Locally recoverable codes

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Let $\mathbb{F}_q$ be a finite field with q elements.

A linear code $\mathcal{C}$ is an $\mathbb{F}_q$ subspace of $\mathbb{F}_q^n$ of dimension k.

The parameters of a code:

1. length n,
2. dimension k and
3. minimum distance d (Hamming distance).

Singleton bound: $d \leq n - k + 1$.

Singleton defect: $n + 1 - k - d \geq 0$.

A Locally Recoverable Code is a code such that the value of an erased coordinate of a codeword can be recovered from the values of a small subset of size r of other coordinates.

$[n, k, d; r]$-code

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$$[n, k, d; r]\text{-code}$$

In 2012, Gopalan, Huang, Simitci and Huseyin proved a bound for LRC codes.

Let $\mathcal{C}$ be an $[n, k, d; r]$-code, then

$$d \leq n - k - \left\lceil \frac{k}{r} \right\rceil + 2; \tag{1}$$

When $r = k$ we have the Singleton bound.

A code that achieves equality in (1) is called an optimal LRC code.

Some constructions of Optimal LRC codes:

1. using particular types of polynomials over $\mathbb{F}_q[x]$(Tamo and Barg 2014),
2. cyclic codes(Luo, Xing, and Yuan 2019),
3. over the rational function field $\mathbb{F}_q(x)$ (of genus 0)(Jin, Ma, Xing, 2019).
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However, the local repair may not be performed when some of the r coordinates are also erased.

We can work with δ non overlapping repair sets of size no more than r_i for a coordinate.

Definition

The i-th coordinate, where $1 \leq i \leq n$, of an $[n, k, d]$ linear code $\mathcal{C}$ whose generator matrix is $(g_1, \ldots, g_n)$ is said to have $(r_1, \ldots, r_\delta)$-locality if there exist pairwise disjoint repair sets $R_1^{(i)}, \ldots, R_\delta^{(i)} \in \{1, \ldots, n\} \setminus \{i\}$ such that for each $1 \leq j \leq \delta$

i) $\#R_j^{(i)} = r_i$;

ii) $g_i \in \langle g_\ell \mid \ell \in R_j^{(i)} \rangle$.

A linear code $\mathcal{C}$ with length n, dimension k, minimum distance d, and $(r_1, \ldots, r_\delta)$-locality is denoted by

$$[n, k, d; r_1, r_2, \ldots, r_\delta].$$

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Bounds on the min. dist. for $[n, k, d; r_1, r_2, \ldots, r_\delta]$ codes with more than one recoverability

For $\delta \geq 1$ and $r = r_1 = \cdots = r_\delta$:

In 2014 we have a result from A. Wang, and Z. Zhang:

$$d \leq n - k + 2 - \left\lceil \frac{(k-1)\delta + 1}{(r-1)\delta + 1} \right\rceil . \tag{2}$$

From Tamo, Barg and Frolov in 2017:

$$d \leq n - \sum_{i=0}^{\delta} \left\lfloor \frac{k-1}{r^i} \right\rfloor \tag{3}$$

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In our work we obtain a bound on the minimum distance for a $[n, k, d; r_1, r_2, \dots, r_\delta]$- code.

$$d \leq n - k - \left\lceil \frac{(k-1)\delta + 1}{1 + \sum_{i=1}^{\delta} r_i} \right\rceil + 2 \tag{4}$$

For $\delta = 1$ we have $d \leq n - k - \left\lceil \frac{k}{1+r} \right\rceil + 2$

Mac Willians Lemma: if $G = (g_1, \dots, g_n)$ is a generator matrix for an $[n, k, d]$ code, then

$$d = n - max\{\#N \ : \ N \subset \{1, \dots, n\}, \ rank(\langle g_j \mid j \in N \rangle) < k\}.$$

Obtain a lower bound for the max using the repairing sets.

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Relative defect for an $[n, k, d; r_1, \dots, r_\delta]$-code

$$\Delta(\mathcal{C}) = \frac{1}{n}\left(n - k - d + 2 - \left\lceil \frac{(k-1)\delta + 1}{1 + \sum_{i=1}^{\delta} r_i} \right\rceil\right).$$

Algebraic geometry codes

Function field $\mathcal{F}|\mathbb{F}_q$.

For a divisor D in $\mathcal{F}$ the Riemann-Roch $\mathbb{F}_q$-vector space:

$$\mathcal{L}(D) = \{z \in \mathcal{F} \, : \, (z) \geq -D\} \cup \{0\}$$

Let G be a divisor on $\mathcal{F}$ and $P_1, P_2, \ldots, P_n$ be pairwise distinct rational places on $\mathcal{F}$, with $P_i \notin supp(G)$ for all i. Define $D = \sum_{i=1}^{n} P_i$. The linear algebraic geometry code $\mathcal{C}_{\mathcal{L}}(D, G)$ is defined as the image of the evaluation function

$$ev \, : \, \mathcal{L}(G) \to \mathbb{F}_q^n, \, f \mapsto (f(P_1), f(P_2), \ldots, f(P_n)).$$

The $\mathcal{C}_{\mathcal{L}}(D, G)$ code has length n and the classical bound on the minimum distance d is

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Let $a \in \mathbb{F}_{q^2}^*$ of order $u|(q+1)$.

$$\begin{aligned} \mathcal{H}_1 &:= \{(x, y) \mapsto (x + c, y) \mid c \in \mathbb{F}_{q^2}, c^q + c = 0\} < Aut(\mathcal{F}|\mathbb{F}_q), \\ \mathcal{H}_2 &:= \{(x, y) \mapsto (x, a^i y) \mid i = 0, \ldots, u-1\} < Aut(\mathcal{F}|\mathbb{F}_q). \end{aligned}$$

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Let P_∞ be the unique pole of x and $y \in \mathcal{F}$.

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We have $n = q(q^2 - 1)$(*length*) places in $\mathcal{F}$ totally split in $\mathcal{F}/\mathcal{F}^{\mathcal{G}}$: $\{P_1, \ldots P_n\}$

The code will given by:

$$\begin{array}{cccc} e_{\mathcal{P}} : & V & \to & \mathbb{F}_q^n \\ & f & \mapsto & e_{\mathcal{P}}(f) = (f(P_1), \ldots, f(P_n)). \end{array}$$

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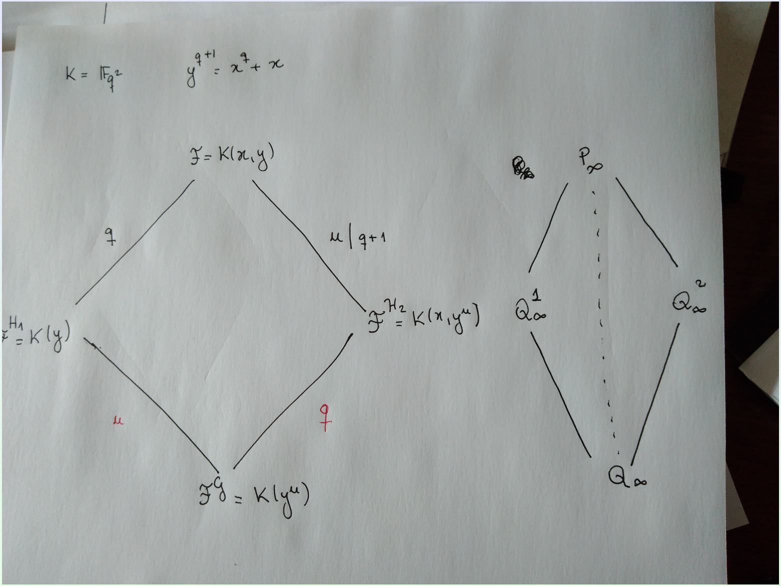

$K = \mathbb{F}_{q^2}$
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u
q
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P_∞
Q_∞^1
Q_∞^2
Q_∞

$$V_1 := \left\{ \sum_{\ell=0}^{q-2} \left(\sum_{j=0}^{t_1} a_{\ell,j} y^j \right) x^\ell \ : \ a_{\ell,j} \in \mathbb{F}_{q^2} \right\} \text{ and}$$

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$\dim_{\mathbb{F}_{q^2}} V_1 = (q-1)(t_1+1)$ and $\dim_{\mathbb{F}_{q^2}} V_2 = (u-1)(t_2+1)$.

$\bullet z \in V_1 \implies \deg(z)_\infty \leq q^2 - q - 2 + t_1 q$

$\bullet z \in V_2 \implies \deg(z)_\infty \leq q((u-1)t_2 + u - 2)$

Choose $t_1, t_2, 1 \leq d \leq n$ such that

$$V := V_1 \cap V_2 \subseteq \mathcal{L}((n-d)P_\infty)$$

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$$\dim_{\mathbb{F}_q} V = q \geq 1$$

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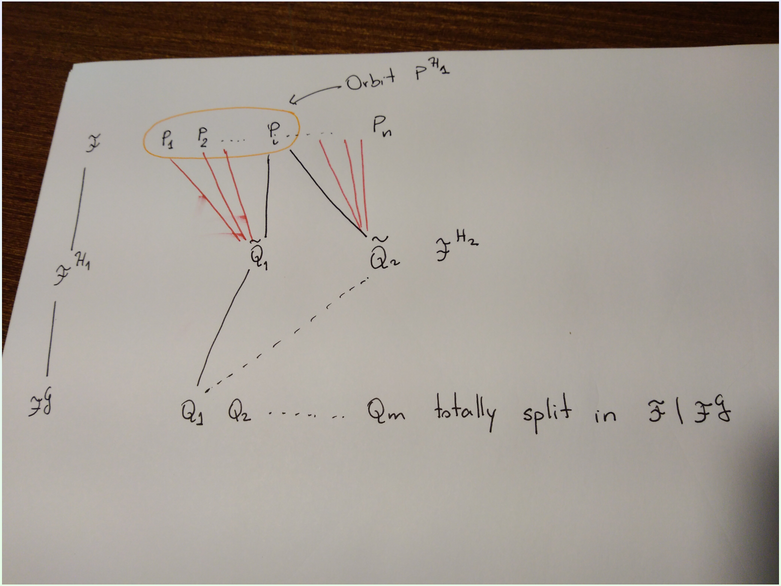

Orbit P^{H_1}
$\mathcal{F}$
P_1 P_2 .... P_i ...... P_n
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Q_1 Q_2 .... Q_m totally split in $\mathcal{F} | \mathcal{F}^{G}$

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$$\begin{array}{rccl} e_{\mathcal{P}} : & V & \to & \mathbb{F}_q^n \\ & f & \mapsto & e_{\mathcal{P}}(f) = (f(P_1), \ldots, f(P_n)). \end{array}$$

For every $1 < u$ a divisor of $q+1$ and $\frac{2q^2-qu-2}{q(u-1)} \leq t_2 < \frac{q^2}{u-1} - 1$ there exists a

$$[q(q^2-1), q, \geq q(q^2-1) - q(ut_2 + u - t_2 - 2); q-1, u-1]$$

recoverable code $\mathcal{C}_1$.

$$\text{Relative defect: } \Delta(\mathcal{C}_1) \leq \frac{q(ut_2 + u - t_2 - 3) + 1}{q(q^2-1)}.$$

For every $1 < u, u(t_2+1) = q+1, t_1 \geq 2$ and $(t_1, t_2) \neq (1, 1)$ there exists a

$$[q(q^2-1), q, \geq q(q^2-1) - (q^2 - q - 2 + t_1 q); q-1, u-1]$$

recoverable code $\mathcal{C}_2$.

$$\Delta(\mathcal{C}_2) \leq= \frac{q^2 - 2q + t_1 q - 1}{q(q^2-1)} \simeq \frac{q+t_1}{q^2-1}$$

The code is given by:

$$
\begin{array}{rcccl}
e_{\mathcal{P}} & : & V & \to & \mathbb{F}_q^n \\
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\end{array}
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Theorem

Let $\mathcal{F}|\mathbb{F}_q$ be a FF of genus g
$\mathcal{H}_i < Aut(\mathcal{F}|\mathbb{F}_q), \#\mathcal{H}_i = r_i + 1$, and $\mathcal{G} \simeq \bigotimes_{i=1}^{s} \mathcal{H}_i$ is isomorphic to the internal direct product of $\mathcal{H}_1, \ldots, \mathcal{H}_s$.

Let $\mathcal{P}$ be a set of places in $\mathcal{F}$ lying over m rational places in the fixed field $\mathcal{F}^{\mathcal{G}}$ that are completely split in the extension $\mathcal{F}|\mathcal{F}^{\mathcal{G}}$.
$n = m\prod_{i=1}^{s}(r_i + 1)$

Suppose that there exists a place P_∞ of $\mathcal{F}$ which is completely ramified in $\mathcal{F}|\mathcal{F}^{\mathcal{G}}$ and let $Q_\infty^{(i)}$ be the unique place in $\mathcal{F}^{\mathcal{H}_i}$ lying under P_∞. For $i = 1, \ldots, s$ suppose further there exist functions z_i, w_i, such that

1. $z_i \in \mathcal{F}^{\mathcal{H}_i}$, $supp((z_i)_\infty) = \{Q_\infty^{(i)}\}$;
2. $w_i \in \mathcal{F} \setminus \mathcal{F}^{\mathcal{H}_i}$, $supp((w_i)_\infty) = \{P_\infty\}$;
3. for any $P \in \{P_1, \ldots, P_n\}$, $P^{\mathcal{H}_i} \cap P^{\mathcal{H}_j} = \{P\}$ for $i \neq j$;
4. $w_i : P^{\mathcal{H}_i} \to \mathbb{F}_q$ is injective.

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Theorem

For each $i = 1, \ldots, s$ *let* $t_i \geq 1$ *be such that*

$$V_i := \left\{ \sum_{\ell=0}^{r_i - 1} \left(\sum_{j=0}^{t_i} a_{\ell j}^{(i)} z_i^j \right) w_i^\ell \in \mathcal{F} \;\middle|\; a_{\ell j}^{(i)} \in \mathbb{F}_q \right\}$$

is contained in $\mathcal{L}((n-d)P_\infty)$ *for some* $1 \leq d \leq n$.

Let $V = \bigcap_{i=1}^{s} V_i \subset \mathcal{L}((n-d)P_\infty)$.

If $\dim_{\mathbb{F}_q}(V) > 0$, *then there exists an*

$$[n, \dim_{\mathbb{F}_q}(V), \geq d; r_1, \ldots, r_s]\text{-recoverable code.}$$

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Codes from algebraic curves with many rational places

Given a function field $\mathcal{F}$ of genus g, by Hasse-Weill we have

$$\#\mathcal{F}(\mathbb{F}_q) \leq q + 1 + 2g\sqrt{q}$$

where $\mathcal{F}(\mathbb{F}_q)$ is the set of $\mathbb{F}_q$ rational places in $\mathcal{F}$.

We say a function field $\mathcal{F}$ over $\mathbb{F}_{q^2}$ is maximal if it attains the HW upper bound

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LRC codes from maximal curves:

• the rational function field.

• Hermitian function field:

$$y^{q+1} = x^q + x, \quad \#\mathcal{F}(\mathbb{F}_q) = q^3 + 1, \quad g = \frac{q(q-1)}{2}.$$

$\mathcal{F} = \mathbb{F}_{q^6}(x, y, z)$ be the function field of the Giullieti-Korchmaros curve

$$\begin{cases} Z^{q^2-q+1} = Y^{q^2} - Y, \\ Y^{q+1} = X^q + X. \end{cases}$$

Maximal over $\mathbb{F}_{q^6}$ with $q^8 - q^6 + q^5 + 1$ rational places and only one place P_∞ at infinity.

$A < \{a \in \mathbb{F}_{q^6} \ : \ a^q + a = 0\}$ and $\eta, \omega \in \mathbb{F}_{q^6}$, with

$ord(\eta) \mid q^3 + 1, ord(\omega) \mid q^2 - q + 1$ and $\gcd(ord(\eta), ord(\omega)) = 1$

Consider the following subgroups of $Aut(\mathcal{F}|K)$:

$\mathcal{H}_1 := \{\sigma_a : (x, y, z) \mapsto (x + a, y, z) \ : \ a \in A\}$;

$\mathcal{H}_2 := \{\sigma_i : (x, y, z) \mapsto (x, \eta^{i(q^2-q+1)}y, \eta^i z) \ : \ i = 0, \dots, ord(\eta) - 1\}$;

$\mathcal{H}_3 := \{\sigma_i : (x, y, z) \mapsto (x, y, \omega^i z) \ : \ i = 0, \dots, ord(\omega) - 1\}$,

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Let $q = p^\ell$, p prime. Then for any

1. i) $a = p^h, 1 \leq h \leq \ell$,
2. ii) $ord(\eta) \mid q^3 + 1$, $ord(\omega) \mid q^2 - q + 1$, with $\gcd(ord(\eta), ord(\omega)) = 1$,
3. iii) $0 < t_1 < q^2(q^2 - 1)(q^3 + 1) - q^3 + 2q^2 - 1, 0 < t_2 < q^3(q^2 - 1) - q, 0 < t_3 < q^3(q^2 - 1) - 1$,
4. iv) $N_1 = \min\{a - 2, t_2, t_3\}$, $M_1 = \min\{t_1, ord(\eta) - 2, ord(\omega) - 2\}$, and
5. v) $S = \max\{(a - 2)(q^3 + 1) + t_1 q, t_2(q^3 + 1) + (ord(\eta) - 2)q, t_3(q^3 + 1) + (ord(\omega) - 2)q\}$,

there exists a

$$[n, (M_1 + 1)(N_1 + 1), \geq n - S; a - 1, ord(\eta) - 1, ord(\omega) - 1]$$

recoverable code $\mathcal{C}_3$ over $\mathbb{F}_{q^6}$, where $n = q^8 - q^6 + q^5 - q^3$.

Generalized Hermitian curve: q odd, $S : y^{q^\ell+1} = x^q + x$ over $\mathbb{F}_{q^{2\ell}}$ with $\ell \geq 1$ odd has $\#S(\mathbb{F}_{q^{2\ell}}) = \mathbb{F}_q^{2\ell+1} + 1$ and $g = q^\ell(q-1)/2$. Consider two subgroups of the automorphism group of the curve given by

$$\begin{aligned} \mathcal{H}_1 &:= \{(x,y) \mapsto (x+a, y) \mid a^q + a = 0 \text{ and } a \in \mathbb{F}_{q^{2\ell}}\}, \text{ and} \\ \mathcal{H}_2 &:= \{(x,y) \mapsto (x, \lambda y) \mid \lambda \in \mathbb{F}_{q^{2\ell}} \text{ and } \lambda^{q^\ell+1} = 1\}, \end{aligned}$$

Proposition

Consider $0 \leq t_i, i = 1, 2$ *satisfying that*

$$S = M_1 q + M_2(q^\ell - 1) \leq q^{2\ell+1} - q$$

for $M_1 = \max\{t_1, q^\ell - 1\}$ *and* $M_2 = \max\{t_2, q-2\}$. *Let* $m_1 = \min\{t_1, q^\ell - 1\}, m_2 = \min\{t_2, q-2\}$, *we get a*

$$[q^{2\ell+1} - q, (m_1+1)(m_2+1), \geq q^{2\ell+1} - q - S; q-1, q^\ell] - code$$

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$$[q^{2\ell+1} - q, (m_1+1)(m_2+1), \geq q^{2\ell+1} - q - S; q-1, q^\ell] - \textit{code}$$

Theorem

Let $\mathcal{F}|\mathbb{F}_q$ be a function field of genus g. Consider s non trivial subgroups $\mathcal{H}_i$ of the automorphism group of $\mathcal{F}|\mathbb{F}_q$ satisfying

(i) $\mathcal{G} \simeq \prod_{i=1}^{s} \mathcal{H}_i$ *is a group; and let*

(ii) $\# \left(\mathcal{H}_i \setminus \left(\bigcup_{j<i} \mathcal{H}_j \right) \right) = r_i$.

$\vdots$

then there exists an

$$[n, \dim_{\mathbb{F}_q}(V), \geq d; r_1, r_2, \ldots, r_s]\text{-recoverable code.}$$

With the same notation as in Theorem 5, changing the hypothesis (*ii*) by

(ii') there exists $m \geq 1$ such that $1 \leq \#\left(\mathcal{H}_i \cap \left(\bigcup_{j \neq i} \mathcal{H}_j\right)\right) \leq m$, with $\#\mathcal{H}_i = r_i + m$.

we also have the existence of an

$$[n, \dim_{\mathbb{F}_q}(V), \geq d; r_1, r_2, \ldots, r_s]\text{-recoverable code.}$$

Table: Examples of the locally recoverable codes over $\mathbb{F}_{2^{12}}$.

Code $\mathcal{C}$	$\Delta(\mathcal{C})$
$[262080, 64, \geq 253952; 63, 4]$	0.03076
$[262080, 910, \geq 257152; 63, 12]$	0.01524
$[62400, 12, \geq 62226; 3, 4, 12]$	0.00259
$[4193280, 1395372, \geq 2738; 1364, 1023]$	0.66630