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Optimal Execution in a General One-Sided Limit-Order Book*

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Abstract. We construct an optimal execution strategy for the purchase of a large number of shares of a financial asset over a fixed interval of time. Purchases of the asset have a nonlinear impact on price, and this is moderated over time by resilience in the limit-order book that determines the price. The limit-order book is permitted to have arbitrary shape. The form of the optimal execution strategy is to make an initial lump purchase and then purchase continuously for some period of time during which the rate of purchase is set to match the order book resiliency. At the end of this period, another lump purchase is made, and following that there is again a period of purchasing continuously at a rate set to match the order book resiliency. At the end of this second period, there is a final lump purchase. Any of the lump purchases could be of size zero. A simple condition is provided that guarantees that the intermediate lump purchase is of size zero.

Key words. optimal execution, limit-order book, price impact

AMS subject classifications. 91B26, 91G80, 49K45, 90C25

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1. Introduction.

We consider optimal execution over a fixed time interval of a large asset purchase in the face of a one-sided limit-order book. We assume that the *ask price* (sometimes called the *best ask price*) for the underlying asset is a continuous martingale that undergoes two adjustments during the period of purchase. The first adjustment is that orders consume a part of the limit-order book, and this increases the ask price for subsequent orders. The second adjustment is that resilience in the limit-order book causes the effect of these prior orders to decay over time. In this paper, there is no permanent effect from the purchase we model. However, the temporary effect requires infinite time to disappear completely.

We assume that there is a fixed shadow limit-order book shape toward which resilience returns the limit-order book. At any time, the actual limit-order book relative to the martingale component of the ask price has this shape but with some left-hand part missing due to prior purchases. An investor is given a period of time and a target amount of asset to be purchased within that period. His goal is to distribute his purchasing over the period in order to minimize the expected cost of purchasing the target. We permit purchases to occur in lumps or to be spread continuously over time. We show that the optimal execution strategy consists of three lump purchases, one or more of which may be of size zero, i.e., does not occur. One of these lump purchases is made at the initial time, one at an intermediate time,

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and one at the final time. Between these lump purchases, the optimal strategy purchases at a constant rate matched to the limit-order book recovery rate so that the ask price minus its martingale component remains constant. We provide a simple condition under which the intermediate lump purchase is of size zero (see Theorem 4.2 and Remark 4.4).

Bouchaud, Farmer, and Lillo [9] provide a survey of the empirical behavior of limit-order books. Dynamic models for optimal execution designed to capture some of this behavior have been developed by several authors, including Bertsimas and Lo [8], Almgren and Chriss [6, 7], Grinold and Kahn [15] (Chapter 16), Almgren [5], Obizhaeva and Wang [10], and Alfonsi, Fruth, and Schied [1, 4]. Trading in [8] is on a discrete-time grid, and the price impact of a trade is linear in the size of the trade and is permanent. In [8], the expected-cost-minimizing liquidation strategy for an order is to divide the order into equal pieces, one for each trading date. Trading in [6, 7] is also on a discrete-time grid, and there are linear permanent and temporary price impacts. In [6, 7], the variance of the cost of execution is taken into account. This leads to the construction of an efficient frontier of trading strategies. In [15] and [5], trading takes place continuously, and finding the optimal trading strategy reduces to a problem in the calculus of variations.

Other authors focus on the possibility of price manipulation, an idea that traces back to Huberman and Stanzl [16]. Price manipulation is a way of starting with zero shares and using a strategy of buying and selling so as to end with zero shares while generating income. Gatheral, Schied, and Slynko [13] permit continuous trading and use an integral of a kernel with respect to the trading strategy to capture the resilience of the book. In such a model, Gatheral [12] shows that exponential decay of market impact and absence of price manipulation opportunities are compatible only with linear market impact. In [14], this result is reconciled with the nonlinear market impact in models such as [2, 3, 4] and this paper. Alfonsi, Schied, and Slynko [3] discover in a discrete-time version of the model of [13] that, even under conditions that prevent price manipulation, it may still be optimal to execute intermediate sells while trying to execute an overall buy order, and they provide conditions to rule out this phenomenon.

For the type of model we consider in this paper, based on a shadow limit-order book, Alfonsi and Schied [2] show that price manipulation is not possible under very general conditions. Furthermore, it is never advantageous to execute intermediate sells while trying to execute an overall buy order. In [2], trading takes place at finitely many stopping times, and execution is optimized over these stopping times. In the present paper, where trading is continuous, we do not permit intermediate sells. This simplification of the model is justified by Remark 3.1, which argues that intermediate sells cannot reduce the total cost.

The present paper is inspired by Obizhaeva and Wang [10], who explicitly model the one-sided limit-order book as a means of capturing the price impact of order execution. Empirical evidence for the model of [10] and its generalizations by Alfonsi, Fruth, and Schied [1, 4] and Alfonsi and Schied [2] are reported in [1, 2, 4, 10]. In [10] and [1], the limit-order book has a block shape, and in this case the price impact of a purchase is linear, the same as in [8, 7]. However, the change of mindset is important because it focuses attention on the shape of the limit-order book as the determinant of price impact, rather than making assumptions about the price impact directly. This change of mindset was exploited in [2, 4], where more general limit-order book shapes are permitted, subject to the condition discussed in Remark 4.4.

In [2, 4], trading is on a discrete-time grid, and it is shown that for an optimal purchasing strategy all purchases except the first and last ones are of the same size. Furthermore, the size of the intermediate purchases is chosen so that the price impact of each purchase is exactly offset by the order book resiliency before the next purchase. Similar results are obtained in [2], although here trades are executed at stopping times.

In contrast to [2, 4, 10], we permit the order book shape to be completely general. However, in our model all price impact is transient; [4, 10] also include the possibility of a permanent linear price impact. In contrast to [2, 4], we do not assume that the limit-order book has a positive density. It can be discrete or continuous and can have gaps. In contrast to [2, 4, 10], we permit the resilience in the order book to be a function of the adjustments to the martingale component of the ask price. Weiss [18] argues in a discrete-time model that this conforms better to empirical observations.

Finally, we set up our model so as to allow for both discrete-time and continuous-time trading, whereas [4, 10] begin with discrete-time trading and then study the limit of their optimal strategies as trading frequency approaches infinity. The simplicity afforded by a fully continuous model is evident in the analysis below. In particular, we provide constructive proofs of Theorems 4.2 and 4.5 that describe the form of the optimal purchasing strategies.

Section 2 of this paper presents our model. It contains the definition of the cost of purchasing in our more general framework, and that is preceded by a justification of the definition. Section 3 shows that randomness can be removed from the optimal purchasing problem and reformulates the cost function into a convenient form. In section 4, we solve the problem, first in the case that is analogous to the one solved by [4] and then in full generality. Sections 4.1 and 4.3 contain examples.

2. The model. Let T be a positive constant. We assume that the ask price of some asset, in the absence of the large investor modeled by this paper, is a continuous nonnegative martingale A_t, $0 \le t \le T$, relative to some filtration $\{\mathcal{F}_t\}_{0\le t\le T}$ satisfying the usual conditions. We assume that

$$\mathbb{E}\left[\max_{0\le t\le T} A_t\right] < \infty. \tag{2.1}$$

We show below that for the optimal execution problem of this paper one can assume without loss of generality that this martingale is identically zero. We make this assumption beginning in section 3 in order to simplify the presentation.

For some extended positive real number M, let μ be an infinite measure on $[0, M)$ that is finite on each compact subset of $[0, M)$. Denote the associated left-continuous cumulative distribution function by

$$F(x) \triangleq \mu\big([0, x)\big), \quad x \ge 0.$$

This is the *shadow limit-order book*, in the sense described below. We assume $F(x) > 0$ for every $x > 0$. If B is a measurable subset of $[0, M)$, then, in the absence of the large investor modeled in this paper, at time $t \ge 0$ the number of limit orders with prices in $B + A_t \triangleq \{b + A_t; b \in B\}$ is $\mu(B)$.

There is a strictly positive constant $\overline{X}$ such that our large investor must purchase $\overline{X}$ shares over the time interval $[0, T]$. His *purchasing strategy* is a nondecreasing right-continuous adapted process X with $X_T = \overline{X}$. We interpret X_t to be the cumulative amount of purchasing done by time t. We adopt the convention $X_{0-} = 0$, so that $X_0 = \Delta X_0$ is the number of shares purchased at time zero. Here and elsewhere, we use the notation ΔX_t to denote the jump $X_t - X_{t-}$ in X at time t.

The effect of the purchasing strategy X on the limit-order book is determined by a *resilience function* h, a strictly increasing, locally Lipschitz function defined on $[0, \infty)$ and satisfying

$$h(0) = 0, \quad h(\infty) \triangleq \lim_{x\to\infty} h(x) > \frac{\overline{X}}{T}. \tag{2.2}$$

The function h together with X determines the *volume effect process*[1] E satisfying

$$E_t = X_t - \int_0^t h(E_s)\, ds, \quad 0 \le t \le T. \tag{2.3}$$

It is shown in Appendix A that there is a unique nonnegative right-continuous finite-variation adapted process E satisfying (2.3). As with X, we adopt the convention $E_{0-} = 0$. We note that $\Delta X_t = \Delta E_t$ for $0 \le t \le T$.

Let B be a measurable subset of $[0, M)$. The interpretation of E is that, in the presence of the large investor using strategy X, at time $t \ge 0$ the number of limit orders with prices in $B + A(t)$ is $\mu_t(B)$, where μ_t is the σ-finite infinite measure on $[0, M)$ with left-continuous cumulative distribution function $(F(x) - E_t)^+$, $x \ge 0$. In other words, E_t units of mass have been removed from the shadow limit-order book μ. In any interval in which no purchases are made, (2.3) implies $\frac{d}{dt}E_t = -h(E_t)$. Hence, in the absence of purchases, the volume effect process decays toward zero and the limit-order book tends toward the shadow limit-order book μ, displaced by the ask price A.

To calculate the cost to the investor of using the strategy X, we introduce the following notation. We first define the left-continuous inverse of F,

$$\psi(y) \triangleq \sup\{x \ge 0 | F(x) < y\}, \quad y > 0.$$

We set $\psi(0) \triangleq \psi(0+) = 0$, where the second equality follows from the assumption that $F(x) > 0$ for every $x > 0$. The *ask price* in the presence of the large investor is defined to be $A_t + D_t$, where

$$D_t \triangleq \psi(E_t), \quad 0 \le t \le T. \tag{2.4}$$

This is the price after any lump purchases by the investor at time t (see Figure 1). We give some justification for calling $A_t + D_t$ the ask price after the following three examples.

Example 2.1 (block order book). Let q be a fixed positive number. If q is the density of shares available at each price, then for each $x \ge 0$ the quantity available at prices in $[0, x]$ is $F(x) = qx$. This is the *block order book* considered by [10]. In this case, $\psi(y) = y/q$ and $F(\psi(y)) = y$ for all $y \ge 0$.

[1]The case that resilience is based on price rather than volume is also considered in [2, 4].

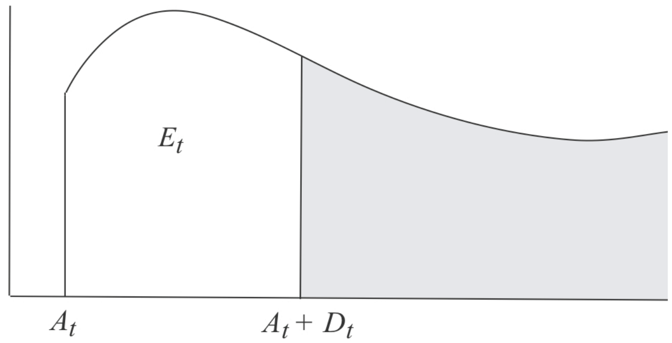

Figure 1. *Limit-order book at time t. The shaded region corresponds to the remaining shares. The white area E_t corresponds to the amount of shares missing from the order book at time t. The current ask price is $A_t + D_t$.*

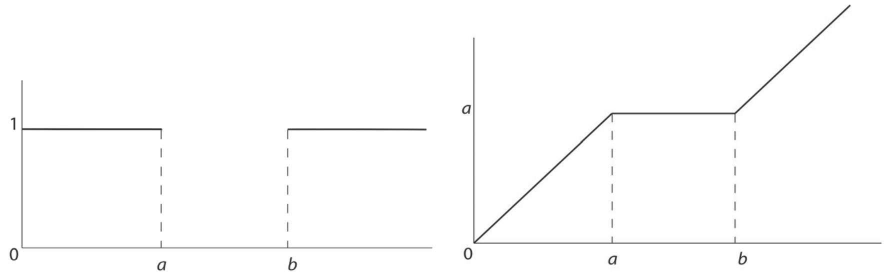

Figure 2. *Density and cumulative distribution of the modified block order book.*

Example 2.2 (modified block order book). Let $0 < a < b < \infty$ be given, and suppose

$$F(x) = \begin{cases} x, & 0 \le x \le a, \\ a, & a \le x \le b, \\ x - (b-a), & b \le x < \infty. \end{cases} \tag{2.5}$$

This is a block order book, except that the orders with prices between a and b are not present (see Figure 2). In this case,

$$\psi(y) = \begin{cases} y, & 0 \le y \le a, \\ y + b - a, & a < y < \infty. \end{cases} \tag{2.6}$$

We have $F(\psi(y)) = y$ for all $y \ge 0$.

Example 2.3 (discrete order book). Suppose that

$$F(x) = \sum_{i=0}^{\infty} \mathbb{I}_{(i,\infty)}(x), \quad x \ge 0, \tag{2.7}$$

which corresponds to an order of size 1 at each of the nonnegative integers (see Figure 3). Then

$$\psi(y) = \sum_{i=1}^{\infty} \mathbb{I}_{(i,\infty)}(y), \quad y \ge 0. \tag{2.8}$$

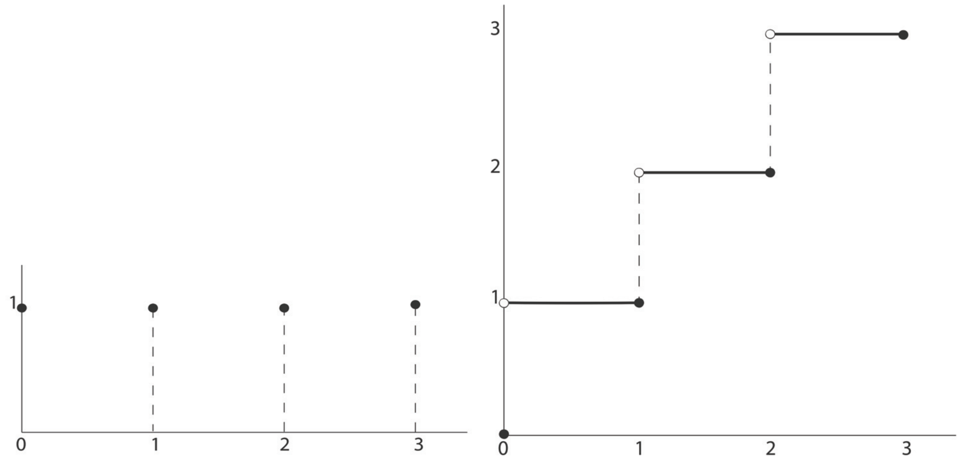

Figure 3. *Measure and cumulative distribution function of the discrete order book.*

For every nonnegative integer j, we have $F(j) = j$, $F(j+) = j + 1$, $\psi(j + 1) = j$, $\psi(j+) = j$, $F(\psi(j)+) = j$, and $\psi(F(j)+) = j$.

We return to the definition of the ask price as $A_t + D_t$ to provide some justification, leading up to Definition 2.4, for the total cost of a purchasing strategy. Suppose, as in Example 2.2, F is constant on an interval $[a, b]$ but strictly increasing to the left of a and to the right of b. Let $y = F(x)$ for $a \le x \le b$. Then $\psi(y) = a$ and $\psi(y+) = b$. Suppose, at time t, we have $E_t = y$. Then $D_t = a$, but the measure μ_t assigns mass zero to $[a, b)$. The ask price is $A_t + D_t$, but there are no shares for sale at this price, nor in an interval to the right of this price. Nonetheless, it is reasonable to call $A_t + D_t$ the ask price for an infinitesimal purchase because if the agent will wait an infinitesimal amount of time before making this purchase, shares will appear at the price $A_t + D_t$ due to resilience. We make this argument more precise.

Suppose the agent wishes to purchase a small number $\varepsilon > 0$ shares at time t at the ask price $A_t + D_t$. This purchase can be approximated by first purchasing zero shares in the time interval $[t, t + \delta]$, where δ is chosen so that $\int_t^{t+\delta} h(E_s)\, ds = \varepsilon$ and

$$E_s = X_t - \int_0^s h(E_u)\, du, \quad t \le s < t + \delta.$$

In other words, E_s for $t \le s < t + \delta$ is given by (2.3) with X held constant (no purchases) over this interval. With δ chosen this way, $E_{(t+\delta)-} = E_t - \varepsilon$. Resilience in the order book has created ε shares. Suppose the investor purchases these shares at time $t + \delta$, which means that $\Delta X_{t+\delta} = \Delta E_{t+\delta} = \varepsilon$ and $E_{t+\delta} = E_t$. Immediately before the purchase, the ask price is $A_{t+\delta} + \psi(E_t - \varepsilon)$; immediately after the purchase, the ask price is $A_{t+\delta} + \psi(E_t) = A_{t+\delta} + a$. The cost of purchasing these shares is

$$\varepsilon A_{t+\delta} + \int_{[\psi(E_t-\varepsilon),a]} \xi\, d\big(F(\xi) - E_t + \varepsilon\big)^+. \tag{2.9}$$

Because $\int_{[\psi(E_t-\varepsilon),a]} d\big(F(\xi) - E_t + \varepsilon\big)^+ = \varepsilon$, the integral in (2.9) is bounded below by $\varepsilon\psi(E_t - \varepsilon)$ and bounded above by εa. But $a = \psi(E_t) = D_t$ and ψ is left continuous, so the cost per share obtained by dividing (2.9) by ε converges to $A_t + a = A_t + D_t$ as ε (and hence δ) converges down to zero.

On the other hand, an impatient agent who does not wait before purchasing shares could choose a different method of approximating an infinitesimal purchase at time t that leads to a limiting cost per share $A_t + b$. In particular, it is not the case that our definition of ask price is consistent with all limits of discrete-time purchasing strategies. Our definition is designed to capture the limit of discrete-time purchasing strategies that seek to minimize cost.

To simplify calculations of the type just presented, we define the functions

$$\varphi(x) = \int_{[0,x)} \xi \, dF(\xi), \quad x \geq 0, \tag{2.10}$$

$$\Phi(y) = \varphi\big(\psi(y)\big) + \big[y - F\big(\psi(y)\big)\big]\psi(y), \quad y \geq 0. \tag{2.11}$$

We note that $\Phi(0) = 0$, and we extend Φ to be zero on the negative half-line. In the absence of the large investor, the cost one would pay to purchase all the shares available at prices in the interval $[A(t), A(t) + x)$ at time t would be $A(t) + \varphi(x)$. The function $\Phi(y)$ captures the cost, in excess of A_t, of purchasing y shares in the absence of the large investor. The first term on the right-hand side of (2.11) is the cost less A_t of purchasing all the shares with prices in the interval $[A_t, A_t + \psi(y))$. If F has a jump at $\psi(y)$, this might be fewer than y shares. The difference, $y - F(\psi(y))$ shares, can be purchased at price $A_t + \psi(y)$, and this explains the second term on the right-hand side of (2.11). We present these functions in the three examples considered earlier.

Example 2.1 (block order book, continued). We have simply $\varphi(x) = q \int_0^x \xi \, d\xi = \frac{q}{2}x^2$ for all $x \geq 0$, and $\Phi(y) = \frac{q}{2}\psi^2(y) = \frac{1}{2q}y^2$ for all $y \geq 0$. Note that Φ is convex and $\Phi'(y) = \psi(y)$ for all $y \geq 0$, including at $y = 0$ because we define Φ to be identically zero on the negative half-line.

Example 2.2 (modified block order book, continued). With F and ψ given by (2.5) and (2.6), we have

$$\varphi(x) = \begin{cases} \dfrac{1}{2}x^2, & 0 \leq x \leq a, \\[2ex] \dfrac{1}{2}a^2, & a \leq x \leq b, \\[2ex] \dfrac{1}{2}(x^2 + a^2 - b^2), & b \leq x < \infty, \end{cases}$$

and

$$\Phi(y) = \begin{cases} \dfrac{1}{2}y^2, & 0 \leq y \leq a, \\[2ex] \dfrac{1}{2}\big((y + b - a)^2 + a^2 - b^2\big), & a \leq y < \infty. \end{cases}$$

Note that Φ is convex with subdifferential

$$\partial\Phi(y) = \begin{cases} \{y\}, & 0 \leq y < a, \\ [a, b], & y = a, \\ \{y + b - a\}, & a < y < \infty. \end{cases} \tag{2.12}$$

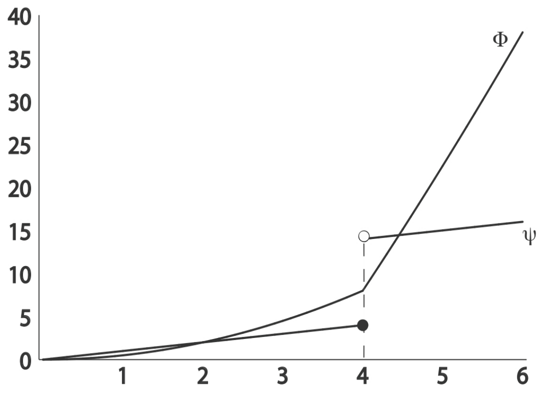

Figure 4. *Functions Φ and ψ for the modified block order book with parameters $a = 4$ and $b = 14$.*

In particular, $\partial\Phi(y) = [\psi(y), \psi(y+)]$ for all $y \geq 0$ (see Figure 4).

Example 2.3 (discrete order book, continued). With F given by (2.7), we have $\varphi(x) = \sum_{i=0}^{\infty} i\mathbb{I}_{(i,\infty)}(x)$. In particular, $\varphi(0) = 0$, and for integers $k \geq 1$ and $k - 1 < x \leq k$,

$$\varphi(x) = \sum_{i=0}^{k-1} i = \frac{k(k-1)}{2}.$$

For $0 \leq y \leq 1$, $\psi(y) = 0$ and hence $\varphi(\psi(y)) = 0$, $[y - F(\psi(y))]\psi(y) = 0$, and $\Phi(y) = 0$. For integers $k \geq 1$ and $k < y \leq k + 1$, (2.8) gives $\psi(y) = k$, and hence $\varphi(\psi(y)) = \frac{k(k-1)}{2}$. Finally, for y in this range, $[y - F(\psi(y))]\psi(y) = k(y - k)$. We conclude that

$$\Phi(y) = \sum_{k=1}^{\infty} k\left(y - \frac{1}{2}k - \frac{1}{2}\right)\mathbb{I}_{(k,k+1]}(y). \tag{2.13}$$

For each positive integer k, $\Phi(k-) = \Phi(k+) = \frac{1}{2}k(k-1)$, so Φ is continuous. Furthermore, $\partial\Phi(k) = [k-1, k] = [\psi(k), \psi(k+)]$. For nonnegative integers k and $k < y < k+1$, $\Phi'(y)$ is defined and is equal to $\psi(y) = k$. Furthermore, $\Phi'(0) = \psi(0) = 0$. Once again we have $\partial\Phi(y) = [\psi(y), \psi(y+)]$ for all $y \geq 0$, and because ψ is nondecreasing, Φ is convex (see Figure 5).

We decompose the purchasing strategy X into its continuous and pure jump parts $X_t = X_t^c + \sum_{0 \leq s \leq t} \Delta X_s$. The investor pays price $A_t + D_t$ for infinitesimal purchases at time t, and hence the total cost of these purchases is $\int_0^T (A_t + D_t)\, dX_t^c$. On the other hand, if $\Delta X_t > 0$, the investor makes a lump purchase of size $\Delta X_t = \Delta E_t$ at time t. Because mass E_{t-} is missing in the shadow order book immediately prior to time t, the cost of this purchase is the difference between purchasing E_t and purchasing E_{t-} from the shadow order book, i.e., the difference in what the costs of these purchases would be in the absence of the large investor. Therefore, the cost of the purchase ΔX_t at time t is $A_t\Delta X_t + \Phi(E_t) - \Phi(E_{t-})$. These considerations lead to the following definition.

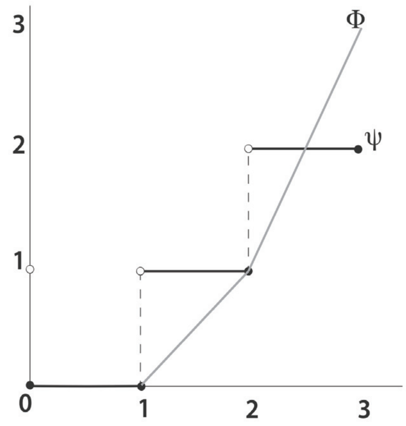

Figure 5. *Functions Φ and ψ for the discrete order book.*

Definition 2.4. *The* total cost *incurred by the investor using purchasing strategy X over the interval $[0,T]$ is*

$$C(X) \triangleq \int_0^T (A_t + D_t)\, dX_t^c + \sum_{0\le t\le T} \big[A_t \Delta X_t + \Phi(E_t) - \Phi(E_{t\_})\big]$$

$$= \int_0^T D_t\, dX_t^c + \sum_{0\le t\le T} \big[\Phi(E_t) - \Phi(E_{t-})\big] + \int_{[0,T]} A_t\, dX_t. \tag{2.14}$$

Our goal is to determine the purchasing strategy X that minimizes $\mathbb{E}C(X)$.

3. Problem simplifications. To compute the expectation of $C(X)$ defined by (2.14), we invoke the integration by parts formula

$$\int_{[0,T]} A_t\, dX_t = A_T X_T - A_0 X_{0-} - \int_0^T X_t\, dA_t$$

for the bounded variation process X and the continuous martingale A. Our investor's strategies must satisfy $0 = X_{0-} \le X_t \le X_T = \overline{X}$, $0 \le t \le T$, and hence $\mathbb{E}\int_0^T X_t\, dA_t = 0$ (see Appendix B) and $\mathbb{E}\int_0^T A_t\, dX_t = \overline{X}\mathbb{E}A_T = \overline{X}A_0$. It follows that

$$\mathbb{E}C(X) = \mathbb{E}\int_0^T D_t\, dX_t^c + \mathbb{E}\sum_{0\le t\le T} \big[\Phi(E_t) - \Phi(E_{t-})\big] + \overline{X}A_0. \tag{3.1}$$

Since the third term on the right-hand side of (3.1) does not depend on X, minimization of $\mathbb{E}C(X)$ is equivalent to minimization of the first two terms. But the first two terms do not depend on A, and hence we may assume without loss of generality that A is identically zero. Under this assumption, the cost of using strategy X is

$$C(X) = \int_0^T D_t\, dX_t^c + \sum_{0\le t\le T} \big[\Phi(E_t) - \Phi(E_{t-})\big]. \tag{3.2}$$

But, with $A \equiv 0$, there is no longer a source of randomness in the problem. Consequently, without loss of generality we may restrict the search for an optimal strategy to nonrandom functions of time. Once we find a nonrandom purchasing strategy minimizing (3.2), then even if A is a continuous nonzero nonnegative martingale, we have found a purchasing strategy that minimizes the expected value of (2.14) over all (possibly random) purchasing strategies.

Remark 3.1. We do not allow our agent to make intermediate sells in order to achieve the ultimate goal of purchasing $\overline{X}$ shares because doing so would not decrease the cost, at least when the total amount of buying and selling is bounded. Indeed, in addition to the purchasing strategy X, suppose the agent has a selling strategy Y, which we take to be a nondecreasing right-continuous adapted process with $Y_{0-}=0$. We assume that both X and Y are bounded. For each t, X_t represents the number of shares bought by time t and Y_t is the number of shares sold. These processes must be chosen so that $X_T - Y_T = \overline{X}$. We have not modeled the limit-buy-order book, but if we did so in a way analogous to the model of the limit-sell-order book, then the bid price at each time t would be less than or equal to A_t. Therefore, the net cost of executing the strategy (X,Y) would satisfy

$$C(X,Y) \geq \int_0^T D_t\,dX_t^c + \sum_{0\leq t\leq T}\big[\Phi(E_t)-\Phi(E_{t-})\big] + \int_{[0,T]} A_t\,dX_t - \int_{[0,T]} A_t\,dY_t.$$

The integration by parts formula implies

$$\begin{aligned}\int_{[0,T]} A_t\,dX_t - \int_{[0,T]} A_t\,dY_t &= A_T(X_T-Y_T) - A_0(X_{0-}-Y_{0-}) - \int_0^T (X_t-Y_t)\,dA_t\\ &= A_T\overline{X} - \int_0^T (X_t-Y_t)\,dA_t.\end{aligned}$$

Because we can apply Lemma B.1 to both X and Y, the expectation of $\int_0^T (X_t-Y_t)\,dA_t$ is zero and

$$\mathbb{E}C(X,Y) \geq \mathbb{E}\int_0^T D_t\,dX_t^c + \mathbb{E}\sum_{0\leq t\leq T}\big[\Phi(E_t)-\Phi(E_{t-})\big] + \overline{X}A_0. \tag{3.3}$$

The right-hand side of (3.3) is the formula (3.1) obtained for the cost of using the purchasing strategy X alone, but the X in inequality (3.3) makes a total purchase of $X_T = \overline{X} + Y_T \geq \overline{X}$. If we replace X by $\min\{X, \overline{X}\}$, we obtain a feasible purchasing strategy whose total cost is less than or equal to the right-hand side of (3.3).

Theorem 3.2. *Under the assumption (made without loss of generality) that A is identically zero, the cost* (3.2) *associated with a nonrandom nondecreasing right-continuous function X_t, $0 \leq t \leq T$, satisfying $X_{0-}=0$ and $X_T = \overline{X}$ is equal to*

$$C(X) = \Phi(E_T) + \int_0^T D_t h(E_t)\,dt. \tag{3.4}$$

Proof. The proof proceeds in two steps. In Step 1 we show that, as we have seen in the examples, Φ is a convex function with subdifferential

$$\partial\Phi(y) = [\psi(y), \psi(y+)], \quad y \geq 0. \tag{3.5}$$

In Step 2 we justify the integration formula

$$\Phi(E_T) = \int_0^T D^-\Phi(E_t)\,dE_t^c + \sum_{0\le t\le T}\big[\Phi(E_t) - \Phi(E_{t-})\big], \tag{3.6}$$

where $D^-\Phi(E_t)$ denotes the left-hand derivative $\psi(E_t) = D_t$ of Φ at E_t, and E^c is the continuous part of E: $E_t^c = E_t - \sum_{0\le s\le t}\Delta E_s$. From (2.3) and (3.6) we have immediately that

$$\Phi(E_T) = \int_{[0,T]} D_t\,dX_t^c - \int_0^T D_t h(E_t)\,dt + \sum_{0\le t\le T}\big[\Phi(E_t) - \Phi(E_{t-})\big],$$

and (3.4) follows from (3.2).

Step 1. Using the integration by parts formula $xF(x) = \int_{[0,x)}\xi\,dF(\xi) + \int_0^x F(\xi)\,d\xi$, we write

$$\begin{aligned}\Phi(y) &= \int_{[0,\psi(y))}\xi\,dF(\xi) + [y - F(\psi(y))]\psi(y)\\ &= \int_0^{\psi(y)}\big(y - F(\xi)\big)\,d\xi\\ &= \int_0^{\psi(y)}\int_{F(\xi)}^y d\eta\,d\xi\\ &= \int_0^y\int_0^{\psi(\eta)} d\xi d\eta,\end{aligned}$$

where the last step follows from the fact that the symmetric difference of the sets $\{(\eta,\xi)|\xi\in[0,\psi(y)],\eta\in[F(\xi),y]\}$ and $\{(\eta,\xi)|\eta\in[0,y],\xi\in[0,\psi(\eta)]\}$ is an at most countable union of line segments and thus has two-dimensional Lebesgue measure 0. Therefore,

$$\Phi(y) = \int_0^y \psi(\eta)\,d\eta, \tag{3.7}$$

and by Problem 3.6.20 on p. 213 of [17], with ψ and Φ extended to be 0 for the negative reals, we conclude that Φ is convex and that $\partial\Phi(y) = [\psi(y),\psi(y+)]$, as desired.

Step 2. We mollify ψ, taking ρ to be a nonnegative C^∞ function with support on $[-1,0]$ and integral 1, defining $\rho_n(\eta) = n\rho(n\eta)$, and defining

$$\psi_n(y) = \int_{\mathbb{R}}\psi(y+\eta)\rho_n(\eta)\,d\eta = \int_{\mathbb{R}}\psi(\zeta)\,\rho_n(\zeta - y)\,d\zeta.$$

Then each ψ_n is a C^∞ function satisfying $0\le\psi_n(y)\le\psi(y)$ for all $y\ge 0$. Furthermore, $\psi(y) = \lim_{n\to\infty}\psi_n(y)$ for every $y\in\mathbb{R}$. We set $\Phi_n(y) = \int_0^y\psi_n(\eta)\,d\eta$, so that each Φ_n is also a C^∞ function and $\lim_{n\to\infty}\Phi_n'(y) = D^-\Phi(y)$.

Because $\Phi_n(E_{0-}) = \Phi(0) = 0$, we have

$$\Phi_n(E_T) = \int_0^T \Phi_n'(E_t)\,dE_t^c + \sum_{0\le t\le T}[\Phi_n(E_t) - \Phi_n(E_{t-})]; \tag{3.8}$$

see, e.g., [11, p. 78]. The function E_t, $0 \le t \le T$, is bounded. Letting $n \to \infty$ in (3.8) and using the bounded convergence theorem, we obtain

$$\Phi(E_T) = \int_0^T D^-\Phi(E_t)\,dE_t^c + \lim_{n\to\infty} \sum_{0\le t\le T} \big[\Phi_n(E_t) - \Phi_n(E_{t-})\big]. \tag{3.9}$$

To conclude the proof of (3.6), we divide the sum in (3.9) into two parts. Given $\delta > 0$, we define $S_\delta = \{t \in [0,T] : 0 < \Delta E_t \le \delta\}$ and $S'_\delta = \{t \in [0,T] : \Delta E_t > \delta\}$. The sum in (3.9) is over $t \in S_\delta \cup S'_\delta$, and because E has finite variation, $\sum_{t\in S_\delta \cup S'_\delta} \Delta E_t < \infty$. Let $\varepsilon > 0$ be given. We choose $\delta > 0$ so small that $\sum_{t\in S_\delta} \Delta E_t \le \varepsilon$. Because ψ (and hence each ψ_n) is bounded on $[0, E_T]$, the function Φ and each Φ_n is Lipschitz continuous on $[0, E_T]$ with the same Lipschitz constant $L = \psi(E_T)$. It follows that

$$\sum_{t\in S_\delta} \big[\Phi(E_t) - \Phi(E_{t-})\big] \le L \sum_{t\in S_\delta} \Delta E_t \;\le\; L\varepsilon,$$
$$\sum_{t\in S_\delta} \big[\Phi_n(E_t) - \Phi_n(E_{t-})\big] \le L \sum_{t\in S_\delta} \Delta E_t \;\le\; L\varepsilon, \quad n = 1, 2, \dots .$$

Hence the difference between $\sum_{t\in S_\delta} \big[\Phi(E_t) - \Phi(E_{t-})\big]$ and any limit point as $n \to \infty$ of $\sum_{t\in S_\delta} \big[\Phi_n(E_t) - \Phi_n(E_{t-})\big]$ is at most $2L\varepsilon$. On the other hand, the set S'_δ contains only finitely many elements, and thus

$$\lim_{n\to\infty} \sum_{t\in S'_\delta} \big[\Phi_n(E_t) - \Phi_n(E_{t-})\big] = \sum_{t\in S'_\delta} \big[\Phi(E_t) - \Phi(E_{t-})\big].$$

Since $\varepsilon > 0$ is arbitrary, (3.9) reduces to (3.6). ■

4. Solution of the optimization problem.

In view of Theorem 3.2, we want to minimize $\Phi(E_T) + \int_0^T D_t h(E_t)\,dt$ over the set of deterministic purchasing strategies. The main result of this paper is that there exists an optimal strategy X under which the trader buys a lump quantity $X_0 = E_0$ of shares at time 0, then buys at a constant rate $dX_t = h(E_0)\,dt$ up to time t_0 (so as to keep $E_t = E_0$ for $t \in [0, t_0)$), then buys another lump quantity of shares at time t_0, subsequently trades again at a constant rate $dX_t = h(E_{t_0})\,dt$ until time T (so as to keep $E_t = E_{t_0}$ for $t \in [t_0, T)$), and finally buys the remaining shares at time T. We shall call this strategy a *Type B strategy*. We further show that if the nonnegative function

$$g(y) \triangleq y\psi\big(h^{-1}(y)\big) \tag{4.1}$$

is convex, then the purchase at time t_0 consists of 0 shares (so X has jumps only at times 0 and T). We call such a strategy a *Type A strategy*. Clearly the latter is a special case of the former.

Although g is naturally defined on $[0, h(\infty))$ by (4.1), we will want it to be defined on a compact set. Therefore we set

$$\overline{Y} = \max\left\{h(\overline{X}), \frac{\overline{X}}{T}\right\} \tag{4.2}$$

and note that, because of assumption (2.2), h^{-1} is defined on $[0,\overline{Y}]$. We specify the domain of the function g to be $[0,\overline{Y}]$. For future reference, we make three observations about the function g. First,

$$\lim_{y\downarrow 0} g(y) = g(0) = 0. \tag{4.3}$$

Second, using the definition (2.4) of D_t, we can rewrite the cost function formula (3.4) as

$$C(X) = \Phi(E_T) + \int_0^T g\big(h(E_t)\big)\,dt. \tag{4.4}$$

Lemma A.1(iv) in Appendix A shows that $0 \le E_t \le \overline{X}$, so the domain $[0,\overline{Y}]$ of g is large enough in order for (4.4) to make sense. Because h^{-1} is strictly increasing and continuous and ψ is nondecreasing and left continuous, the function g is nondecreasing and left continuous and hence lower semicontinuous. In particular,

$$g\big(\overline{Y}\big) = \lim_{y\uparrow\overline{Y}} g(y). \tag{4.5}$$

4.1. Convexity and Type A strategies.

Remark 4.1. A Type A strategy X^A can be characterized in terms of the terminal value E_T^A of the process E^A related to X^A by (2.3), and the cost of using a Type A strategy can be written as a function of E_T^A. It is this function of E_T^A we will minimize. To see that this is possible, let X^A be a Type A strategy and let E^A be related to X^A via (2.3), so that $E_t^A = X_0^A$ for $0 \le t < T$. Then

$$X_{T-}^A = E_{T-}^A + \int_0^T h(E_t^A)\,dt \;=\; X_0^A + h(X_0^A)T, \tag{4.6}$$

$$\Delta X_T^A = \overline{X} - X_{T-}^A \;=\; \overline{X} - X_0^A - h(X_0^A)T, \tag{4.7}$$

$$E_T^A = E_{T-}^A + \Delta X_T^A \;=\; \overline{X} - h(X_0^A)T. \tag{4.8}$$

A Type A strategy is fully determined by its initial condition X_0^A, and from (4.8) we now see that choosing X_0^A is equivalent to choosing E_T^A. According to (4.4) and (4.8), the cost of this strategy

$$C(X^A) = \Phi(E_T^A) + Tg\big(h(X_0^A)\big) = \Phi(E_T^A) + Tg\left(\frac{\overline{X} - E_T^A}{T}\right) \tag{4.9}$$

can be written as a function of E_T^A.

We conclude this remark by determining the range of values that E_T^A can take for a Type A strategy. We must choose X_0^A so that $X_0^A \ge 0$ and X_{T-}^A given by (4.6) does not exceed $\overline{X}$. The function $k(x) \triangleq x + h(x)T$ is strictly increasing and continuous on $[0,\infty)$, and $k(\overline{X}) > \overline{X}$. Therefore, there exists a unique $\overline{e} \in (0,\overline{X})$ such that $k(\overline{e}) = \overline{X}$, i.e.,

$$\overline{e} + h(\overline{e})T = \overline{X}. \tag{4.10}$$

The constraint on the initial condition of Type A strategies that guarantees that the strategy is feasible is $0 \le X_0^A \le \overline{e}$. From (4.8) and (4.10) we see that the corresponding feasibility condition on E_T^A for Type A strategies is

$$\overline{e} \le E_T^A \le \overline{X}. \tag{4.11}$$

Theorem 4.2. *If g given by (4.1) is convex on $[0, \overline{Y}]$, then there exists a Type A purchasing strategy that minimizes $C(X)$ over all purchasing strategies X. If g is strictly convex, this is the unique optimal strategy.*

Proof. Assume that g is convex, and let X be a purchasing strategy. Jensen's inequality applied to (4.4) yields the lower bound

$$C(X) = \Phi(E_T) + T \int_0^T g\big(h(E_t)\big)\frac{dt}{T} \ge \Phi(E_T) + Tg\left(\int_0^T h(E_t)\frac{dt}{T}\right).$$

From (2.3) we further have $\int_0^T h(E_t)\,dt = \overline{X} - E_T$, and thus the lower bound can be rewritten as

$$C(X) \ge \Phi(E_T) + Tg\left(\frac{\overline{X} - E_T}{T}\right). \tag{4.12}$$

Recall that $0 \le E_T \le \overline{X}$, so the argument of g in (4.12) is in $[0, \overline{Y}]$.

This leads us to consider minimization of the function

$$G(e) \triangleq \Phi(e) + Tg\left(\frac{\overline{X} - e}{T}\right)$$

over $e \in [0, \overline{X}]$. By assumption, the function g is convex on $[0, \overline{Y}]$ and hence continuous on $(0, \overline{Y})$. Equations (4.3) and (4.5) show that g is also continuous at the endpoints of its domain. Because Φ has the integral representation (3.7), it also is convex and continuous on $[0, \overline{X}]$. Therefore, G is a convex continuous function on $[0, \overline{X}]$, and hence the minimum is attained.

We show next that the minimum of G over $[0, \overline{X}]$ is attained in $[\overline{e}, \overline{X}]$. For this, we first observe that, because g is convex,

$$D^+g(y) \ge \frac{g(y) - g(0)}{y} = \psi\big(h^{-1}(y)\big), \quad 0 < y \le \overline{Y}.$$

This inequality together with (3.5) and (4.10) implies that

$$D^-G(\overline{e}) = \psi(\overline{e}) - D^+g(y)\Big|_{y=\frac{\overline{X}-\overline{e}}{T}} \le \psi(\overline{e}) - \psi\left(h^{-1}\left(\frac{\overline{X} - \overline{e}}{T}\right)\right) = 0. \tag{4.13}$$

Therefore, the minimum of the convex function G over $[0, \overline{X}]$ is obtained in $[\overline{e}, \overline{X}]$.

Let $e^* \in [\overline{e}, \overline{X}]$ attain the minimum of G over $[0, \overline{X}]$. The Type A strategy X^A with initial condition $X_0^A = h^{-1}(\frac{\overline{X}-e^*}{T})$ satisfies $E_T^A = e^*$ (see (4.8)), and hence the strategy is feasible (see (4.11)). The cost associated with this strategy is less than or equal to the right-hand side of (4.12) (see (4.9)). This strategy is therefore optimal.

If g is strictly convex at the point $\frac{\overline{X}-e^*}{T}$, where e^* minimizes G, then G is strictly convex at e^*, and this point is thus the unique minimizer of G. Therefore, every optimal strategy strategy must satisfy $E_T = e^*$. By strict convexity of g, a strategy that does not keep $h(E)$ equal to $\frac{\overline{X}-e^*}{T}$ almost everywhere in $(0, T)$ would result in strict inequality in (4.12). Since h is strictly increasing, we conclude that the only optimal strategy is the Type A strategy constructed above. ■

If g is not strictly convex at the point $\frac{\overline{X}-e^*}{T}$ found in the proof of Theorem 4.2, then G might still be strictly convex at e^*, in which case there would be only one optimal strategy of Type A, but there could be optimal strategies that are not of Type A. We demonstrate this phenomenon with an example.

Example 4.3 (nonuniqueness of optimal purchasing strategy). Suppose

$$F(x) = \begin{cases} x, & 0 \le x \le 2, \\ \frac{4}{4-x}, & 2 \le x \le 3, \\ 4 + \frac{1}{8}(x-3), & x \ge 3. \end{cases}$$

This function is continuous and strictly increasing, and hence

$$\psi(y) = \begin{cases} y, & 0 \le y \le 2, \\ 4 - \frac{4}{y}, & 2 \le y \le 4, \\ 8y - 29, & y \ge 4, \end{cases}$$

is also continuous and strictly increasing. This implies that

$$\Phi(y) = \int_0^y \psi(\eta) d\eta = \begin{cases} \frac{1}{2}y^2, & 0 \le y \le 2, \\ 4y - 6 - 4\log\frac{y}{2}, & 2 \le y \le 4, \\ 4y^2 - 29y + 62 - 4\log 2, & y \ge 4. \end{cases}$$

We take $h(x) = x$, so that

$$g(y) = y\psi(y) = \begin{cases} y^2, & 0 \le y \le 2, \\ 4y - 4, & 2 \le y \le 4, \\ 8y^2 - 29y, & y \ge 4, \end{cases}$$

and

$$g'(y) = \begin{cases} 2y, & 0 \le y \le 2, \\ 4, & 2 \le y < 4, \\ 16y - 29, & y > 4. \end{cases}$$

Note that g' is nondecreasing, so g is convex, but g is not strictly convex on the interval $[2, 4]$. Finally, we take $\overline{X} = 10\frac{1}{8}$ and $T = 2$.

In the notation of the proof of Theorem 4.2, we have $e^* = 4\frac{1}{8}$ and hence $\frac{\overline{X}-e^*}{T} = 3$. Indeed, $G'\left(4\frac{1}{8}\right) = \psi\left(4\frac{1}{8}\right) - g'(3) = 0$, and because ψ is strictly increasing, G is strictly convex, and hence $4\frac{1}{8}$ is the unique minimizer of G.

The Type A strategy with $E_T^A = 4\frac{1}{8}$ begins with an initial purchase of $X_0^A = 3$ and then consumes at rate 3 over the interval $[0, 2]$, so that $E_t^A = 3$ for $0 \le t < T$. At the final time

$T = 2$, there is an additional lump purchase of $1\frac{1}{8}$, so that $E_T^A = 4\frac{1}{8}$. The total cost of this strategy is

$$\Phi(E_T^A) + \int_0^T g(E_t^A)\,dt = \Phi\left(4\frac{1}{8}\right) + \int_0^2 \left(4E_t^A - 4\right) dt = \Phi\left(4\frac{1}{8}\right) + 16.$$

In particular, $\int_0^2 E_t^A\,dt = 6$.

In fact, any policy that satisfies $2 \le E_t \le 4$, $0 \le t < 2$, and $\int_0^2 E_t\,dt = 6$ will result in the same cost. Indeed, for such a policy, we will have

$$E_T = X_T - \int_0^T E_t\,dt = 10\frac{1}{8} - 6 = 4\frac{1}{8} = E_T^A$$

and

$$\int_0^T g(E_t)\,dt = \int_0^T (4E_t - 4)\,dt = 16 = \int_0^T g(E_t^A)\,dt,$$

so $\Phi(E_T) + \int_0^T g(E_t)\,dt = \Phi(E_T^A) + \int_0^T g(E_t^A)\,dt$. There are infinitely many policies like this. One such policy is to make an initial lump purchase of size 2 and then purchase at rate 2 up to time $\frac{1}{2}$ so that $E_t = 2$, $0 \le t < \frac{1}{2}$, make a lump purchase of size 1 at time $\frac{1}{2}$ and then purchase at rate 3 up to time $\frac{3}{2}$ so that $E_t = 3$, $\frac{1}{2} \le t < \frac{3}{2}$, make a lump purchase of size 1 at time $\frac{3}{2}$ and then purchase at rate 4 up to time 2 so that $E_t = 4$, $\frac{3}{2} \le t < 2$, and conclude with a lump purchase of size $\frac{1}{8}$ at time 2 so that $E_2 = 4\frac{1}{8}$.

Remark 4.4. Alfonsi, Fruth, and Schied [4] consider the case that the measure μ has a strictly positive density f. In this case, the function $F(x) = \int_0^x f(\xi)\,d\xi$ is strictly increasing and continuous with derivative $F'(x) = f(x)$, and its inverse ψ is likewise strictly increasing and continuous with derivative $\psi'(y) = 1/f(\psi(y))$. Furthermore, in [4], the resilience function is $h(x) = \rho x$, where ρ is a positive constant. In this case,

$$g'(y) = \psi(y/\rho) + \frac{y/\rho}{f\big(\psi(y/\rho)\big)},$$

and Theorem 4.2 guarantees the existence of a Type A strategy under the assumption that g' is nondecreasing. This is equivalent to the condition that

$$\psi(y) + \frac{y}{f\big(\psi(y)\big)}$$

is nondecreasing.

Alfonsi, Fruth, and Schied [4] obtain a discrete-time version of a Type A strategy under the assumption that

$$h_1(y) \triangleq \psi(y) - e^{-\rho\tau}\psi(e^{-\rho\tau}y)$$

is strictly increasing, where τ is the time between trading dates. In order to study the limit of their model as $\tau \downarrow 0$, they observe that

$$\lim_{\tau\downarrow 0} \frac{h_1(y)}{1 - e^{-\rho\tau}} = \psi(y) + \frac{y}{f\big(\psi(y)\big)},$$

which is thus nondecreasing. Thus g given by (4.1) is convex in their model.

To find a simpler formulation of the hypothesis of Theorem 4.2 under the assumption that μ has a strictly positive density f and $h(x) = \rho x$ for a positive constant ρ, we compute

$$\frac{d}{dy}\left(\psi(y) + \frac{y}{f\big(\psi(y)\big)}\right) = \frac{2}{f\big(\psi(y)\big)} - \frac{yf'\big(\psi(y)\big)}{f^3\big(\psi(y)\big)}.$$

This is nonnegative if and only if $2f^2\big(\psi(y)\big) \geq yf'\big(\psi(y)\big)$. Replacing y by $F(x)$, we obtain the condition

$$2f^2(x) \geq F(x)f'(x), \quad x \geq 0.$$

This is clearly satisfied under the assumption of [10] that f is a positive constant.

Example 2.1 (block order book, continued). In the case of the block order book with $h(x) = \rho x$, where ρ is a strictly positive constant,

$$g(y) = \frac{yh^{-1}(y)}{q} = \frac{y^2}{\rho q},$$

which is strictly convex. Theorem 4.2 implies that there is an optimal strategy of Type A, and this is the unique optimal strategy. From the formula $\Phi(e) = \frac{1}{2q}e^2$, we have

$$G(e) = \frac{e^2}{2q} + \frac{(\overline{X} - e)^2}{\rho q T}.$$

The minimizer is $e^* = \frac{2\overline{X}}{2+\rho T}$, which lies between $\overline{e} = \frac{\overline{X}}{1+\rho T}$ and $\overline{X}$, as expected. According to Remark 4.1, the optimal strategy of Type A is to make an initial purchase of size

$$X_0^A = h^{-1}\left(\frac{\overline{X} - e^*}{T}\right) = \frac{\overline{X}}{2+\rho T},$$

then purchase continuously at rate $dX_t^A = h(X_0^A)\,dt = \frac{\rho\overline{X}}{2+\rho T}\,dt$ over the time interval $[0,T]$, and conclude with a lump purchase

$$e^* - X_0^A = \frac{\overline{X}}{2+\rho T}$$

at the final time T. In particular, the initial and final lump purchases are of the same size, and there is no intermediate lump purchase.

4.2. Type B strategies.

Theorem 4.5. *In the absence of the assumption that g given by* (4.1) *is convex, there exists a Type B purchasing strategy that minimizes $C(X)$ over all purchasing strategies X.*

The proof of Theorem 4.5 depends on the following lemma, whose proof is given in Appendix C.

Lemma 4.6. *The convex hull of g, defined by*

$$\widehat{g}(y) \triangleq \sup\left\{\ell(y) : \ell \textit{ is an affine function and } \ell(\eta) \leq g(\eta)\ \forall \eta \in [0,\overline{Y}]\right\}, \tag{4.14}$$

is the largest convex function defined on $[0,\overline{Y}]$ *that is dominated by* g *there. It is continuous and nondecreasing on* $[0,\overline{Y}]$, $\widehat{g}(0)=g(0)=0$, *and* $\widehat{g}(\overline{Y})=g(\overline{Y})$. *If* $y^*\in(0,\overline{Y})$ *satisfies* $\widehat{g}(y^*)<g(y^*)$, *then there exists a unique affine function* ℓ *lying below* g *on* $[0,\overline{Y}]$ *and agreeing with* $\widehat{g}$ *at* y^*. *In addition, there exist numbers* α *and* β *satisfying*

$$0\le\alpha<y^*<\beta\le\overline{Y}, \tag{4.15}$$

$$\ell(\alpha)=\widehat{g}(\alpha)=g(\alpha),\quad \ell(\beta)=\widehat{g}(\beta)=g(\beta), \tag{4.16}$$

$$\ell(y)=\widehat{g}(y)<g(y),\quad \alpha<y<\beta. \tag{4.17}$$

Proof of Theorem 4.5. Using $\widehat{g}$ in place of g in (4.4), we define the *modified cost function*

$$\widehat{C}(X)\triangleq\Phi(E_T)+\int_0^T\widehat{g}\big(h(E_t)\big)\,dt.$$

For any purchasing strategy X, we obviously have $\widehat{C}(X)\le C(X)$. Analogously to (4.12), for any purchasing strategy X, the lower bound

$$\widehat{C}(X)\ge\Phi(E_T)+T\widehat{g}\left(\frac{\overline{X}-E_T}{T}\right)$$

holds. This leads us to consider minimization of the function

$$\widehat{G}(e)\triangleq\Phi(e)+T\widehat{g}\left(\frac{\overline{X}-e}{T}\right) \tag{4.18}$$

over $e\in[0,\overline{X}]$. As in the proof of Theorem 4.2, this function attains its minimum at some $e^*\in[0,\overline{X}]$.

For the remainder of the proof, we use the notation

$$y^*=\frac{\overline{X}-e^*}{T},\quad x^*=h^{-1}(y^*), \tag{4.19}$$

where it is assumed without loss of generality that e^* is the largest minimizer of $\widehat{G}$ in $[0,\overline{X}]$. There are two cases. In both cases, we construct a strategy that satisfies $E_T^B=e^*$ and

$$C(X^B)=\widehat{G}(e^*). \tag{4.20}$$

In the first case, the strategy is a Type A strategy, and it is Type B in the second case. In both cases, we exhibit the strategy explicitly.

Case I. $\widehat{g}(y^*)=g(y^*)$. It is tempting to claim that we are now in the situation of Theorem 4.2 with the convex function $\widehat{g}$ replacing g. However, the proof needed here that $e^*\ge\overline{e}$, where $\overline{e}$ is determined by (4.10), cannot follow the proof of Theorem 4.2. In the proof of Theorem 4.2, this inequality was a consequence of (4.13), which ultimately depended on the definition (4.1) of $g(\overline{e})$. But we have only $\widehat{g}(\overline{e})\le\overline{e}\psi(h^{-1}(\overline{e}))$; we do not have an equation analogous to (4.1) for $\widehat{g}$. We thus provide a different proof, which depends on e^* being the largest minimizer of $\widehat{G}$ in $[0,\overline{X}]$.

If $x^*=0$, then $y^*=0$, $e^*=\overline{X}$, and $\widehat{G}(e^*)=G(e^*)$. The Type A strategy that waits until the final time T and then purchases $\overline{X}$ is optimal. In particular, this strategy satisfies the initial condition $X_0^A=x^*$.

If $x^*>0$, we must consider two subcases. It could be that $0<x^*\le F(0+)$. In this subcase, $\widehat{g}(y^*)=g(y^*)=y^*\psi(x^*)=0$ because $\psi\equiv 0$ on $[0,F(0+)]$. But $\widehat{g}(0)=0$ and $\widehat{g}$ is nondecreasing, so $\widehat{g}\equiv 0$ on $[0,y^*]$. Furthermore, x^* is positive, so $e^*<\overline{X}$. For $e\in(e^*,\overline{X})$, the number $\frac{\overline{X}-e}{T}$ is in $(0,y^*)$, and by (3.5), $D^+\widehat{G}(e)=D^+\Phi(e)=\psi(e+)$. On the other hand, e^* is the largest minimizer of $\widehat{G}$ in $[0,\overline{X}]$, which implies that $D^+\widehat{G}(e)>0$. This shows that $\psi(e+)>0$ for every $e\in(e^*,\overline{X})$, which implies that $\psi(e)>0$ for every $e\in(e^*,\overline{X})$ and further implies that $e>F(0+)$ for every $e\in(e^*,\overline{X})$. We conclude that $e^*\ge F(0+)$. Applying h to this inequality and using the subcase assumption $x^*\le F(0+)$, we obtain

$$h(e^*)\ge h\big(F(0+)\big)\ge h(x^*)=\frac{\overline{X}-e^*}{T}. \tag{4.21}$$

In other words, $e^*+h(e^*)T\ge\overline{X}$, and by the defining equation (4.10) of $\overline{e}$ we conclude that $e^*\ge\overline{e}$. The corresponding optimal strategy, which is Type A, satisfies $X_0^A=x^*$ and $E_T^A=e^*$. The proof of optimality of this strategy follows the proof of Theorem 4.2 with $\widehat{g}$ replacing g.

Finally, we consider the subcase $x^*>F(0+)$. Because $y^*=h(x^*)$ is positive, the left-hand derivative of $\widehat{g}$ at y^* is defined, and it satisfies

$$D^-\widehat{g}(y^*)\ge\frac{\widehat{g}(y^*)-\widehat{g}(0)}{y^*}=\frac{g(y^*)}{y^*}=\psi(x^*). \tag{4.22}$$

In fact, the inequality in (4.22) is strict. It it were not, the affine function

$$\ell(y)=\psi(x^*)\big(y-y^*\big)+\widehat{g}(y^*)=y\psi(x^*)$$

would describe a tangent line to the graph of $\widehat{g}$ at $(y^*,\widehat{g}(y^*))$ lying below $\widehat{g}(y)$, and hence below $g(y)$, for all $y\in[0,\overline{Y}]$. But the resulting inequality $y\psi(x^*)\le g(y)=y\psi(h^{-1}(y))$ yields $\psi(x^*)\le\psi(h^{-1}(y))$ for all $y\in(0,\overline{Y}]$, and letting $y\downarrow 0$ we would conclude that $\psi(x^*)=0$. This violates the subcase assumption $x^*>F(0+)$. We conclude that $D^-\widehat{g}(y^*)>\psi(x^*)$. The strict inequality, the fact that e^* minimizes $\widehat{G}$, and (3.5) further imply that

$$0\le D^+\widehat{G}(e^*)=D^+\Phi(e^*)-D^-\widehat{g}(y^*)<\psi(e^*+)-\psi(x^*).$$

But $\psi(x^*)<\psi(e^*+)$ implies that $x^*\le e^*$. Consequently, $h(e^*)\ge h(x^*)=\frac{\overline{X}-e^*}{T}$. This is the essential part of inequality (4.21), and we conclude as above, constructing an optimal Type A strategy with $X_0^A=x^*$ and $E_T^A=e^*$.

Case II. $\widehat{g}(y^*)<g(y^*)$. Recall from Lemma 4.6 that this case can occur only if $0<y^*<\overline{Y}$. In particular, $x^*>0$. We let ℓ be the affine function and α and β be numbers as described in Lemma 4.6, and we construct a Type B strategy. To do this, we define $t_0\in(0,T)$ by

$$t_0=\frac{(\beta-y^*)T}{\beta-\alpha}, \tag{4.23}$$

so that $\alpha t_0 + \beta(T - t_0) = y^* T$. Consider the Type B strategy that makes an initial purchase $X_0^B = h^{-1}(\alpha)$, then purchases at rate $dX_t^B = \alpha\, dt$ for $0 \le t < t_0$ (so $E_t^B = h^{-1}(\alpha)$ for $0 \le t < t_0$), then follows this with a purchase $\Delta X_{t_0}^B = h^{-1}(\beta) - h^{-1}(\alpha)$ at time t_0, thereafter purchases at rate $dX_t^B = \beta\, dt$ for $t_0 \le t < T$ (so $E_t^B = h^{-1}(\beta)$ for $t_0 \le t < T$), and makes a final purchase $\overline{X} - X_{T-}^B$ at time T. According to (2.3),

$$X_t^B = \begin{cases} h^{-1}(\alpha) + \alpha t, & 0 \le t < t_0, \\ h^{-1}(\beta) + \alpha t_0 + \beta(t - t_0), & t_0 \le t < T, \\ \overline{X}, & t = T. \end{cases}$$

In particular,

$$\Delta X_T^B = \overline{X} - h^{-1}(\beta) - \alpha t_0 - \beta(T - t_0) = \overline{X} - h^{-1}(\beta) - y^* T = e^* - h^{-1}(\beta). \tag{4.24}$$

We show at the end of this proof that

$$h^{-1}(\beta) \le e^*. \tag{4.25}$$

This will ensure that ΔX_T^B is nonnegative, and since X^B is obviously nondecreasing on $[0, T)$, this will establish that X^B is a feasible purchasing strategy.

Accepting (4.25) for the moment, we note that (4.24) implies that

$$E_T^B = E_{T-}^B + \Delta E_T^B = h^{-1}(\beta) + \Delta X_T^B = e^*. \tag{4.26}$$

Using (4.4), (4.26), (4.16), the affine property of ℓ, and (4.17) in that order, we compute

$$\begin{aligned} C(X^B) &= \Phi(E_T^B) + \int_0^T g\big(h(E_t^B)\big)\, dt \\ &= \Phi(e^*) + g(\alpha) t_0 + g(\beta)(T - t_0) \\ &= \Phi(e^*) + \ell(\alpha) t_0 + \ell(\beta)(T - t_0) \\ &= \Phi(e^*) + T\ell\left(\frac{\alpha t_0 + \beta(T - t_0)}{T}\right) \\ &= \Phi(e^*) + T\ell(y^*) \\ &= \Phi(e^*) + T\widehat{g}(y^*) \\ &= \widehat{G}(e^*). \end{aligned}$$

This is (4.20).

Finally, we turn to the proof of (4.25). Because e^* is the largest minimizer of the convex function $\widehat{G}$ in $[0, \overline{X}]$ and $e^* < \overline{X}$ (because $x^* > 0$), the right-hand derivative of $\widehat{G}$ at e^* must be nonnegative. Indeed, for all $e \in (e^*, \overline{X})$, this right-hand derivative must in fact be strictly positive. For e greater than but sufficiently close to e^*, $\frac{\overline{X} - e}{T}$ is in (α, y^*), where $\widehat{g}$ is linear with slope $\frac{g(\beta)-g(\alpha)}{\beta-\alpha}$. For such e,

$$\begin{aligned}
0 &< D^+\widehat{G}(e) \\
&= D^+\Phi(e) - D^-\widehat{g}(y)\Big|_{y=\frac{X-e}{T}} \\
&= \psi(e+) - \frac{g(\beta)-g(\alpha)}{\beta-\alpha} \\
&= \psi(e+) - \frac{\beta\psi\big(h^{-1}(\beta)\big) - \alpha\psi\big(h^{-1}(\alpha)\big)}{\beta-\alpha} \\
&\le \psi(e+) - \frac{\beta\psi\big(h^{-1}(\beta)\big) - \alpha\psi\big(h^{-1}(\beta)\big)}{\beta-\alpha} \\
&= \psi(e+) - \psi\big(h^{-1}(\beta)\big).
\end{aligned}$$

This inequality $\psi\big(h^{-1}(\beta)\big) < \psi(e+)$ for all e greater than but sufficiently close to e^* implies (4.25). ■

Remark 4.7 (uniqueness). In Case I of the proof of Theorem 4.5, when $\widehat{g}(y^*) = g(y^*)$, strict convexity of $\widehat{g}$ at y^* implies uniqueness of the optimal purchasing strategy. The proof is similar to the uniqueness proof in Theorem 4.2.

However, in Case II, $\widehat{g}$ is not strictly convex at y^*. In this case, if ψ is strictly increasing at e^* and if the affine function ℓ of Lemma 4.6 agrees with g only at α and β, then the optimal purchasing strategy is unique. Indeed, if ψ is strictly increasing at e^*, then Φ (and hence $\widehat{G}$) is strictly convex at e^*, which implies that e^* is the unique minimizer of $\widehat{G}$. In order to be optimal, a purchasing strategy must satisfy the two inequalities

$$\int_0^T g\big(h(E_t)\big)\,dt \ge \int_0^T \widehat{g}\big(h(E_t)\big)\,dt \ge T\widehat{g}\left(\int_0^T h(E_t)\frac{dt}{T}\right) \tag{4.27}$$

with equality, as we explain below, and must also satisfy $E_T = e^*$. When the inequalities (4.27) hold, we can use (2.3) to obtain a lower bound on the cost of an arbitrary purchasing strategy X by the relations

$$\begin{aligned}
C(X) &= \Phi(E_T) + \int_0^T g(h(E_t))dt \\
&\ge \Phi(E_T) + T\widehat{g}\left(\int_0^T h(E_t)\frac{dt}{T}\right) \\
&= \Phi(E_T) + T\widehat{g}\left(\frac{\overline{X} - E_T}{T}\right) \\
&= \widehat{G}(E_T).
\end{aligned}$$

The minimal cost is $\widehat{G}(e^*) = \Phi(e^*) + T\widehat{g}(\frac{\overline{X}-e^*}{T}) = \Phi(e^*) + T\widehat{g}(y^*)$, and hence optimality of a strategy requires that equality hold in both parts of (4.27). The second inequality in (4.27) is Jensen's inequality, and equality holds if and only if $h(E_t)$, $0 \le t < T$, stays in the region in which $\widehat{g}$ is affine. But the average value of $h(E_t)$, $\frac{1}{T}\int_0^T h(E_t)dt$, is equal to y^*, and hence

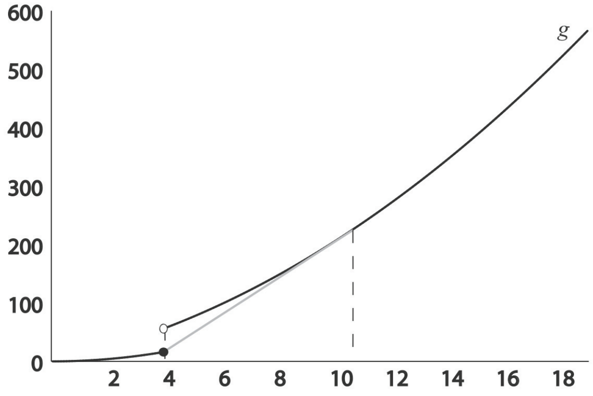

Figure 6. *Function g for the modified block order book with parameters $a = 4$ and $b = 14$. The convex hull $\widehat{g}$ is constructed by replacing a part $\{g(y)\ , y \in (a, \beta)\}$ by a straight line connecting $g(a)$ and $g(\beta)$. Here $\beta = 10.3246$.*

we cannot have $h(E_t) < y^*$ for all $t \in [0, T)$, nor can we have $h(E_t) > y^*$ for all $t \in [0, T)$. Hence the region in which $h(E_t)$ stays must be the region in which $\widehat{g}$ agrees with ℓ. To get an equality in the first inequality in (4.27), $h(E_t)$, $0 \le t < T$, must stay in the region where $\widehat{g}$ agrees with g. If ℓ agrees with g only at the two points α and β, then $h(E_t)$, $0 \le t < T$, must stay in the two-point set $\{\alpha, \beta\}$. Because $\Delta E_t = \Delta X_t \ge 0$ for all t, there must be some initial time interval $[0, t_0)$ on which $h(E_t) = \alpha$ and there must be some final time interval $[t_0, T)$ on which $h(E_t) = \beta$. In order to achieve this and to also have $\frac{1}{T}\int_0^T h(E_t) = y^*$, t_0 must be given by (4.23).

4.3. Examples of Type B optimal strategies.

Example 2.2 (modified block order book, continued). We continue Example 2.2 under the simplifying assumptions $T = 1$ and $h(x) = x$ for all $x \ge 0$, so $h^{-1}(y) = y$ for all $y \ge 0$ and $\overline{Y} = \overline{X}$. Recalling (2.6) and (4.1), we see that

$$g(y) = \begin{cases} y^2, & 0 \le y \le a, \\ y^2 + (b-a)y, & a < y < \infty. \end{cases}$$

The convex hull of g over $[0, \infty)$, given by (4.14), is

$$\widehat{g}(y) = \begin{cases} y^2, & 0 \le y \le a, \\ (2\beta + b - a)(y - a) + a^2, & a \le y \le \beta, \\ y^2 + (b-a)y, & \beta \le y < \infty, \end{cases}$$

where

$$\beta = a + \sqrt{a(b-a)} \tag{4.28}$$

(see Figure 6). We take $\overline{X} = \overline{Y} > \beta$, so that this is also the convex hull of g over $[0, \overline{Y}]$.

For $a < y^* < \beta$, we have $\widehat{g}(y^*) < g(y)$. For constants α and β from the statement of Lemma 4.6 (see (C.1)–(C.2) in Appendix C), we have that α of (C.1) is a, and β of (C.2) is given by (4.28). In order to illustrate a case in which a Type B purchasing strategy is optimal, we assume

$$a + 2\beta < \overline{X} < 3\beta. \tag{4.29}$$

The function $\widehat{G}$ of (4.18) is minimized over $[0, \overline{X}]$ at e^* if and only if

$$0 \in \partial \widehat{G}(e^*) = \partial \Phi(e^*) - \partial \widehat{g}(\overline{X} - e^*),$$

which is equivalent to $\partial \Phi(e^*) \cap \partial \widehat{g}(\overline{X} - e^*) \neq \emptyset$. We show below that the largest value of e^* satisfying this condition is $e^* = 2\beta$. According to (4.29), $e^* = 2\beta$ is in $(\overline{X} - \beta, \overline{X} - a)$. Because $\beta > a$, e^* is also in (a, ∞). We compute (recall (2.12))

$$\partial \Phi(e) = \begin{cases} \{e\}, & 0 \le e < a, \\ [a, b], & e = a, \\ \{e + b - a\}, & a < e < \infty, \end{cases}$$

$$\partial \widehat{g}(\overline{X} - e) = \begin{cases} \{2(\overline{X} - e) + b - a\}, & 0 \le e \le \overline{X} - \beta, \\ \{2\beta + b - a\}, & \overline{X} - \beta \le e < \overline{X} - a, \\ [2a, 2\beta + b - a], & e = \overline{X} - a, \\ \{2(\overline{X} - e)\}, & \overline{X} - a < e \le \overline{X}, \end{cases}$$

and then evaluate

$$\partial \Phi(e^*) = \{e^* + b - a\} = \{2\beta + b - a\} = \partial \widehat{g}(\overline{X} - e^*).$$

Therefore, $\widehat{G}$ attains its minimum at e^*.

To see that there is no $e \in (2\beta, \overline{X}]$ where $\widehat{G}$ attains its minimum, we observe that for $e \in (2\beta, \overline{X} - a)$, $\partial \Phi(e) \cap \partial \widehat{g}(\overline{X} - e) = \{e + b - a\} \cap \{2\beta + b - a\} = \emptyset$. For $e \in [\overline{X} - a, \overline{X}]$, all points in $\partial \widehat{g}(\overline{X} - e)$ lie in the interval $[0, 2a]$, whereas the only point in $\partial \Phi(e)$, which is $e + b - a$, lies in the interval $[\overline{X} + b - 2a, \overline{X} + b - a]$. Because of (4.29), we have $2a < \overline{X} + b - 2a$, and hence $\partial \Phi(e) \cap \partial \widehat{g}(\overline{X} - e) = \emptyset$ for $e \in [\overline{X} - a, \overline{X}]$.

As in the proof of Theorem 4.5, we set $y^* = \overline{X} - e^* = \overline{X} - 2\beta$, $x^* = h^{-1}(y^*) = \overline{X} - 2\beta$. Condition (4.29) is equivalent to $a < y^* < \beta$, which in turn is equivalent to $\widehat{g}(y^*) < g(y^*)$. The first inequality in (4.29) shows that $x^* > 0$, and we are thus in Case II of the proof of Theorem 4.5. In this case, we define

$$t_0 = \frac{\beta - y^*}{\beta - a} = \frac{3\beta - \overline{X}}{\beta - a}.$$

The optimal purchasing strategy is

$$X_t^B = \begin{cases} a(t + 1), & 0 \le t < t_0, \\ a t_0 + \beta(t + 1 - t_0), & t_0 \le t < 1, \\ \overline{X}, & t = 1. \end{cases}$$

In particular, $\Delta X_0 = a$, $\Delta X_{t_0} = \beta - a$, $\Delta X_1 = \beta$ (see (4.24) for the last equality). The corresponding E^B process is

$$E_t^B = \begin{cases} a, & 0 \le t < t_0, \\ \beta, & t_0 \le t < 1, \\ 2\beta, & t = 1. \end{cases}$$

The initial lump purchase moves the ask price to the left endpoint a of the gap in the order book. Purchasing is done to keep the ask price at a until time t_0, when another lump purchase moves the ask price to β beyond the right endpoint b of the gap in the order book. Purchasing is done to keep the ask price at β until the final time, when another lump purchase is executed.

Example 2.3 (discrete order book, continued). We continue Example 2.3 under the simplifying assumptions that $T = 1$ and $h(x) = x$ for all $x \ge 0$, so that $h^{-1}(y) = y$ for all $y \ge 0$ and $\overline{Y} = \overline{X}$. From (2.8) and (4.1) we see that $g(0) = 0$, and $g(y) = ky$ for integers $k \ge 0$ and $k < y \le k+1$. In particular, $g(k) = (k-1)k$ for nonnegative integers k. The convex hull of g interpolates linearly between the points $(k, (k-1)k)$ and $(k+1, k(k+1))$, i.e., $\widehat{g}(y) = k(2y - (k+1))$ for $k \le y \le k+1$, where k ranges over the nonnegative integers (see Figure 7).

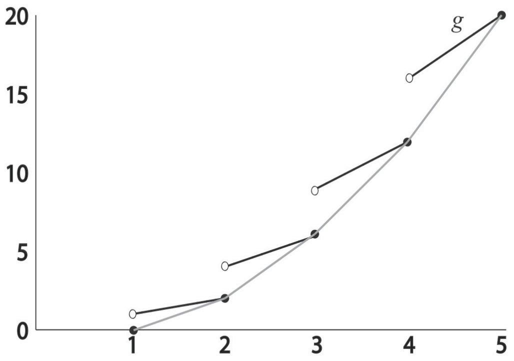

Figure 7. *Function g for the discrete order book. The convex hull $\widehat{g}$ interpolates linearly between the points $(k, (k-1)k)$ and $(k+1, k(k+1))$.*

Therefore,

$$\partial\widehat{g}(y) = \begin{cases} \{0\}, & y = 0, \\ [2(k-1), 2k], & y = k \text{ and } k \text{ is a positive integer}, \\ \{2k\}, & k < y < k+1 \text{ and } k \text{ is a nonnegative integer}. \end{cases}$$

Recall from the discussion following (2.13) that

$$\partial\Phi(y) = \begin{cases} \{0\}, & y = 0, \\ [k-1, k], & y = k \text{ and } k \text{ is a positive integer}, \\ \{k\}, & k < y < k+1 \text{ and } k \text{ is a nonnegative integer}. \end{cases}$$

We seek the largest number $e^* \in [0, \overline{X}]$ for which $\partial\Phi(e^*) \cap \partial\widehat{g}(\overline{X} - e^*) \neq \emptyset$. This is the largest minimizer of $\widehat{G}(e) = \Phi(e) + \widehat{g}(\overline{X} - e)$ in $[0, \overline{X}]$. We define k^* to be the largest integer less than or equal to $\frac{\overline{X}}{3}$, so that

$$3k^* \leq \overline{X} < 3k^* + 3.$$

We divide the analysis into three cases:

Case A. $3k^* \leq \overline{X} \leq 3k^* + 1$.

Case B. $3k^* + 1 < \overline{X} < 3k^* + 2$.

Case C. $3k^* + 2 \leq \overline{X} < 3k^* + 3$.

We show below that in Cases A and B the optimal strategy makes an initial lump purchase of size k^*, which executes the orders at prices $0, 1, \ldots, k^* - 1$. In Case A, the optimal strategy then purchases at rate k^* over the interval $(0, 1)$ and at time 1 makes a final lump purchase of size $\overline{X} - 2k^*$, which is in the interval $[k^*, k^* + 1]$. This is a Type A strategy. In Case B, there is an intermediate lump purchase of size 1 at time $3k^* + 2 - \overline{X}$. Before this intermediate purchase, the rate of purchase is k^*, and after this purchase the rate of purchase is $k^* + 1$. In Case B, at time 1, there is a final lump purchase of size k^*. In Case B, we have a Type B strategy. In Case C, the optimal strategy makes a lump purchase of size $k^* + 1$ at time 0, which executes the orders at prices $0, 1, \ldots, k^* - 1, k^*$. The optimal strategy then purchases continuously at rate $k^* + 1$ over the interval $(0, 1)$ and at time 1 makes a final lump purchase of size $\overline{X} - 2k^* - 2$, which is in the interval $[k^*, k^* + 1)$. This is a Type A strategy.

Case A. $3k^* \leq \overline{X} \leq 3k^* + 1$. We define $e^* = \overline{X} - k^*$, so that $2k^* \leq e^* \leq 2k^* + 1$ and $k^* = \overline{X} - e^*$. Then $2k^* \in \partial\Phi(e^*)$ and $\partial\widehat{g}(\overline{X} - e^*) = [2(k^* - 1), 2k^*]$, so the intersection of $\partial\Phi(e^*)$ and $\partial\widehat{g}(\overline{X} - e^*)$ is nonempty, as desired. On the other hand, if $e > e^*$, then $\partial\Phi(e) \subset [2k^*, \overline{X}]$ and $\partial\widehat{g}(\overline{X} - e) \subset [0, 2(k^* - 1)]$, so the intersection of these two sets is empty.

In this case, y^* and x^* defined by (4.19) are both equal to k^*, and hence $\widehat{g}(y^*) = g(y^*)$. If $k^* = 0$, we are in the first subcase of Case I of the proof of Theorem 4.5. The optimal purchasing strategy is to do nothing until time 1 and then make a lump purchase of size $\overline{X}$. If $k^* = 1$, which is equal to $F(0+)$, we are in the second subcase of Case I of the proof of Theorem 4.5. We should make an initial purchase of size $x^* = 1$, purchase continuously over the time interval $(0, 1)$ at rate 1 so that $E_t \equiv 1$ and $D_t \equiv 0$, and make a final purchase of size $\overline{X} - 2$. If $k^* \geq 2$, we are in the third subcase of Case I of the proof of Theorem 4.5. We should make an initial purchase of size k^*, purchase continuously over the time interval $(0, 1)$ at rate k^* so that $E_t \equiv k^*$ and $D_t \equiv k^* - 1$, and make a final purchase of size $\overline{X} - 2k^*$.

Case B. $3k^* + 1 < \overline{X} < 3k^* + 2$. We define $e^* = 2k^* + 1$, so that $k^* < \overline{X} - e^* < k^* + 1$. Then $\partial\Phi(e^*) = [2k^*, 2k^* + 1]$ and $\partial\widehat{g}(\overline{X} - e^*) = \{2k^*\}$, so the intersection of $\partial\Phi(e^*)$ and $\partial\widehat{g}(\overline{X} - e^*)$ is nonempty, as desired. On the other hand, if $e > e^*$, then $\partial\Phi(e) \subset [2k^* + 1, \overline{X}]$ and $\partial\widehat{g}(\overline{X} - e) \subset [0, 2k^*]$, so the intersection of these two sets is empty.

In this case, y^* and x^* defined by (4.19) are both equal to $\overline{X} - e^*$. Hence $k^* < y^* < k^* + 1$, $\widehat{g}(y^*) < g(y^*)$, and we are in Case II of the proof of Theorem 4.5 with $\alpha = k^*$ and $\beta = k^* + 1$ (see (4.14)–(4.17) and (C.1)–(C.2)). The optimal purchasing strategy is Type B. In particular, with $t_0 = \beta - y^* = k^* + 1 - x^* = 3k^* + 2 - \overline{X}$, the optimal purchasing strategy makes an initial lump purchase $\alpha = k^*$, which executes the orders at prices 0, $1, \ldots, k^* - 1$, then purchases continuously over the interval $(0, t_0)$ at rate k^* so that $E_t \equiv k^*$ and $D_t \equiv k^* - 1$, at time t_0 makes a lump purchase of size $\beta - \alpha = 1$, which consumes the order at price k^*, then purchases continuously over the interval $(t_0, 1)$ at rate $k^* + 1$ so that $E_t \equiv k^* + 1$ and $D_t \equiv k^*$, and finally executes a lump purchase of size $e^* - \beta = k^*$ (see (4.24)) at time 1. The total quantity purchased is

$$k^* + k^* t_0 + 1 + (k^* + 1)(1 - t_0) + k^* = \overline{X},$$

as required.

Case C. $3k^* + 2 \leq \overline{X} < 3k^* + 3$. We define $e^* = \overline{X} - k^* - 1$, so that $2k^* + 1 \leq e^* < 2k^* + 2$ and $\overline{X} - e^* = k^* + 1$. Then $2k^* + 1 \in \partial\Phi(e^*)$ and $\widehat{g}(\overline{X} - e^*) = [2k^*, 2k^* + 2]$, and the intersection of $\partial\Phi(e^*)$ and $\partial\widehat{g}(\overline{X} - e^*)$ is nonempty, as desired. On the other hand, if $e > e^*$, then $\partial\Phi(e) \subset [2k^* + 1, \overline{X}]$ and $\partial\widehat{g}(\overline{X} - e) \subset [0, 2k^*]$, so the intersection of these two sets is empty. In this case, y^* and x^* are both equal to $k^* + 1$. The optimal purchasing strategy falls into either the second (if $k^* = 0$) or third (if $k^* \geq 1$) subcase of Case I of the proof of Theorem 4.5.

Appendix A. The process E. In this appendix we prove that there exists a unique adapted process E satisfying (2.3) pathwise, and we provide a list of its properties.

Lemma A.1. *Let h be a nondecreasing, real-valued, locally Lipschitz function defined on $[0, \infty)$ such that $h(0) = 0$. Let X be a purchasing strategy. Then there exists a unique bounded adapted process E depending pathwise on X such that (2.3) is satisfied. Furthermore, the following hold:*

(i) *E is right continuous with left limits;*
(ii) *$\Delta E_t = \Delta X_t$ for all t;*
(iii) *E has finite variation on $[0, T]$;*
(iv) *E takes values in $[0, \overline{X}]$.*

Proof. Because we do not know a priori that E is nonnegative, we extend h to all of $\mathbb{R}$ by defining $h(x) = 0$ for $x < 0$. This extended h is nondecreasing and locally Lipschitz.

In section 2 we introduced the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$. The purchasing strategy X is right continuous and adapted to this filtration and hence is an optional process; i.e., $(t, \omega) \mapsto X_t(\omega)$ is measurable with respect to the optional σ-algebra, the σ-algebra generated by the right-continuous adapted processes. For any bounded optional process Y, $h(Y)$ and $\int_0^{\cdot} h(Y_s)\, ds$ are also bounded optional processes. Optional processes are adapted, and hence $\int_0^t h(Y_s)\, ds$ is $\mathcal{F}_t$-measurable for each $t \in [0, T]$.

We first prove uniqueness. If E and $\widehat{E}$ are bounded processes satisfying (2.3), then there is a local Lipschitz constant K, chosen taking the bounds on E and $\widehat{E}$ into account, such that

$$|E_t - \widehat{E}_t| = \left| \int_0^t \big(h(E_s) - h(\widehat{E}_s)\big)\, ds \right| \leq K \int_0^t |E_s - \widehat{E}_s|\, ds.$$

Gronwall's inequality implies $E = \widehat{E}$.

For the existence part of the proof, we assume for the moment that h is globally Lipschitz with Lipschitz constant K, and we construct E as a limit of a recursion. Let $E_t^0 \equiv X_0$. For $n = 1, 2, \ldots$, define recursively

$$E_t^n = X_t - \int_0^t h(E_s^{n-1})\, ds, \quad 0 \leq t \leq T.$$

Since X is bounded and optional, each E^n is bounded and optional. For $n = 1, 2, \ldots,$ let $Z_t^n = \sup_{0 \le s \le t} |E_s^n - E_s^{n-1}|$. A proof by induction shows that

$$Z_t^n \le \frac{K^{n-1} t^{n-1}}{(n-1)!} \max\left\{\overline{X}, Th(X_0) + X_0\right\}.$$

Because this sequence of nonrandom bounds is summable, E^n converges uniformly in $t \in [0,T]$ and ω to a bounded optional process E that satisfies (2.3). In particular, E_t is $\mathcal{F}_t$-measurable for each t, and since X is nondecreasing and right continuous with left limits and the integral in (2.3) is continuous, (i), (ii), and (iii) hold.

It remains to prove (iv). For $\varepsilon > 0$, let $X_t^\varepsilon = X_t + \varepsilon t$ and define $t_0^\varepsilon = \inf\{t \in [0,T] : E_t^\varepsilon < 0\}$. Assume this set is not empty. Then the right continuity of E^ε combined with the fact that E^ε has no negative jumps implies that $E_{t_0^\varepsilon}^\varepsilon = 0$. Let $t_n^\varepsilon \downarrow t_0^\varepsilon$ be such that $E_{t_n^\varepsilon}^\varepsilon < 0$ for all n. Then

$$\int_{t_0^\varepsilon}^{t_n^\varepsilon} h(E_s^\varepsilon)\, ds = X_{t_n^\varepsilon}^\varepsilon - X_{t_0^\varepsilon}^\varepsilon - (E_{t_n^\varepsilon}^\varepsilon - E_{t_0^\varepsilon}^\varepsilon) > X_{t_n^\varepsilon}^\varepsilon - X_{t_0^\varepsilon}^\varepsilon \ge \varepsilon(t_n^\varepsilon - t_0^\varepsilon).$$

But, since

$$\int_{t_0^\varepsilon}^{t_n^\varepsilon} h(E_s^\varepsilon)\, ds \le K \left(\max_{t_0^\varepsilon \le s \le t_n^\varepsilon} E_s^\varepsilon \right) (t_n^\varepsilon - t_0^\varepsilon),$$

there must exist $s_n^\varepsilon \in (t_0^\varepsilon, t_n^\varepsilon)$ such that $E_{s_n^\varepsilon}^\varepsilon \ge \frac{\varepsilon}{K}$. This contradicts the right continuity of E^ε at t_0^ε. Consequently, the set $\{t \in [0,T] : E_t^\varepsilon < 0\}$ must be empty. We conclude that $E_t^\varepsilon \ge 0$ for all $t \in [0,T]$.

Now notice that for $0 \le t \le T$,

$$E_t^\varepsilon - E_t = \varepsilon t - \int_0^t (h(E_s^\varepsilon) - h(E_s))\, ds,$$

and hence

$$|E_t^\varepsilon - E_t| \le \varepsilon t + K \int_0^t |E_s^\varepsilon - E_s|\, ds.$$

Gronwall's inequality implies that $E^\varepsilon \to E$ as $\varepsilon \downarrow 0$. Since $E_t^\varepsilon \ge 0$, we must have $E_t \ge 0$ for all t. Equation (2.3) now implies that $E_t \le X_t$, and therefore $E_t \le \overline{X}$. The proof of (iv) is complete.

When h is locally but not globally Lipschitz, we let $\tilde{h}$ be equal to h on $[0, \overline{X}]$, $\tilde{h}(x) = 0$ for $x < 0$, and $\tilde{h}(x) = h(\overline{X})$ for $x > \overline{X}$. We apply the previous arguments to $\tilde{h}$, and we observe that the resulting $\tilde{E}$ satisfies the equation corresponding to h. ■

Remark A.2. The pathwise construction of E in the proof of Lemma A.1 shows that if X is deterministic, then so is E.

Appendix B. $\mathbb{E} \int_0^T X_t\, dA_t = 0$.

Lemma B.1. *Under the assumptions that* $0 \le X_t \le \overline{X}$, $0 \le t \le T$, *and that the continuous nonnegative martingale* A *satisfies* (2.1), *we have* $\mathbb{E} \int_0^T X_t\, dA_t = 0$.

Proof. The Burkholder–Davis–Gundy inequality implies that the continuous local martingale $M_t = \int_0^t X_s \, dA_s$ satisfies

$$\begin{aligned}\mathbb{E}\left[\max_{0\le t\le T}|M_t|\right] &\le C\mathbb{E}\big[\langle M\rangle_T^{1/2}\big]\\ &= C\mathbb{E}\left[\left(\int_0^T X_t^2 \, d\langle A\rangle_t\right)^{1/2}\right]\\ &\le C\overline{X}\mathbb{E}\big[\langle A\rangle_T^{1/2}\big]\\ &\le C'\overline{X}\mathbb{E}\left[\max_{0\le t\le T} A_t\right],\end{aligned}$$

where C and C' are positive constants. By virtue of being a local martingale, M has the property that $\mathbb{E}M_{\tau_n} = 0$ for a sequence of stopping times $\tau_n \uparrow T$. The dominated convergence theorem implies that $\mathbb{E}M_T = 0$. ■

Appendix C. Convex hull of g.

Proof of Lemma 4.6. Recall the definition

$$\widehat{g}(y) \triangleq \sup\left\{\ell(y) : \ell \text{ is an affine function and } \ell(\eta) \le g(\eta)\ \forall \eta \in [0,\overline{Y}]\right\} \tag{4.14}$$

of the convex hull of g, defined for $y \in [0,\overline{Y}]$. The function $\widehat{g}$ is the largest convex function defined on $[0,\overline{Y}]$ that is dominated by g there.

For each $0 \le y < \overline{Y}$, the supremum in (4.14) is obtained by the support line of $\widehat{g}$ at y. For $y = 0$, the zero function is such a support line, and hence $0 \le \widehat{g}(0) \le g(0) = 0$ (recall (4.3)). At $y = \overline{Y}$ the only support line might be vertical, in which case the supremum in (4.14) is not attained. Because $\widehat{g}(0) = 0$, $\widehat{g}$ is nonnegative, and $\widehat{g}$ is convex, we know that $\widehat{g}$ is also nondecreasing. Being convex, $\widehat{g}$ is continuous on $(0,\overline{Y})$ and upper semicontinuous on $[0,\overline{Y}]$, and we have continuity at 0 because of (4.3). We also have continuity of $\widehat{g}$ at $\overline{Y}$, as we now show. Given $\varepsilon > 0$, the definition of $\widehat{g}$ implies that there exists an affine function $\ell \le g$ such that $\ell(\overline{Y}) \ge \widehat{g}(\overline{Y}) - \varepsilon$. But $\widehat{g} \ge \ell$, and thus $\liminf_{y\uparrow\overline{Y}} \widehat{g}(y) \ge \lim_{y\uparrow\overline{Y}} \ell(y) = \ell(\overline{Y}) \ge \widehat{g}(\overline{Y}) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we must in fact have $\liminf_{y\uparrow\overline{Y}} \widehat{g}(y) \ge \widehat{g}(\overline{Y})$. Coupled with the upper semicontinuity of $\widehat{g}$ at $\overline{Y}$, this gives us continuity.

We next argue that $\widehat{g}(\overline{Y}) = g(\overline{Y})$. Suppose, on the contrary, we had $\widehat{g}(\overline{Y}) < g(\overline{Y})$. The function g is continuous at $\overline{Y}$ (see (4.5)), and $\widehat{g}$ is upper semicontinuous. Therefore, there is a one-sided neighborhood $[\gamma,\overline{Y}]$ of $\overline{Y}$ (with $\gamma < \overline{Y}$) on which $g - \widehat{g}$ is bounded away from zero by a positive number ε. The function

$$\widehat{g}(y) + \frac{\varepsilon(y-\gamma)}{\overline{Y}-\gamma}, \quad 0 \le y \le \overline{Y},$$

is convex, lies strictly above $\widehat{g}$ at $\overline{Y}$, and lies below g everywhere. This contradicts the fact that $\widehat{g}$ is the largest convex function dominated by g. We must therefore have $\widehat{g}(\overline{Y}) = g(\overline{Y})$.

Finally, we describe the situation when for some $y^* \in [0,\overline{Y}]$ we have $\widehat{g}(y^*) < g(y^*)$. We have shown that this can happen only if $0 < y^* < \overline{Y}$. Let ℓ be a support line of $\widehat{g}$ at y^*, which is an affine function that attains the maximum in (4.14) at the point y^*. In particular, $\ell \le \widehat{g} \le g$ and $\ell(y^*) = \widehat{g}(y^*)$. Define

$$\alpha = \sup\{\eta \in [0, y^*] : g(\eta) - \ell(\eta) = 0\}, \tag{C.1}$$

$$\beta = \inf\{\eta \in [y^*, \overline{Y}] : g(\eta) - \ell(\eta) = 0\}. \tag{C.2}$$

Because g is lower semicontinuous, the minimum of $g - \ell$ over $[0, \overline{Y}]$ is attained. This minimum cannot be a positive number ε, for then $\ell + \varepsilon$ would be an affine function lying below g. Therefore, either the supremum in (C.1) or the infimum in (C.2) is taken over a nonempty set. In the former case, we must have $g(\alpha) = \ell(\alpha)$, whereas in the latter case $g(\beta) = \ell(\beta)$.

Let us consider first the case that $g(\alpha) = \ell(\alpha)$. Define $\gamma = \frac{1}{2}(\alpha + y^*)$. Like α, γ is strictly less than y^*. The function $g - \ell$ attains its minimum over $[\gamma, \overline{Y}]$. If this minimum were a positive number ε, then the affine function

$$\ell(y) + \frac{\varepsilon\big(y - \gamma\big)}{\overline{Y} - \gamma}, \quad 0 \le y \le \overline{Y},$$

would lie below g but have a larger value at y^* than ℓ, violating the choice of ℓ. It follows that $g - \ell$ attains the minimum value zero on $[\gamma, \overline{Y}]$, and since this function is strictly positive on $[\gamma, y^*]$, the minimum is attained to the right of y^*. This implies that $g(\beta) = \ell(\beta)$. Similarly, if we begin with the assumption that $g(\beta) = \ell(\beta)$, we can argue that $g(\alpha) = \ell(\alpha)$.

In conclusion, α and β defined by (C.1) and (C.2) satisfy (4.15) and (4.16). Finally, (4.16) shows that ℓ restricted to $[\alpha, \beta]$ is the largest affine function lying below g on this interval, and hence (4.17) holds.

Because of (4.16), every affine function lying below g on $[0, \overline{Y}]$ must lie below ℓ on $[\alpha, \beta]$. If such an affine function agrees with $\widehat{g}$ and hence with ℓ at y^*, it must in fact agree with ℓ everywhere. Hence, ℓ is the only function lying below g on $[0, \overline{Y}]$ and agreeing with $\widehat{g}$ at y^*. ■

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