Last updated: April 21, 2009.

MATH 1002 Practice Problems - Sequences and Series.

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Sequences.

Definition. Let f be a real-valued function and k an integer. The *ordered* set of values of f at the integers $k,\ k+1,\ k+2, ...,$ is called a **sequence**, denoted by $\{f(k),\ f(k+1),\ f(k+2), ...\}$ or simply by $\Big\{f(n)\Big\}_{n=k}^{\infty}$. It is customary to write a_n for $f(n)$, so the sequence is expressed as $\{a_k,\ a_{k+1},\ a_{k+2}, ...\}$ or as $\Big\{a_n\Big\}_{n=k}^{\infty}$.

Definition. $\lim\limits_{n\to\infty} a_n = L$ iff for any $\epsilon > 0$ there exists a real number N such that $|a_n - L| < \epsilon$ for all $n \geq N$.

Definition. If $\lim\limits_{n\to\infty} a_n = L$ (finite), then the sequence is said to converge. Otherwise, it diverges.

Definition. $\lim\limits_{n\to\infty} a_n = \infty$ means that for any number $K > 0$ there exists a number $N > 0$ such that $a_n \geq K$ for all $n \geq N$.

Theorem. $\lim\limits_{n\to\infty} a_n = 0$ iff $\lim\limits_{n\to\infty} |a_n| = 0$.

Theorem. If $\lim\limits_{x\to\infty} f(x) = L$ and $a_n = f(n)$, then $\lim\limits_{n\to\infty} a_n = L$. The converse is not true.

PRACTICE:

Section 16: 16.3, 16.6 (a-d);

Section 17: 17.5.

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Definition. $\{a_n\}$ is **increasing** for $n \geq k$ if $a_{n+1} \geq a_n$ for all $n \geq k$. It is **decreasing** for $n \geq k$ if $a_n \geq a_{n+1}$ for all $n \geq k$. $\{a_n\}$ is **monotone** for $n \geq k$ if it is either increasing or decreasing for all $n \geq k$.

Definition. $\{a_n\}$ is **bounded above** if $\exists$ a number $M \in \mathbb{R}$ such that $M \geq a_n$ for all n. It is **bounded below** if $\exists$ a number $K \in \mathbb{R}$ such that $a_n \geq K$ for all n. $\{a_n\}$ is **bounded** if it is bounded above and bounded below.

Theorem. If $\{a_n\}$ is increasing and bounded above for all $n \geq k$, then $\{a_n\}$ converges.

Theorem. If $\{a_n\}$ is decreasing and bounded below for all $n \geq k$, then $\{a_n\}$ converges.

Theorem (Monotone Convergence). If $\{a_n\}$ is monotone and bounded for all $n \geq k$, then $\{a_n\}$ converges.

PRACTICE:

Section 18: 18.1 (a, b);

18.3 (a) Hint: show that the sequence is bounded above by 2.
18.3 (b) Hint: show that the sequence is bounded below by 1.
18.3 (c) Hint: show that the sequence is bounded above by 3.
18.3 (d) Hint: show that the sequence is bounded above by 3.

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Series.

Definition. Let $\left\{a_n\right\}_{n=1}^{\infty}$ be a sequence. We define another sequence $\left\{s_n\right\}_{n=1}^{\infty}$ as follows:
$s_1 = a_1$;
$s_2 = a_1 + a_2$;
$s_3 = a_1 + a_2 + a_3$;
.
.
$s_n = a_1 + a_2 + a_3 + ... + a_n = \sum_{k=1}^{n} a_k$.

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ is denoted by $\sum_{n=1}^{\infty} a_n$, and called a **series**. The series *converges* is the sequence $\left\{s_n\right\}$ of *partial sums* converges. Otherwise, the series diverges. If $\lim_{n\to\infty} s_n = s$, we write $\sum_{n=1}^{\infty} a_n = s$ and call s the *sum* of the series.

Example. Consider the **geometric series** $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ...$, where a and r are constants. It is divergent if $|r| \geq 1$. It is convergent if $|r| < 1$ and its sum is

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

Theorem. If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. (The converse is not true in general).

Theorem (The Test for Divergence). If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

PRACTICE:

Section 32: 32.1, 32.2, 32.4 (b,d), 32.5 (a, b, c, d, e, g, h, i, j), 32.9.

Theorem (The p-series). The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $1 \geq p$.

Theorem (The Comparison Test). Let $b_n \geq a_n \geq 0$ for all $n \geq k$.

(a) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Theorem (The Limit Comparison Test). Suppose that $a_n \geq 0$ and $b_n \geq 0$ for all $n \geq k$.

(a) If $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$ (and finite), then either both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, or they both diverge.

(b) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(c) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

PRACTICE:

1. Determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{\infty} ne^{-n^2}$.

(b) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$.

(c) $\sum_{n=1}^{\infty} \frac{3}{n^3+5}$.

(d) $\sum_{n=1}^{\infty} \frac{5}{3+2^n}$.

(e) $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n\sqrt{n}}$.

(f) $\sum_{n=1}^{\infty} \frac{n+1}{n^3}$.

(g) $\sum_{n=1}^{\infty} \frac{4+3^n}{2^n}$.

(h) $\sum_{n=1}^{\infty} \frac{n^2+1}{n^4+1}$.

(i) $\sum_{n=1}^{\infty} \frac{1}{n^3 - n}$.

(g) $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$.

Alternating Series. Absolute Convergence.

Definition. An **alternating series** is a series of the form $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 ...$, with $b_n > 0$ for all n.

The Alternating Series Test. If $b_n \geq b_{n+1}$ for all n, and $\lim_{n\to\infty} b_n = 0$, then the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges.

Definition. A series $\sum_{n=1}^{\infty} a_n$ is said to converge **absolutely**, if $\sum_{n=1}^{\infty} |a_n|$ converges. If $\sum_{n=1}^{\infty} a_n$ converges but not absolutely, it is said to converge **conditionally**.

The Ratio Test. Let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$, $L \geq 0$.

If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

If $L = 1$, then the test is inconclusive.

The Root Test. Let $L = \lim_{n\to\infty} |a_n|^{1/n}$, $L \geq 0$.

If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

If $L = 1$, then the test is inconclusive.

PRACTICE:

Section 33: 33.3 (a,d,e,m,n,o), 33.5 (a - h).

Power Series.

Definition. Let a and $c_n, (n = 0, 1, 2, ...)$ be real numbers. Then a series of the form $\sum\limits_{n=0}^{\infty} c_n(x-a)^n$ is called a **power series about the point** a. If it converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$, then R is called its **radius of convergence**.

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|.$$

PRACTICE:

Section 34: 34.1, 34.3 (a,b,c,d,f,j,k), 34.5 (a,b,c).

Taylor and Maclaurin Series.

If f has a power series representation at $x = a$, that is, if

$$\sum_{n=0}^{\infty} c_n(x-a)^n, \qquad |x-a| < R,$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

This power series is also called the **Taylor series of** f **about** a. When $x = 0$, it is called the **Maclaurin series of** f.

Corollary. The Taylor series of f about a is unique.

Taylor's Theorem . Let f and its first n derivatives be continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2), and let $a \in [x_1, x_2]$. Then for each $x \in [x_1, x_2]$ with $x \neq a$ there exists a point c between x_1 and x_2 such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!} + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} = T_n(x) + R_n(x).$$

$T_n(x)$ is called the n**-th degree Taylor polynomial of** f **about** a, $R_n(x)$ is called the **remainder**, or error.

The series converges to f iff $\lim\limits_{n\to\infty} R_n = 0$.

PRACTICE:

1. Find the Maclaurin series and the radius of convergence of f:

(a) $(1+x)^{-3}$. (b) $\ln(1+x)$.

2. Find the Taylor series and the radius of convergence for $f(x) = 1 + x + x^2$ about $a = 2$:

3. Find the 3-d degree Taylor polynomial for $f(x) = \sqrt{x}$ about $a = 4$.

4. Evaluate the indefinite integral as an infinite series.

(a) $\displaystyle\int \frac{\sin x}{x}\,dx$. (b) $\displaystyle\int e^{x^3}\,dx$.

5. Use series to approximate the definite integral within $|\text{error}| < 0.001$.

(a) $\displaystyle\int_0^1 \sin x^2\,dx$ (b) $\displaystyle\int_0^{1/2} x^2\,e^{-x^2}\,dx$.

6. Use series to evaluate the limit.

(a) $\displaystyle\lim_{x\to 0} \frac{x - Arctan(x)}{x^3}$. (b) $\displaystyle\lim_{x\to 0} \frac{\sin(x) - x + \frac{1}{6}x^3}{x^5}$.

Answers:

1. (a) $$1 - 3x + \frac{4\cdot 3}{2!}x^2 - \frac{5\cdot 4\cdot 3}{3!}x^3 + \frac{6\cdot 5\cdot 4\cdot 3}{4!}x^4 - \ldots = 1 - 3x + \frac{4\cdot 3\cdot 2}{2\cdot 2!}x^2 - \frac{5\cdot 4\cdot 3\cdot 2}{2\cdot 3!}x^3 +$$
$$+\frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{2\cdot 4!}x^4 - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)!\,x^n}{2(n!)} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+1)\,x^n}{2}. \quad (R = 1.)$$

1. (b) $$x - \frac{1}{2}x^2 + \frac{2}{6}x^3 - \frac{6}{24}x^4 + \ldots = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^n. \quad (R = 1.)$$

2. $7 + 5(x-2) + (x-2)^2$. Since $a_n = 0$ for large n, $R = \infty$.

3. $$2 + \frac{x-4}{2^2} - \frac{(x-4)^2}{2^6} + \frac{(x-4)^3}{2^9}.$$

4. (a) $\displaystyle C + \sum_{n=0}^{\infty} \frac{(-1)^n\,x^{2n+1}}{(2n+1)(2n+1)!}$. (b) $\displaystyle C + \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)n!}$.

5. (a) $\displaystyle\frac{13}{42} \approx 0.3095$. (b) ≈ 0.0354.

6. (a) $\displaystyle\frac{1}{3}$. (b) $\displaystyle\frac{1}{120}$.

Binomial Series.

If α is any real number and $|x| < 1$, then

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}\,x^2 + \frac{\alpha(\alpha-1)(\alpha-3)}{3!}\,x^3 + \ldots$$

This power series is a special case of the Maclaurin series and is called the **binomial series** of f.

PRACTICE:

1. Use the binomial series to expand the function as a power series (give the first four terms):

(a) $\dfrac{1}{(1+x)^4}$. (b) $\dfrac{1}{(2+x)^3}$. (c) $(1+x^2)^{1/3}$.

2. Use the binomial series to find the Maclaurin series of $f(x) = \dfrac{1}{\sqrt{1+x^3}}$. Then use the series to evaluate $f^{(9)}(0)$.

Answers:

1. (a) $1 - 4x + 10x^2 - 20x^3 + \ldots$

(b) $\dfrac{1}{8} - \dfrac{3}{8}\,x + \dfrac{3}{8}\,x^2 - \dfrac{5}{32}\,x^3 + \ldots$

(c) $1 + \dfrac{1}{3}\,x^2 - \dfrac{1}{9}\,x^4 + \dfrac{5}{51}\,x^6 - \ldots$

2. $1 + \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, 1\cdot 3\cdot 5\cdot \ldots (2n-1)}{2^n \cdot n!}\,x^{3n}$.

The coefficient of x^9 (corresponding to $n=3$) is $\dfrac{f^{(9)}(0)}{9!}$, so $f^{(9)}(0) = -\dfrac{9!\cdot 5}{8\cdot 2} = -113,400$.

Parametric Curves.

PRACTICE:

1. Sketch the curve by using parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases. Eliminate the parameter to find a Cartesian equation of the curve.

(a) $x = t^2, \quad y = 6 - 3t.$

(b) $x = \sqrt{t}, \quad y = 1 - t.$

(c) $x = 3\cos t, \quad y = 4\sin t, \quad \frac{\pi}{2} \geq t \geq -\frac{\pi}{2}$

(d) $x = e^t, \quad y = e^{-t}.$

2. Find the length of the curve.

(a) $x = e^t \cos t, \quad y = e^t \sin t, \quad \pi \geq t \geq 0.$ $(\sqrt{2}(e^\pi - 1))$

(b) $x = e^t + e^{-t}, \quad y = 5 - 2t, \quad 3 \geq t \geq 0.$ $(e^3 - e^{-3})$

(c) $x = e^t - t, \quad y = 4e^{t/2}, \quad 3 \geq t \geq -8.$ $(e^3 - e^{-8} + 11)$

3. Find the area above the x-axis and below the parametric curve

$$x = 1 + e^t, \quad y = t^2, \quad 2 \geq t \geq 0. \qquad (10e^2 - 2)$$