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# Oscillation criteria for difference equations with non-monotone arguments

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## Abstract

This paper is concerned with the oscillatory behavior of first-order retarded [advanced] difference equation of the form

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n \in \mathbb{N}_0 \quad [\nabla x(n) - q(n)x(\sigma(n)) = 0, \quad n \in \mathbb{N}],$$

where  $(p(n))_{n \geq 0}$  [ $(q(n))_{n \geq 1}$ ] is a sequence of nonnegative real numbers and  $\tau(n)$  [ $\sigma(n)$ ] is a non-monotone sequence of integers such that  $\tau(n) \leq n - 1$ , for  $n \in \mathbb{N}_0$  and  $\lim_{n \rightarrow \infty} \tau(n) = \infty$  [ $\sigma(n) \geq n + 1$ , for  $n \in \mathbb{N}$ ]. Sufficient conditions, involving  $\limsup$ , which guarantee the oscillation of all solutions are established. These conditions improve all previous well-known results in the literature. Also, using algorithms on MATLAB software, examples illustrating the significance of the results are given.

**MSC:** 39A10; 39A21

**Keywords:** difference equations; non-monotone arguments; retarded arguments; advanced arguments; oscillation; Grönwall inequality

## 1 Introduction

The paper deals with the difference equation with a single variable retarded argument of the form

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n \in \mathbb{N}_0, \tag{E}$$

and the (dual) difference equation with a single variable advanced argument of the form

$$\nabla x(n) - q(n)x(\sigma(n)) = 0, \quad n \in \mathbb{N}, \tag{E'}$$

where  $\mathbb{N}_0$  and  $\mathbb{N}$  are the sets of nonnegative integers and positive integers, respectively.

Equations (E) and (E') are studied under the following assumptions: everywhere  $(p(n))_{n \geq 0}$  and  $(q(n))_{n \geq 1}$  are sequences of nonnegative real numbers,  $(\tau(n))_{n \geq 0}$  is a sequence of integers such that

$$\tau(n) \leq n - 1, \quad \forall n \in \mathbb{N}_0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \tau(n) = \infty, \tag{1.1}$$

and  $(\sigma(n))_{n \geq 1}$  is a sequence of integers such that

$$\sigma(n) \geq n + 1, \quad \forall n \in \mathbb{N}. \tag{1.1'}$$

Here,  $\Delta$  denotes the forward difference operator  $\Delta x(n) = x(n+1) - x(n)$  and  $\nabla$  corresponds to the backward difference operator  $\nabla x(n) = x(n) - x(n-1)$ .

Set

$$w = -\min_{n \geq 0} \tau(n).$$

Clearly,  $w$  is a finite positive integer if (1.1) holds.

By a *solution* of (E), we mean a sequence of real numbers  $(x(n))_{n \geq -w}$  which satisfies (E) for all  $n \geq 0$ . It is clear that, for each choice of real numbers  $c_{-w}, c_{-w+1}, \dots, c_{-1}, c_0$ , there exists a unique solution  $(x(n))_{n \geq -w}$  of (E) which satisfies the initial conditions  $x(-w) = c_{-w}, x(-w + 1) = c_{-w+1}, \dots, x(-1) = c_{-1}, x(0) = c_0$ .

By a solution of (E'), we mean a sequence of real numbers  $(x(n))_{n \geq 0}$  which satisfies (E') for all  $n \geq 1$ .

A solution  $(x(n))_{n \geq -w}$  (or  $(x(n))_{n \geq 0}$ ) of (E) (or (E')) is called *oscillatory*, if the terms  $x(n)$  of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory*. An equation is *oscillatory* if all its solutions oscillate.

In the last few decades, the oscillatory behavior and the existence of positive solutions of difference equations with deviating arguments have been extensively studied; see, for example, papers [1–19] and the references cited therein. Most of these papers concern the special case where the arguments are nondecreasing, while a small number of these papers are dealing with the general case where the arguments are non-monotone. See, for example, [1–3, 11, 15] and the references cited therein. By the consideration of non-monotone arguments of other than the pure mathematical interest, it approximates the natural phenomena described by equation of the type (E) or (E'). That is because there are always natural disturbances (e.g. noise in communication systems) that affect all the parameters of the equation and therefore the fair (from a mathematical point of view) monotone arguments become non-monotone almost always.

### 1.1 Retarded difference equations

In 2008 Chatzarakis, Koplatadze and Stavroulakis [4, 5] proved that, if

$$\limsup_{n \rightarrow \infty} \sum_{j=h(n)}^n p(j) > 1, \tag{1.2}$$

where  $h(n) = \max_{0 \leq s \leq n} \tau(s)$ , or

$$\liminf_{n \rightarrow \infty} \sum_{j=\tau(n)}^{n-1} p(j) > \frac{1}{e}, \tag{1.3}$$

then all solutions of (E) oscillate.

It is obvious that there is a gap between the conditions (1.2) and (1.3) when the limit

$$\lim_{n \rightarrow \infty} \sum_{j=\tau(n)}^{n-1} p(j)$$

does not exist. How to fill this gap is an interesting problem which has been investigated by several authors. For example, in 2009, Chatzarakis, Philos and Stavroulakis [6] proved that if

$$\limsup_{n \rightarrow \infty} \sum_{j=h(n)}^n p(j) > 1 - \frac{1 - a - \sqrt{1 - 2a - a^2}}{2}, \tag{1.4}$$

where  $a = \liminf_{n \rightarrow \infty} \sum_{j=\tau(n)}^{n-1} p(j)$ , then all solutions of (E) oscillate.

In 2011, Braverman and Karpuz [3] proved that if

$$\limsup_{n \rightarrow \infty} \sum_{j=h(n)}^n p(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1 - p(i)} > 1, \tag{1.5}$$

then all solutions of (E) oscillate, while, in 2014, Stavroulakis [15], improved (1.5) to

$$\limsup_{n \rightarrow \infty} \sum_{j=h(n)}^n p(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1 - p(i)} > 1 - \frac{1 - a - \sqrt{1 - 2a - a^2}}{2}. \tag{1.6}$$

In 2015, Braverman, Chatzarakis, and Stavroulakis [2] proved that if for some  $r \in \mathbb{N}$

$$\limsup_{n \rightarrow \infty} \sum_{j=h(n)}^n p(j) a_r^{-1}(h(n), \tau(j)) > 1, \tag{1.7}$$

or

$$\limsup_{n \rightarrow \infty} \sum_{j=h(n)}^n p(j) a_r^{-1}(h(n), \tau(j)) > 1 - \frac{1 - a - \sqrt{1 - 2a - a^2}}{2}, \tag{1.8}$$

where

$$a_1(n, k) = \prod_{i=k}^{n-1} [1 - p(i)], \quad a_{r+1}(n, k) = \prod_{i=k}^{n-1} [1 - p(i) a_r^{-1}(i, \tau(i))], \tag{1.9}$$

then all solutions of (E) oscillate.

Recently, Asteris and Chatzarakis [1] proved that if for some  $\ell \in \mathbb{N}$

$$\limsup_{n \rightarrow \infty} \sum_{i=h(n)}^n p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1 - p_\ell(j)} > 1, \tag{1.10}$$

where

$$p_\ell(n) = p(n) \left[ 1 + \sum_{i=\tau(n)}^{n-1} p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1 - p_{\ell-1}(j)} \right] \tag{1.11}$$

with  $p_0(n) = p(n)$ , then all solutions of (E) oscillate.

### 1.2 Advanced difference equations

In 2012, Chatzarakis and Stavroulakis [7] proved that, if

$$\limsup_{n \rightarrow \infty} \sum_{i=n}^{\rho(n)} q(i) > 1, \tag{1.12}$$

or

$$\limsup_{n \rightarrow \infty} \sum_{i=n}^{\rho(n)} q(i) > 1 - (1 - \sqrt{1 - b})^2, \tag{1.13}$$

where  $\rho(n) = \min_{s \geq n} \sigma(s)$  and  $b = \liminf_{n \rightarrow \infty} \sum_{i=n+1}^{\sigma(n)} p(i)$ , then all solutions of (E') oscillate.

In 2015, Braverman, Chatzarakis, and Stavroulakis [2] proved that if for some  $r \in \mathbb{N}$

$$\limsup_{n \rightarrow \infty} \sum_{j=n}^{\rho(n)} q(j) b_r^{-1}(\rho(n), \sigma(j)) > 1, \tag{1.14}$$

or

$$\limsup_{n \rightarrow \infty} \sum_{j=n}^{\rho(n)} q(j) b_r^{-1}(\rho(n), \sigma(j)) > 1 - \frac{1 - b - \sqrt{1 - 2b - b^2}}{2}, \tag{1.15}$$

where

$$b_1(n, k) = \prod_{i=n+1}^k [1 - q(i)], \tag{1.16}$$

$$b_{r+1}(n, k) = \prod_{i=n+1}^k [1 - q(i) b_r^{-1}(i, \sigma(i))]$$

and  $b = \liminf_{n \rightarrow \infty} \sum_{i=n+1}^{\sigma(n)} p(i)$ , then all solutions of (E') oscillate.

Recently, Asteris and Chatzarakis [1] proved that if for some  $\ell \in \mathbb{N}$

$$\limsup_{n \rightarrow \infty} \sum_{i=n}^{\rho(n)} q(i) \prod_{j=\rho(n)+1}^{\sigma(i)} \frac{1}{1 - q_\ell(j)} > 1, \tag{1.17}$$

where

$$q_\ell(n) = q(n) \left[ 1 + \sum_{i=n+1}^{\rho(n)} q(i) \prod_{j=\rho(n)+1}^{\sigma(i)} \frac{1}{1 - q_{\ell-1}(j)} \right] \tag{1.18}$$

with  $q_0(n) = q(n)$ , then all solutions of (E') oscillate.

In this paper we study further (E) and (E') and derive new sufficient oscillation conditions. Examples illustrate cases when the results of the present paper imply oscillation while previously known results fail.

## 2 Main results

### 2.1 Retarded difference equations

We study further (E) and derive a new sufficient oscillation condition, involving  $\limsup$ , which essentially improves all the previous results.

Let

$$h(n) = \max_{0 \leq s \leq n} \tau(s), \quad n \geq 0. \tag{2.1}$$

Clearly, the sequence  $h(n)$  is nondecreasing and  $\tau(n) \leq h(n) \leq n - 1$  for all  $n \geq 0$ .

The proof of our main result is essentially based on the following lemmas.

**Lemma 1** *Assume that (1.1) holds and*

$$0 < a := \liminf_{n \rightarrow \infty} \sum_{j=\tau(n)}^{n-1} p(j).$$

*Then we have*

$$\liminf_{n \rightarrow \infty} \sum_{j=h(n)}^{n-1} p(j) = \liminf_{n \rightarrow \infty} \sum_{j=\tau(n)}^{n-1} p(j) = a, \tag{2.2}$$

where  $h(n)$  is defined by (2.1).

*Proof* Since  $h(n)$  is nondecreasing and  $\tau(n) \leq h(n) \leq n - 1$  for all  $n \geq 0$ , we have

$$\sum_{j=h(n)}^{n-1} p(j) \leq \sum_{j=\tau(n)}^{n-1} p(j).$$

Therefore

$$\liminf_{n \rightarrow \infty} \sum_{j=h(n)}^{n-1} p(j) \leq \liminf_{n \rightarrow \infty} \sum_{j=\tau(n)}^{n-1} p(j).$$

If (2.2) does not hold, then there exist  $a' > 0$  and a subsequence  $(\theta(n))$  such that  $\theta(n) \rightarrow \infty$  as  $n \rightarrow \infty$  and

$$\lim_{n \rightarrow \infty} \sum_{j=h(\theta(n))}^{\theta(n)-1} p(j) \leq a' < a.$$

But  $h(\theta(n)) = \max_{0 \leq s \leq \theta(n)} \tau(s)$ , hence there exists  $\theta'(n) \leq \theta(n)$ ,  $\theta'(n) \in \mathbb{N}_0$  such that  $h(\theta(n)) = \tau(\theta'(n))$ , and consequently

$$\sum_{j=h(\theta(n))}^{\theta(n)-1} p(j) = \sum_{j=\tau(\theta'(n))}^{\theta(n)-1} p(j) \geq \sum_{j=\tau(\theta'(n))}^{\theta'(n)-1} p(j).$$

It follows that  $(\sum_{j=\tau(\theta'(n))}^{\theta'(n)-1} p(j))_{n=1}^\infty$  is a bounded sequence having a convergent subsequence, say

$$\sum_{j=\tau(\theta'(n_k))}^{\theta'(n_k)-1} p(j) \rightarrow c \leq a', \quad \text{as } k \rightarrow \infty$$

which implies that

$$\liminf_{n \rightarrow \infty} \sum_{j=\tau(n)}^{n-1} p(j) \leq a' < a.$$

This contradicts (2.2).

The proof of the lemma is complete. □

**Lemma 2** ([6], Lemma 2.1) *In addition to hypothesis (1.1), assume that  $h(n)$  is defined by (2.1),*

$$0 < a := \liminf_{n \rightarrow \infty} \sum_{j=\tau(n)}^{n-1} p(j) \leq \frac{1}{e} \tag{2.3}$$

and  $x(n)$  is an eventually positive solution of (E). Then

$$\liminf_{n \rightarrow \infty} \frac{x(n+1)}{x(h(n))} \geq \frac{1 - a - \sqrt{1 - 2a - a^2}}{2}. \tag{2.4}$$

**Theorem 1** *Assume that (1.1) holds,  $h(n)$  is defined by (2.1) and  $a$  by (2.3). If for some  $\ell \in \mathbb{N}$*

$$\limsup_{n \rightarrow \infty} \sum_{i=h(n)}^n p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1 - p_\ell(j)} > 1 - \frac{1 - a - \sqrt{1 - 2a - a^2}}{2}, \tag{2.5}$$

where  $p_\ell(n)$  is defined by (1.11), then all solutions of (E) oscillate.

*Proof* Assume, for the sake of contradiction, that  $(x(n))_{n \geq -w}$  is a nonoscillatory solution of (E). Then it is either eventually positive or eventually negative. As  $(-x(n))_{n \geq -w}$  is also a solution of (E), we may restrict ourselves only to the case where  $x(n) > 0$  for all large  $n$ . Let  $n_1 \geq -w$  be an integer such that  $x(n) > 0$  for all  $n \geq n_1$ . Then, there exists  $n_2 \geq n_1$  such that  $x(\tau(n)) > 0, \forall n \geq n_2$ . In view of this, equation (E) becomes

$$\Delta x(n) = -p(n)x(\tau(n)) \leq 0, \quad \forall n \geq n_2,$$

which means that the sequence  $(x(n))$  is eventually decreasing.

Therefore, since  $\tau(n) < n$ , (E) implies

$$\Delta x(n) + p(n)x(n) \leq 0. \tag{2.6}$$

Applying the discrete Grönwall inequality, we obtain

$$x(k) \geq x(n) \prod_{i=k}^{n-1} \frac{1}{1-p(i)}, \quad \text{for all } n \geq k \geq n_2. \tag{2.7}$$

Summing up (E) from  $\tau(n)$  to  $n-1$ , we have

$$x(n) - x(\tau(n)) + \sum_{i=\tau(n)}^{n-1} p(i)x(\tau(i)) = 0. \tag{2.8}$$

Since  $\tau(i) \leq h(i) \leq h(n)$ , (2.7) and (2.8) give

$$x(n) - x(\tau(n)) + x(h(n)) \sum_{i=\tau(n)}^{n-1} p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-p(j)} \leq 0.$$

Multiplying the last inequality by  $p(n)$ , we get

$$p(n)x(n) - p(n)x(\tau(n)) + p(n)x(h(n)) \sum_{i=\tau(n)}^{n-1} p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-p(j)} \leq 0,$$

which, in view of (E), becomes

$$\Delta x(n) + p(n)x(n) + p(n)x(h(n)) \sum_{i=\tau(n)}^{n-1} p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-p(j)} \leq 0.$$

Since  $h(n) < n$ , the last inequality gives

$$\Delta x(n) + p(n)x(n) + p(n)x(n) \sum_{i=\tau(n)}^{n-1} p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-p(j)} \leq 0,$$

or

$$\Delta x(n) + p(n) \left[ 1 + \sum_{i=\tau(n)}^{n-1} p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-p(j)} \right] x(n) \leq 0.$$

Therefore

$$\Delta x(n) + p_1(n)x(n) \leq 0,$$

where

$$p_1(n) = p(n) \left[ 1 + \sum_{i=\tau(n)}^{n-1} p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-p(j)} \right].$$

Repeating the above argument leads to a new estimate,

$$\Delta x(n) + p_2(n)x(n) \leq 0,$$

where

$$p_2(n) = p(n) \left[ 1 + \sum_{i=\tau(n)}^{n-1} p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-p_1(j)} \right].$$

Continuing by induction, for sufficiently large  $n$  we get

$$\Delta x(n) + p_\ell(n)x(n) \leq 0,$$

where

$$p_\ell(n) = p(n) \left[ 1 + \sum_{i=\tau(n)}^{n-1} p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-p_{\ell-1}(j)} \right].$$

Clearly, by the Grönwall inequality, we have

$$x(\tau(i)) \geq x(h(n)) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-p_\ell(j)}. \tag{2.9}$$

Summing up (E) from  $h(n)$  to  $n$ , we have

$$x(n+1) - x(h(n)) + \sum_{i=h(n)}^n p(i)x(\tau(i)) = 0. \tag{2.10}$$

Combining (2.9) and (2.10), we have

$$x(n+1) - x(h(n)) + x(h(n)) \sum_{i=h(n)}^n p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-p_\ell(j)} \leq 0,$$

or

$$\sum_{i=h(n)}^n p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-p_\ell(j)} \leq 1 - \frac{x(n+1)}{x(h(n))}.$$

Therefore

$$\limsup_{n \rightarrow \infty} \sum_{i=h(n)}^n p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-p_\ell(j)} \leq 1 - \liminf_{n \rightarrow \infty} \frac{x(n+1)}{x(h(n))}. \tag{2.11}$$



By combining Lemmas 1 and 2, it becomes obvious that inequality (2.4) is fulfilled. So, (2.11) leads to

$$\limsup_{n \rightarrow \infty} \sum_{i=h(n)}^n p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-p_\ell(j)} \leq 1 - \frac{1-a-\sqrt{1-2a-a^2}}{2},$$

which contradicts (2.5). The proof of the theorem is complete. □

**Example 1** Consider the retarded difference equation

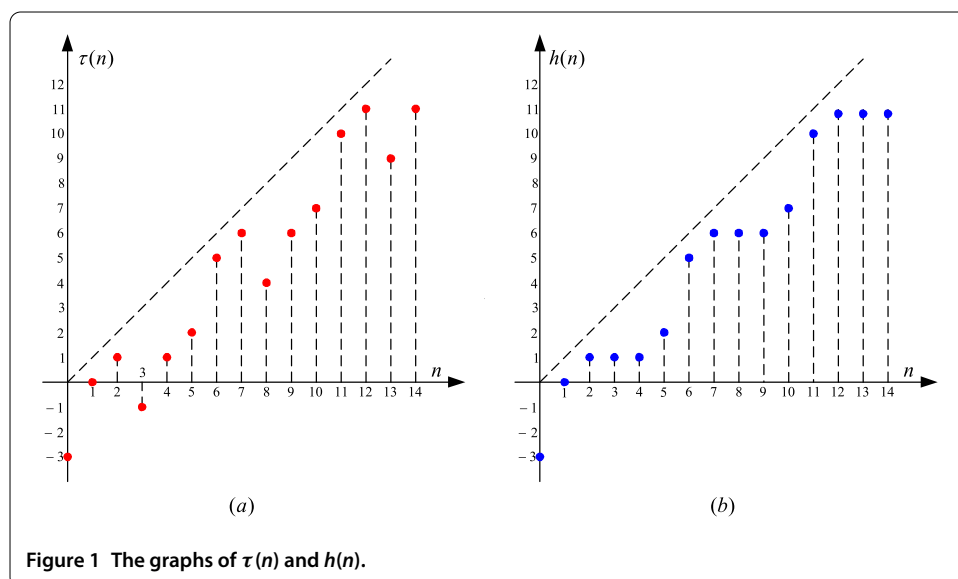
$$\Delta x(n) + \frac{179}{1,000} x(\tau(n)) = 0, \quad n \in \mathbb{N}_0, \tag{2.12}$$

with (see Figure 1(a))

$$\tau(n) = \begin{cases} n-3, & \text{if } n = 5\mu, \\ n-1, & \text{if } n = 5\mu + 1, \\ n-1, & \text{if } n = 5\mu + 2, \\ n-4, & \text{if } n = 5\mu + 3, \\ n-3, & \text{if } n = 5\mu + 4, \end{cases} \quad \mu \in \mathbb{N}_0.$$

By (2.1), we see (Figure 1(b)) that

$$h(n) = \max_{0 \leq s \leq n} \tau(s) = \begin{cases} n-3, & \text{if } n = 5\mu, \\ n-1, & \text{if } n = 5\mu + 1, \\ n-1, & \text{if } n = 5\mu + 2, \\ n-2, & \text{if } n = 5\mu + 3, \\ n-3, & \text{if } n = 5\mu + 4, \end{cases} \quad \mu \in \mathbb{N}_0.$$



**Figure 1** The graphs of  $\tau(n)$  and  $h(n)$ .

Observe that the function  $F : \mathbb{N}_0 \rightarrow \mathbb{R}_+$  defined as

$$F_\ell(n) = \sum_{i=h(n)}^n p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-p_\ell(j)}$$

attains its maximum at  $n = 5\mu + 4$ ,  $\mu \in \mathbb{N}_0$ , for every  $\ell \in \mathbb{N}$ . Specifically, by using an algorithm of Matlab software, we obtain

$$\begin{aligned} F_1(n) &= \sum_{i=h(n)}^n p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-p_1(j)} \\ &= \sum_{i=h(n)}^n p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-p(j)[1 + \sum_{k=\tau(j)}^{j-1} p(k) \prod_{m=\tau(k)}^{h(j)-1} \frac{1}{1-p(m)}]} \\ &= \sum_{i=5\mu+1}^{5\mu+4} \frac{179}{1,000} \prod_{j=\tau(i)}^{3\mu-2} \frac{1}{1 - \frac{179}{1,000} [1 + \sum_{k=\tau(j)}^{j-1} \frac{179}{1,000} \prod_{m=\tau(k)}^{h(j)-1} \frac{1}{1 - \frac{179}{1,000}}]} \\ &\simeq 0.9913. \end{aligned}$$

Thus

$$\limsup_{n \rightarrow \infty} F_1(n) = \limsup_{n \rightarrow \infty} \sum_{i=h(n)}^n p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-p_1(j)} \simeq 0.9913.$$

Since

$$a = \liminf_{n \rightarrow \infty} \sum_{j=\tau(n)}^{n-1} p(j) = \liminf_{n \rightarrow \infty} \sum_{j=5\mu}^{5\mu} p(j) = 0.179,$$

we have

$$0.9913 > 1 - \frac{1-a-\sqrt{1-2a-a^2}}{2} \simeq 0.9799,$$

that is, condition (2.5) of Theorem 1 is satisfied for  $\ell = 1$ . Therefore, all solutions of (2.12) oscillate.

Observe, however, that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sum_{j=h(n)}^n p(j) &= \limsup_{\mu \rightarrow \infty} \sum_{i=5\mu+1}^{5\mu+4} p(j) = 4 \cdot 0.179 = 0.716 < 1, \\ a &= 0.179 < \frac{1}{e}, \\ 0.716 &< 1 - \frac{1-a-\sqrt{1-2a-a^2}}{2} \simeq 0.9799, \end{aligned}$$

$$\begin{aligned}
 & \limsup_{n \rightarrow \infty} \sum_{j=h(n)}^n p(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-p(i)} \\
 &= \limsup_{\mu \rightarrow \infty} \sum_{j=5\mu+1}^{5\mu+4} 0.179 \prod_{i=\tau(j)}^{5\mu} \frac{1}{1-0.179} \\
 &= 0.179 \cdot \limsup_{\mu \rightarrow \infty} \left\{ \prod_{i=\tau(5\mu+1)}^{5\mu} \frac{1}{1-0.179} + \prod_{i=\tau(5\mu+2)}^{5\mu} \frac{1}{1-0.179} \right. \\
 &\quad \left. + \prod_{i=\tau(5\mu+3)}^{5\mu} \frac{1}{1-0.179} + \prod_{i=\tau(5\mu+4)}^{5\mu} \frac{1}{1-0.179} \right\} \\
 &= 0.179 \cdot \limsup_{\mu \rightarrow \infty} \left\{ \prod_{i=5\mu}^{5\mu} \frac{1}{1-0.179} + \prod_{i=5\mu+1}^{5\mu} \frac{1}{1-0.179} + \prod_{i=5\mu-1}^{5\mu} \frac{1}{1-0.179} \right. \\
 &\quad \left. + \prod_{i=5\mu+1}^{5\mu} \frac{1}{1-0.179} \right\} \\
 &= 0.179 \cdot \left\{ \frac{1}{1-0.179} + 1 + \left( \frac{1}{1-0.179} \right)^2 + 1 \right\} \simeq 0.8416 < 1, \\
 &0.8416 < 1 - \frac{1-a-\sqrt{1-2a-a^2}}{2} \simeq 0.9799, \\
 &0.9913 < 1.
 \end{aligned}$$

That is, conditions (1.2), (1.3), (1.4), (1.5)  $\equiv$  (1.7) (for  $r = 1$ ), (1.6)  $\equiv$  (1.8) (for  $r = 1$ ) and (1.10) are not satisfied.

Figure 2, illustrates that the three different solutions of (2.12), as predicted, appear to be oscillatory.

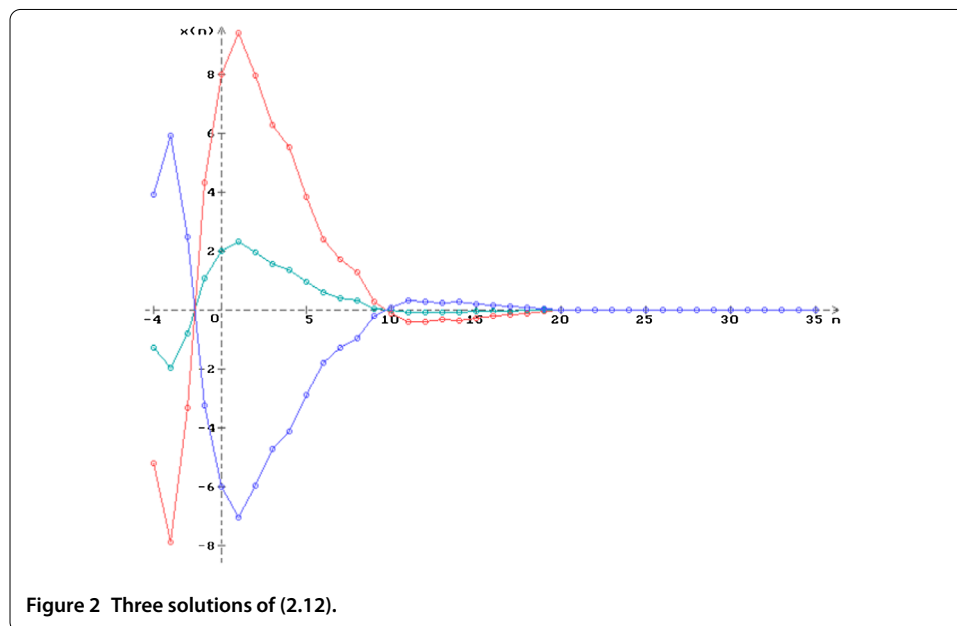


Figure 2 Three solutions of (2.12).

**Notation** It is worth noting that the improvement of condition (2.5) to the corresponding condition (1.4) is significant, approximately 38.45%, if we compare the values on the left-hand side of these conditions. Also, the improvement compared to condition (1.5) (or (1.6) or (1.7) or (1.8)) is very satisfactory, around 17.8%. Also, observe that conditions (1.7), (1.8), and (1.10) do not lead to oscillation for the first iteration. On the contrary, condition (2.5) is satisfied from the first iteration. This means that our condition is better and much faster than (1.7), (1.8), and (1.10).

**2.2 Advanced difference equations**

A similar oscillation theorem for the (dual) advanced difference equation (E') can be derived easily. The proof of this theorem is omitted, since it is quite similar to the proof for a retarded equation.

Let

$$\rho(n) = \min_{s \geq n} \sigma(s), \quad n \geq 1. \tag{2.13}$$

Clearly, the sequence  $\rho(n)$  is nondecreasing and  $\sigma(n) \geq \rho(n) \geq n + 1$  for all  $n \geq 1$ .

**Theorem 2** Assume that (1.1') holds and  $\rho(n)$  is defined by (2.13). If for some  $\ell \in \mathbb{N}$

$$\limsup_{n \rightarrow \infty} \sum_{i=n}^{\rho(n)} q(i) \prod_{j=\rho(n)+1}^{\sigma(i)} \frac{1}{1 - q_\ell(j)} > 1 - \frac{1 - b - \sqrt{1 - 2b - b^2}}{2}, \tag{2.14}$$

where  $q_\ell(n)$  is defined by (1.18) and  $0 < b = \liminf_{n \rightarrow \infty} \sum_{i=n+1}^{\sigma(n)} p(i) \leq 1/e$ , then all solutions of (E') oscillate.

**Example 2** Consider the advanced difference equation

$$\nabla x(n) - \frac{341}{2,000} x(\sigma(n)) = 0, \quad n \in \mathbb{N}_0, \tag{2.15}$$

with (see Figure 3(a))

$$\sigma(n) = \begin{cases} n + 1, & \text{if } n = 5\mu, \\ n + 5, & \text{if } n = 5\mu + 1, \\ n + 1, & \text{if } n = 5\mu + 2, \\ n + 1, & \text{if } n = 5\mu + 3, \\ n + 2, & \text{if } n = 5\mu + 4, \end{cases} \quad \mu \in \mathbb{N}_0.$$

By (2.13), we see (Figure 3(b)) that

$$\rho(n) = \min_{s \geq n} \sigma(s) = \begin{cases} n + 1, & \text{if } n = 5\mu, \\ n + 2, & \text{if } n = 5\mu + 1, \\ n + 1, & \text{if } n = 5\mu + 2, \\ n + 1, & \text{if } n = 5\mu + 3, \\ n + 2, & \text{if } n = 5\mu + 4, \end{cases} \quad \mu \in \mathbb{N}_0.$$

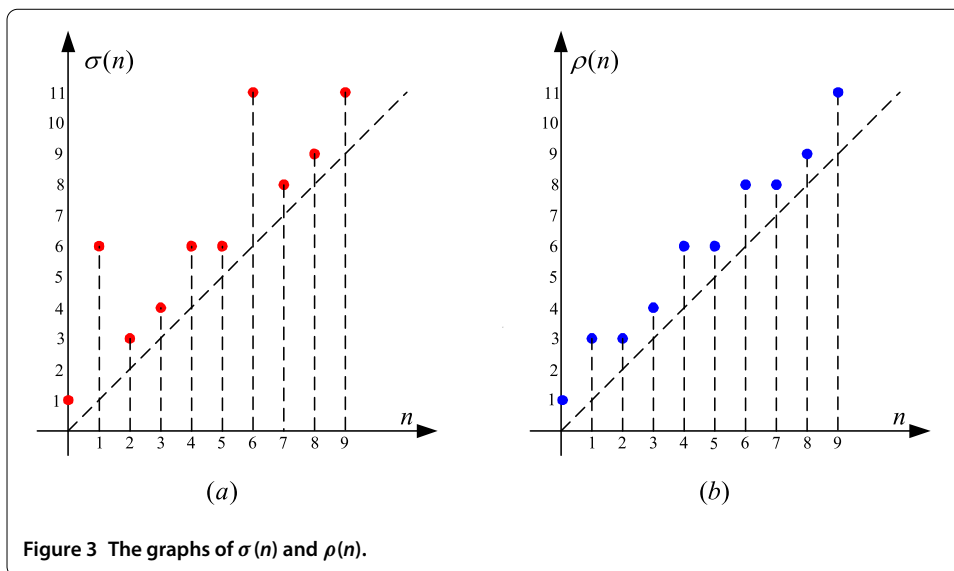


Figure 3 The graphs of  $\sigma(n)$  and  $\rho(n)$ .

Observe that the function  $F : \mathbb{N}_0 \rightarrow \mathbb{R}_+$  defined as

$$F_\ell(n) = \sum_{i=n}^{\rho(n)} q(i) \prod_{j=\rho(n)+1}^{\sigma(i)} \frac{1}{1 - q_\ell(j)}$$

attains its maximum at  $n = 5\mu + 1, \mu \in \mathbb{N}_0$ , for every  $\ell \in \mathbb{N}$ . Specifically, by using an algorithm of Matlab software, we obtain

$$\begin{aligned} F_1(n) &= \sum_{i=n}^{\rho(n)} q(i) \prod_{j=\rho(n)+1}^{\sigma(i)} \frac{1}{1 - q_1(j)} \\ &= \sum_{i=n}^{\rho(n)} q(i) \prod_{j=\rho(n)+1}^{\sigma(i)} \frac{1}{1 - q(j)[1 + \sum_{k=j+1}^{\sigma(j)} q(k) \prod_{m=\rho(j)+1}^{\sigma(k)} \frac{1}{1 - q(m)}]} \\ &= \sum_{i=5\mu+1}^{5\mu+3} \frac{341}{2,000} \prod_{j=5\mu+4}^{\sigma(i)} \frac{1}{1 - \frac{341}{2,000} [1 + \sum_{k=j+1}^{\sigma(j)} \frac{341}{2,000} \prod_{m=\rho(j)+1}^{\sigma(k)} \frac{1}{1 - \frac{341}{2,000}}]} \simeq 0.9872. \end{aligned}$$

Thus

$$\limsup_{n \rightarrow \infty} F_1(n) = \limsup_{n \rightarrow \infty} \sum_{i=n}^{\rho(n)} q(i) \prod_{j=\rho(n)+1}^{\sigma(i)} \frac{1}{1 - q_1(j)} \simeq 0.9872.$$

Since

$$b = \liminf_{n \rightarrow \infty} \sum_{i=n+1}^{\sigma(n)} q(i) = \liminf_{n \rightarrow \infty} \sum_{i=5\mu+1}^{5\mu+1} q(i) = \frac{341}{2,000} = 0.1705,$$

we have

$$0.9872 > 1 - \frac{1 - b - \sqrt{1 - 2b - b^2}}{2} \simeq 0.9821$$

that is, condition (2.14) of Theorem 2 is satisfied for  $\ell = 1$ . Therefore, all solutions of (2.15) oscillate.

Observe, however, that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sum_{i=n}^{\rho(n)} q(i) &= \limsup_{\mu \rightarrow \infty} \sum_{i=5\mu+1}^{5\mu+3} q(i) = 3 \cdot \frac{341}{2,000} = 0.5115 < 1, \\ 0.5115 &< 1 - (1 - \sqrt{1-b})^2 \simeq 0.9920, \\ \limsup_{n \rightarrow \infty} \sum_{j=n}^{\rho(n)} q(j) b_1^{-1}(\rho(n), \sigma(j)) &= \limsup_{\mu \rightarrow \infty} \sum_{j=5\mu+1}^{5\mu+3} \frac{341}{2,000} \prod_{i=5\mu+4}^{\sigma(j)} \frac{1}{1 - \frac{341}{2,000}} \\ &= \frac{341}{2,000} \cdot \limsup_{\mu \rightarrow \infty} \left\{ \prod_{i=5\mu+4}^{\sigma(5\mu+1)} \frac{1}{1 - \frac{341}{2,000}} + \prod_{i=5\mu+4}^{\sigma(5\mu+2)} \frac{1}{1 - \frac{341}{2,000}} + \prod_{i=5\mu+4}^{\sigma(5\mu+3)} \frac{1}{1 - \frac{341}{2,000}} \right\} \\ &= \frac{341}{2,000} \cdot \limsup_{\mu \rightarrow \infty} \left\{ \prod_{i=5\mu+4}^{5\mu+6} \frac{1}{1 - \frac{341}{2,000}} + \prod_{i=5\mu+4}^{5\mu+3} \frac{1}{1 - \frac{341}{2,000}} + \prod_{i=5\mu+4}^{5\mu+4} \frac{1}{1 - \frac{341}{2,000}} \right\} \\ &= \frac{341}{2,000} \cdot \left\{ \frac{1}{(1 - \frac{341}{2,000})^3} + 1 + \frac{1}{1 - \frac{341}{2,000}} \right\} \simeq 0.6748 < 1, \\ 0.6748 &< 1 - \frac{1-b - \sqrt{1-2b-b^2}}{2} \simeq 0.9821, \\ 0.9872 &< 1. \end{aligned}$$

That is, conditions (1.12), (1.13), (1.14), (1.15), and (1.17) are not satisfied.

Figure 4, illustrates three different solutions of (2.15), as predicted, appeared to be oscillatory.

**Notation** It is worth noting that the improvement of condition (2.14) to the corresponding condition (1.12) is significant, approximately 93%, if we compare the values on the left-side of these conditions. Also, the improvement compared to condition (1.14) (or (1.15)) is very satisfactory, around 46.3%. Also, observe that conditions (1.14) and (1.15) do not lead to oscillation for first iteration. On the contrary, condition (2.14) is satisfied from the first iteration. This means that our condition is better and much faster than (1.14) and (1.15).

### 2.3 Deviating difference inequalities

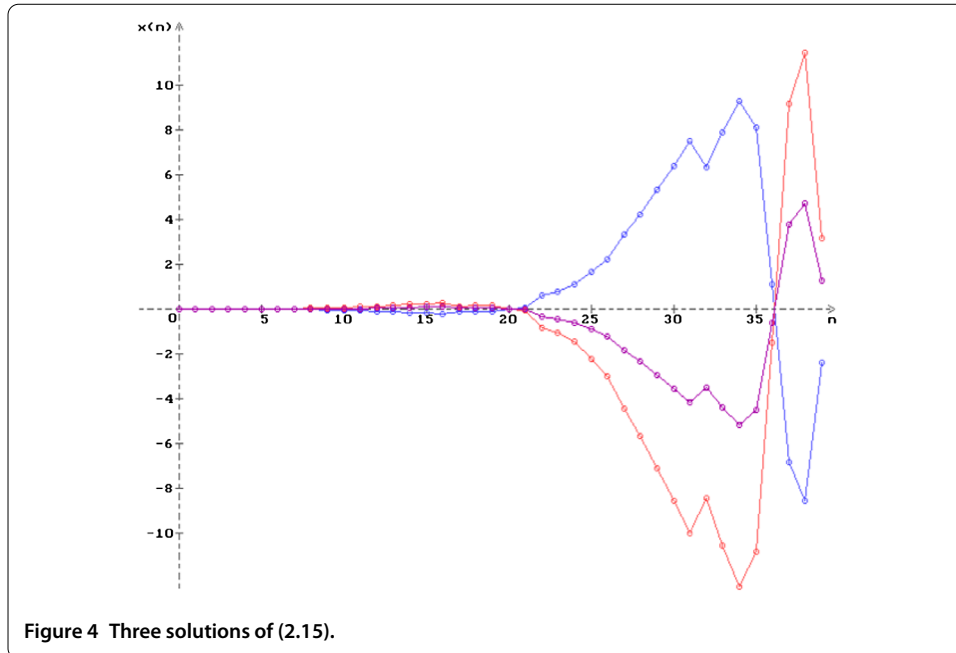
A slight modification in the proofs of Theorems 1 and 2 leads to the following results about deviating difference inequalities.

**Theorem 3** *Assume that all conditions of Theorem 1 hold. Then*

- (i) *the retarded difference inequality*

$$\Delta x(n) + p(n)x(\tau(n)) \leq 0, \quad n \in \mathbb{N}_0,$$

*has no eventually positive solutions;*



(ii) *the retarded difference inequality*

$$\Delta x(n) + p(n)x(\tau(n)) \geq 0, \quad n \in \mathbb{N}_0,$$

*has no eventually negative solutions.*

**Theorem 4** *Assume that all conditions of Theorem 2 hold. Then*

(i) *the advanced difference inequality*

$$\nabla x(n) - q(n)x(\sigma(n)) \geq 0, \quad n \in \mathbb{N},$$

*has no eventually positive solutions;*

(ii) *the advanced difference inequality*

$$\nabla x(n) - q(n)x(\sigma(n)) \leq 0, \quad n \in \mathbb{N},$$

*has no eventually negative solutions.*

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

The authors declare that they have made equal contributions to the paper.

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