mathbf{E}|x(s)|^2 ds \\ & \leq \int_{t_0}^{t_0+h_0} \mathbf{E}V(s) ds + b \int_{t_0-kh_0}^{t_0} \mathbf{E}|x(s)|^2 ds - b \int_{T-kh_0}^T \mathbf{E}|x(s)|^2 ds \\ & \leq h_0 [c_1 C + k|b|(\|\phi\|^2 + C)]. \end{aligned}$$

The statement of Corollary 3.2 follows from here similarly to Theorem 3.2 and Corollary 3.1.

**Theorem 3.3.** Let there exists a nonnegative functional  $V(t) = V(t, x(t), x(t - h_1), x(t - h_2), \dots)$ , which satisfies conditions (10), (15), (19). If  $a > 0$  then each uniformly mean square bounded solution of Eq. (1) is asymptotically mean square quasitrivial. If  $a \leq 0$  then the trivial solution of Eq. (1) is asymptotically mean square quasistable.

**Proof.** It follows from Corollary 1.1 that by conditions (10), (15), (19) each uniformly mean square bounded solution of Eq. (1) is uniformly mean square summable and, therefore,

it is asymptotically mean square quasitrivial. Let us show that if  $a \leq 0$  then the trivial solution of Eq. (1) is stable. Really, if  $a = 0$  then from (19) we have  $b < 0$ . So, it follows from (15) that

$$\mathbf{E}\Delta V(t) \leq b \mathbf{E} |x(t - kh_0)|^2 \leq 0. \quad (20)$$

Rewrite condition (20) in the form  $\mathbf{E}\Delta V(t + jh_0) \leq b \mathbf{E} |x(t + (j - k)h_0)|^2$ ,  $t \geq t_0$ ,  $j = 0, 1, \dots$ . Summing this inequality from  $j = 0$  to  $j = k$ , we obtain

$$\mathbf{E}V(t + (k + 1)h_0) - \mathbf{E}V(t) \leq b \sum_{j=0}^k \mathbf{E} |x(t + (j - k)h_0)|^2.$$

Therefore,

$$|b| \mathbf{E} |x(t)|^2 \leq |b| \sum_{j=0}^k \mathbf{E} |x(t + (j - k)h_0)|^2 \leq \mathbf{E}V(t), \quad t \geq t_0. \quad (21)$$

It follows also from (20) that

$$\mathbf{E}V(t) \leq \mathbf{E}V(t - h_0) \leq \mathbf{E}V(t - 2h_0) \leq \dots \leq \mathbf{E}V(s), \quad t \geq t_0, \quad (22)$$

where  $s = t - \left[ \frac{t - t_0}{h_0} \right] h_0 \in [t_0, t_0 + h_0)$ . From (10) we get

$$\sup_{s \in [t_0, t_0 + h_0)} \mathbf{E}V(s) \leq c_1 \sup_{t \leq t_0 + h_0} \mathbf{E} |x(t)|^2. \quad (23)$$

Using (1)-(4), for  $t \leq t_0 + h_0$ , we have

$$\begin{aligned} \mathbf{E} |x(t)|^2 &= \sum_{l=1}^2 \mathbf{E} |a_l(t - h_0, x(t - h_0), x(t - h_0 - h_1), \\ &\quad x(t - h_0 - h_2), \dots)|^2 \\ &\leq \sum_{l=1}^2 \left( a_{l0} \mathbf{E} |\phi(t - h_0)|^2 + \sum_{j=1}^{\infty} a_{lj} \mathbf{E} |\phi(t - h_0 - h_j)|^2 \right) \\ &\leq A \|\phi\|^2. \end{aligned} \quad (24)$$

From (21)-(24) we obtain  $|b| \mathbf{E} |x(t)|^2 \leq c_1 A \|\phi\|^2$ ,  $t \geq t_0$ . It means that the trivial solution of Eqs. (1), (2) is mean square quasistable.

Let  $a < 0$ . If  $b \leq 0$  then condition (10) follows from (15). So, it follows from Theorem 1.1 that the trivial solution of Eq. (1) is asymptotically mean square quasistable. If  $b > 0$  then condition (15) is a particular case of (13). It follows from here and (10), (19) that the functional  $V(t)$  satisfies the conditions of Theorem 3.1 and, therefore, the trivial solution of Eqs. (1), (2) is asymptotically mean square quasistable. The theorem is proven. ■

**Corollary 3.3.** Let there exists a nonnegative functional  $V(t) = V(t, x(t), x(t - h_1), x(t - h_2), \dots)$ , which satisfies conditions (10) and

$$\mathbf{E}\Delta V(t) = a \mathbf{E} |x(t)|^2 + b \mathbf{E} |x(t - kh_0)|^2, \quad b > 0, \quad t \geq t_0.$$

Then inequality (19) is the necessary and sufficient condition for asymptotic mean square quasistability of the trivial solution of Eq. (1).

**Proof.** A sufficiency follows from Theorem 3.3 and a necessity from Corollary 1.1. ■

**Example 3.1.** Consider the equation

$$x(t + 1) = \alpha x(t) + \beta x(t - k) + \gamma x(t - m) \xi(t + 1). \quad (25)$$

In compliance with the procedure of Lyapunov functionals construction let us consider an auxiliary equation in the form  $y(t + 1) = \alpha y(t)$ . If  $|\alpha| < 1$  then the functional  $v(t) = y^2(t)$  is a Lyapunov functional for this equation, since  $\Delta v(t) = y^2(t + 1) - y^2(t) = (\alpha^2 - 1)y^2(t)$ .

Put  $V_1(t) = x^2(t)$ . Calculating  $\mathbf{E}\Delta V_1(t)$  for Eq. (25), we have

$$\begin{aligned} \mathbf{E}\Delta V_1(t) &= \mathbf{E} [x^2(t + 1) - x^2(t)] \\ &= \mathbf{E} [\alpha x(t) + \beta x(t - k) + \gamma x(t - m) \xi(t + 1)]^2 - \mathbf{E} x^2(t) \\ &= (\alpha^2 - 1) \mathbf{E} x^2(t) + 2\alpha\beta \mathbf{E} x(t) x(t - k) \\ &\quad + \beta^2 \mathbf{E} x^2(t - k) + \gamma^2 \mathbf{E} x^2(t - m) \\ &\leq (\alpha^2 + |\alpha\beta| - 1) \mathbf{E} x^2(t) \\ &\quad + (|\alpha\beta| + \beta^2) \mathbf{E} x^2(t - k) + \gamma^2 \mathbf{E} x^2(t - m). \end{aligned} \quad (26)$$

Choosing an additional functional  $V_2(t)$  in the form

$$V_2(t) = (|\alpha\beta| + \beta^2) \sum_{j=1}^k x^2(t - j) + \gamma^2 \sum_{j=1}^m x^2(t - j),$$

we obtain

$$\begin{aligned} \mathbf{E}\Delta V_2(t) &= (|\alpha\beta| + \beta^2) \sum_{j=1}^k \mathbf{E} [x^2(t + 1 - j) - x^2(t - j)] \\ &\quad + \gamma^2 \sum_{j=1}^m \mathbf{E} [x^2(t + 1 - j) - x^2(t - j)] \\ &= (|\alpha\beta| + \beta^2) \mathbf{E} [x^2(t) - x^2(t - k)] \\ &\quad + \gamma^2 \mathbf{E} [x^2(t) - x^2(t - m)] \\ &= (|\alpha\beta| + \beta^2 + \gamma^2) \mathbf{E} x^2(t) \\ &\quad - (|\alpha\beta| + \beta^2) \mathbf{E} x^2(t - k) - \gamma^2 \mathbf{E} x^2(t - m). \end{aligned} \quad (27)$$

It follows from (26), (27) that if the inequality  $(|\alpha| + |\beta|)^2 + \gamma^2 < 1$  holds then the functional  $V(t) = V_1(t) + V_2(t)$  satisfies the conditions of Theorem 1.1 and, therefore, the trivial

solution of Eq. (25) is asymptotically mean square quasistable.

By virtue of Theorem 3.1 the same result can be obtained only via the functional  $V_1(t)$  without construction of the additional functional  $V_2(t)$ . Really, it follows from (26) that the functional  $V_1(t)$  satisfies conditions (13), (14) with  $a = \alpha^2 + |\alpha\beta| - 1$  and  $b = |\alpha\beta| + \beta^2 + \gamma^2$ .

It follows also from Corollary 3.3 that if  $\beta = 0$  then the inequality  $\alpha^2 + \gamma^2 < 1$  is the necessary and sufficient condition for asymptotic mean square quasistability of the trivial solution of Eq. (25).

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