Algebraic curves through Fernando Torres' lens

Herivelto Borges

Universidade de São Paulo

Carleton Finite Fields eSeminar
A tribute to **Fernando Torres**
Dec 2020

Picture by Cícero Carvalho

Fernando Torres: in the beggingng

- Fernando Torres was born in Tarma-Peru in June 23, 1961.

Fernando Torres: The student

Fernando went to Elementary and Middle school in Tarma, at

- the school Niño Jesús en Praga and San Vicente de Paúl.

He went to high school in Lima, at

- the Santo Tomás de Aquino school.

After finishing high school, Fernando went to the Pontificia Universidad Católica del Perú - PUCP, where he got his

- Undergraduate degree in Mathematics in 1985.
- Masters degree in Mathematics in 1988.

In August of 1988, Fernando came to Brazil, and started his Ph.D at IMPA.

- He finished his Ph.D in 1993.
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- In 1998 he became a faculty in the University of Campinas (Unicamp).
- During his time at Unicamp he went abroad several times, especially to Italy and Spain, where he made many friends and collaborated with many mathematicians.
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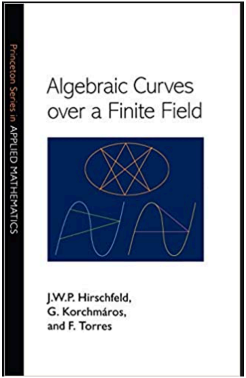
Fernando's Carrer:

- Fernando published over 50 articles, and co-authored, together with G. Korchmáros and J.W.P. Hirschfeld, the famous book



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Weierstrass points and double cover of curves (1993)

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(i) $\mathcal{Y}$ is a hyperelliptic curve;

(ii) there exists a point $P \in \mathcal{Y}$ such that $H(P)$ is hyperelliptic (i.e. $2 \in H(P)$)

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Fernando's thesis

- Let $\mathcal{X}$ be a curve (algebraic, projective, non-singular, irreducible) defined over an algebraically closed field of characteristic zero.

  Fernando's definition: Let γ be a non-negative integer. A curve $\mathcal{X}$ is called γ -hyperelitic curve when $\mathcal{X}$ is a double cover of a genus γ curve.

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In his thesis, Fernando generalized these equivalences for the case of the γ -hypertelitic curves,

Theorem (Torres 1995)

Let $\mathcal{X}$ be a curve of genus $g \geq 6\gamma + 4$, for some $\gamma \in \mathbb{Z}_{\geq 0}$. Then the following are equivalent.

(i) *$\mathcal{X}$ is γ-hyperelliptic*

(ii) *There exists $P \in \mathcal{X}$ such that $H(P)$ is γ-hyperelliptic*

(iii) *There exists a base-point-free complete linear series of $\mathcal{X}$ of projective dimension $2\gamma + 1$ and degree $6\gamma + 2$.*

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Theorem (Torres 1995)

Let $\mathcal{X}$ be a curve of genus g, and let $\gamma \in \mathbb{Z}_{\geq 1}$ be such that

$$g \geq \begin{cases} 30, & \text{if} \quad \gamma = 1 \\ \binom{12\gamma-6}{2} + 1, & \text{if} \quad \gamma \geq 2. \end{cases}$$

Then the following are equivalent.

(i) *$\mathcal{X}$ is γ-hyperelliptic*

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A very influential paper in Fernando's career

- Seminal paper: Stöhr, K.O. and Voloch, J.F.: **Weierstrass points and curves over finite fields**. Proc London Math. Soc $.52, 1 - 19(1986)$

It gives a new proof to the Riemann-Hypothesis for curves over finite fields, and in several circumstances, it improves the famous Hasse-Well bound:

$$N \leq 1 + q + 2g\sqrt{q}. \quad (1)$$

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Let $\mathcal{X}$ be a curve of genus g, and let $\mathcal{D} = g^r_d$ be a base-point-free $\mathbb{F}_q$-linear series on $\mathcal{X}$. Associated to a point $P \in \mathcal{X}$ we have the Hermitian P-invariants

$$j_0(P) = 0 < j_1(P) < \ldots < j_r(P) \leq d$$

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The Stöhr-Voloch divisor

In addition to the order-sequence $\epsilon_0, \epsilon_1, \cdots, \epsilon_r$, there exists another sequence of nonnegative integers

$$\nu_0 < \nu_1 < \cdots < \nu_{r-1},$$

called **Frobenius orders** (of $\mathcal{D}$), which is such that

$$\{\nu_0, ..., \nu_{r-1}\} = \{\epsilon_0, ..., \epsilon_r\} \setminus \{\epsilon_I\},$$

for some suitable $I \in \{1, \ldots, r\}$.

The Stöhr-Voloch theorem

Theorem

Let $\mathcal{X}$ be an irreducible, nonsingular, projective, algebraic curve of genus g defined over $\mathbb{F}_q$, equipped with a base-point-free linear series $\mathcal{D} = g^r_d$ over $\mathbb{F}_q$. If $\mathcal{D}$ has order-sequence $(\epsilon_0, \ldots, \epsilon_r)$, and $\mathbb{F}_q$-Frobenius order-sequence $(\nu_0, \ldots, \nu_{r-1})$, then the number N of $\mathbb{F}_q$-rational points of $\mathcal{X}$ satisfies

$$N \leq \frac{(\nu_1 + \cdots + \nu_{r-1})(2g-2) + (q+r)d}{\sum_{i=1}^{r}(\epsilon_i - \nu_{i-1})}. \tag{2}$$

Maximal Curves

Fernando's main contributions to the theory of Maximal curves are related to the following important problems:

- Determination of the possible genera of maximal curves over $\mathbb{F}_{q^2}$
- Determination of explicit equations for maximal curves over $\mathbb{F}_{q^2}$.
- Classification of maximal curves over $\mathbb{F}_{q^2}$ of a given genus.

The main ingredient of Fernando's approach was the systematic study of the Frobenius linear series

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Let us consider the set

$$\mathbf{M}\left(q^2\right) := \{g \in \mathbf{N} : g \text{ is the genus of an } \mathbb{F}_{q^2}\text{-maximal curve }\}$$

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Theorem (Fuhrmann–Torres (1996))

If the curve $\mathcal{X}$ is an $\mathbb{F}_{q^2}$-maximal of genus g, then either $g = \frac{q(q-1)}{2}$ or

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The important ingredients of their proof was the study of the linear system

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If q is odd, then any $\mathbb{F}_{q^2}$-maximal of genus $g = (q-1)^2/4$ is $\mathbb{F}_{q^2}$-isomorphic to the curve

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$$y^{q/2} + y^{q/4} + \cdots + y = x^{q+1}$$

provided that $q/2$ is a Weierstrass non-gap at some point of the $\mathcal{X}$.

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Theorem (Korchmáros-Torres (2002))

The genus g of an $\mathbb{F}_{q^2}$-maximal curve satisfies either

$$g \leq \left\lfloor \left(q^2 - q + 4\right)/6 \right\rfloor \quad or \quad g = \left\lfloor \frac{(q-1)^2}{4} \right\rfloor \quad or \quad g = (q-1)q/2.$$

To finish this part, I want to point out that one of Fernando's most recent results regarding the set $\mathbf{M}\left(q^2\right)$ can be found in

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Algebraic Geometry Codes

Let $\mathcal{X}$ be a non-singular, projective, geometrically irreducible, algebraic curve of genus g over $\mathbb{F}_q$. Let $D = P_1 + \cdots + P_n$ and G be $\mathbb{F}_q$-rational divisors on $\mathcal{X}$, where $P_1, \ldots, P_n$ are pairwise distinct rational points on $\mathcal{X}$, not in the support of G. Let $\mathcal{L}(G)$ be the Riemann-Roch space associated to G; The $\mathbb{F}_q$-vector space

$$C_{\mathcal{L}}(D, G) := \{(f(P_1), \ldots, f(P_n)) \mid f \in \mathcal{L}(G)\} \subseteq \mathbb{F}_q^n$$

is an $[n, k, d]$-code with parameters

$$k = \ell(G) - \ell(G - D) \quad \text{and} \quad d \geq n - \deg G$$

Some challenging problems are

- Computation of the parameters $[k, d]$.
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Fernando's contributions Finite Geometry

An (n, d)-*arc* $\mathcal{K}$ is a set of n points of the projective plane $PG(2, q)$ over $\mathbb{F}_q$ such that no line meets $\mathcal{K}$ in more than d points and there exists a line that meets $\mathcal{K}$ in exactly d points.

The arc is called **complete** if it is not contained in a $(n+1, d)$-arc; that is to say, if for every point P of $PG(2, q) \setminus \mathcal{K}$ there is a line through P meeting $\mathcal{K}$ in d points.

- FACT: (n, d)-arcs give rise linear codes with parameters $[n, 3, n-d]$

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A few facts:

- Plane curves of degree d in $PG(2,q)$ are natural sources of (n,d)-*arcs*
- The Hermitian curve $y^q + y = x^{q+1}$ gives rise to a complete $(q^3+1, q+1)$-*arc* in $PG(2,q^2)$.
- Non-singular $\mathbb{F}_{q^2}$-maximal curves give rise to complete $(1+q^2+2q, 3)$-*arc* in $PG(2,q^2)$.
- **Question:** Which curves of degree d and with n rational points give rise to a complete (n,d)-*arc* in $PG(2,q)$?

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Other important results by Fernando

- There are many other important results by Fernando that were not reported here.
- The same is true for his endless list of important collaborators: A. Cossidente, R. Pellikaan, M. Bernadini, J. Villanueva, D. Bartoli, M. Montanucci, A. Kazemifard, J.J. Moyano-Fernández, ...

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One of his many happy moments:

We express our profound gratitude for each moment that we were privileged to live and work with you, Fernando.