The Ratio Bound for Erdős-Ko-Rado Type Theorems

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The Question

- A *k-set system* is a collection of subsets from $\{1, 2, \ldots, n\}$ in which each subset has size k.
- A k-set system is *intersecting* if for all sets A, B in the system

$$A \cap B \neq \emptyset.$$

For a fixed k and n what is the largest intersecting set system?

The Solution

Example (A canonically intersecting set system)

$$[1,2,3],\ [1,2,4],\ [1,2,5],\ [1,2,6],\ [1,2,7],$$
$$[1,3,4],\ [1,3,5],\ [1,3,6],\ [1,3,7],\ [1,4,5],$$
$$[1,4,6],\ [1,4,7],\ [1,5,6],\ [1,5,7],\ [1,6,7]$$

Size is $\binom{7-1}{3-1} = \binom{6}{2} = 15$.

Theorem (Erdős-Ko-Rado, 1961)

Let $\mathcal{F}$ be an intersecting k-set system on an n-set. If $n > 2k$, then

1. $|\mathcal{F}| \leq \binom{n-1}{k-1}$,
2. *and $\mathcal{F}$ meets this bound if and only if it is canonically intersecting (all sets contain a common element).*

Kneser Graph

The Kneser graph $K(n, k)$

1. vertices are all k-sets from $\{1, \ldots, n\}$;
2. two k-sets are adjacent if they are disjoint.

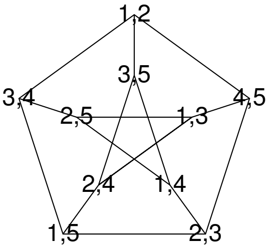


Figure: The Petersen graph $K(5, 2)$.

What is the size of the largest coclique in $K(n, k)$?

Algebraic Graph theory

1. Represent a graph X as a matrix $A(X)$ (Adjacency Matrix).
2. Rows and columns are indexed by the vertices,
3. u, v entry of $A(X)$ is 1 if u and v are adjacent, and 0 otherwise.

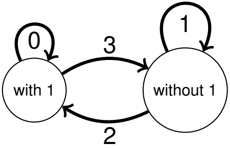
Example for Kneser graph $K(5,2)$:

$$
\begin{array}{c c}
 & \begin{array}{cccccccccc} 12 & 13 & 14 & 15 & 23 & 24 & 25 & 34 & 35 & 45 \end{array} \\
\begin{array}{c} 12 \\ 13 \\ 14 \\ 15 \\ 23 \\ 24 \\ 25 \\ 34 \\ 35 \\ 45 \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

In a d-regular graph, the all ones vector is an eigenvector with eigenvalue d.

Quotient Graphs

$$\begin{array}{cc} & \begin{array}{cccc|cccccc} 12 & 13 & 14 & 15 & 23 & 24 & 25 & 34 & 35 & 45 \end{array} \\ \begin{array}{c} 12 \\ 13 \\ 14 \\ 15 \\ \\ 23 \\ 24 \\ 25 \\ 34 \\ 35 \\ 45 \end{array} & \left(\begin{array}{cccc|cccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

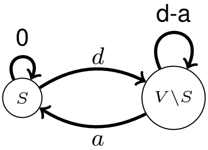


$$\begin{array}{cc} & \begin{array}{cc} \text{with } 1 & \text{without } 1 \end{array} \\ \begin{array}{c} \text{with } 1 \\ \text{without } 1 \end{array} & \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \end{array}$$

This is a *quotient matrix*.

1. The sets of vertices are the orbits of $\mathrm{Sym}(\{1\}) \times \mathrm{Sym}(\{2,3,4,5\})$ on k-sets.
2. If the sets of vertices are orbits of a group action, then they forms an *equitable partition*.

Hoffman's Ratio Bound



If S is a coclique in a d-regular graph

$$\begin{array}{cc} & \begin{array}{cc} S & V\backslash S \end{array} \\ \begin{array}{c} S \\ V\backslash S \end{array} & \begin{pmatrix} 0 & d \\ a & d-a \end{pmatrix} \end{array}$$

This is the quotient graph.

1. Counting edges gives $a = \frac{d|S|}{|V|-|S|}$.
2. The eigenvalues of the quotient graph are d and $-a = -\frac{d|S|}{|V|-|S|}$.
3. If $\{S, V\backslash S\}$ form an equitable partition, the d and $-a$ are also eigenvalues of the adjacency matrix.
4. Eigenvalues of the quotient matrix **interlace eigenvalues** of the adjacency matrix.

Hoffman's Ratio Bound

If τ is the least eigenvalue of the adjacency matrix is τ, by interlacing,

$$\tau \leq -\frac{|S|d}{|V| - |S|}$$

Delsarte/Hoffman/Ratio Bound

If X is a d-regular graph then

$$\alpha(X) \leq \frac{|V(X)|}{1 - \frac{d}{\tau}}$$

and τ is the least eigenvalue for the (weighted) adjacency matrix for X.

Moreover, if equality holds, and v_S is a characteristic vector of a maximum coclique, then

$$v_S - \frac{|S|}{|V(X)|}\mathbf{1}$$

is a τ-eigenvector.

The Eigenvectors

Characteristic vector of a max coclique in $K(5,2)$ (all the sets that contain 1) is:

$$v_{S_1} = (1,\ 1,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0).$$

Balanced characteristic vector is

$$v_{S_1} - \frac{|S|}{|V|}\mathbf{1} = \frac{1}{5}(3,\ 3,\ 3,\ 3,\ -2,\ -2,\ -2,\ -2,\ -2,\ -2).$$

$$\begin{array}{cc}
 & \begin{array}{cccccccccc} 12 & 13 & 14 & 15 & 23 & 24 & 25 & 34 & 35 & 45 \end{array} \\
\begin{array}{c} 12\\13\\14\\15\\23\\24\\25\\34\\35\\45 \end{array} &
\begin{pmatrix}
0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&1&1&0&0&1\\
0&0&0&0&1&0&1&0&1&0\\
0&0&0&0&1&1&0&1&0&0\\
0&0&1&1&0&0&0&0&0&1\\
0&1&0&1&0&0&0&0&1&0\\
0&1&1&0&0&0&0&1&0&0\\
1&0&0&1&0&0&1&0&0&0\\
1&0&0&1&0&1&0&0&0&0\\
1&1&0&0&1&0&0&0&0&0
\end{pmatrix}
\end{array}
\begin{pmatrix} 3\\3\\3\\3\\-2\\-2\\-2\\-2\\-2\\-2 \end{pmatrix}
=
\begin{pmatrix} -6\\-6\\-6\\-6\\4\\4\\4\\4\\4\\4 \end{pmatrix}
= -2
\begin{pmatrix} 3\\3\\3\\3\\-2\\-2\\-2\\-2\\-2\\-2 \end{pmatrix}$$

The Eigenvalues

1. The group $\mathrm{Sym}(n)$ acts transitively on the k-sets,
2. $\mathrm{Sym}(n-k) \times \mathrm{Sym}(k)$ is the stabilizer of a k-set.
3. The action of $\mathrm{Sym}(n)$ on the k-sets is equivalent to the action of $\mathrm{Sym}(n)$ on the cosets $\mathrm{Sym}(n)/(\mathrm{Sym}(n-k) \times \mathrm{Sym}(k))$.
4. The permutation character is, for $g \in \mathrm{Sym}(n)$,

$$\mathrm{fix}(g) = \mathrm{ind}\left(\mathbf{1}_{\mathrm{Sym}(n-k)\times\mathrm{Sym}(k)}\right)^{\mathrm{Sym}(n)} = [n] + [n-1,1] + \cdots + [n-k,k].$$

5. Each eigenspace is a $\mathrm{Sym}(n)$-module as it is invariant under the action of $\mathrm{Sym}(n)$.

Irred. Rep	Eigenvalues	Dimension
$[n]$	$\binom{n-k}{k}$	1
$[n-1,1]$	$-\binom{n-k-1}{k-1}$	$\binom{n}{1}-\binom{n}{0}$
$[n-2,2]$	$\binom{n-k-2}{k-2}$	$\binom{n}{2}-\binom{n}{1}$
...	...	...
$[n-i,i]$	$-1^i\binom{n-k-i}{k-i}$	$\binom{n}{i}-\binom{n}{i-1}$
...	...	...
$[n-k,k]$	$-1^k\binom{n-2k}{0}$	$\binom{n}{k}-\binom{k}{k-1}$

Ratio Bound for the Kneser

By ratio bound the largest coclique is no larger than

$$\frac{|V(K(n,k))|}{1-\frac{d}{\tau}} = \frac{\binom{n}{k}}{1-\frac{\binom{n-k}{k}}{-\binom{n-k-1}{k-1}}} = \binom{n-1}{k-1}.$$

If S is a maximum coclique, then :

1. $v_S - \frac{k}{n}\mathbf{1}$ is a $-\binom{n-k-1}{k-1}$-eigenvector.
2. v_S is in the span of the $\binom{n-k}{k}$-eigenspace and the $-\binom{n-k-1}{k-1}$-eigenspace. (These are isomorphic to the $[n]$ and the $[n-1,1]$-modules.)
3. The stabilizer of a maximum coclique is $\mathrm{Sym}(1) \times \mathrm{Sym}(n-1)$

$$\mathrm{ind}\left(\mathbf{1}_{\mathrm{Sym}(1)\times\mathrm{Sym}(n-1)}\right)^{\mathrm{Sym}(n)} = [n] + [n-1,1].$$

4. The vectors v_{S_i} for $i = 1, \dots, n$ (where S_i is a canonical coclique) is a spanning set for the $[n]$ and the $[n-1,1]$-modules.
5. If S is a maximum coclique, then v_S is a linear combination of the v_{S_i}.

t-intersecting k-subsets

What is the largest set of k-subsets so any two subsets have at least t elements in common?

Example (canonical t-intersecting system)

$$[1,2,3],\ [1,2,4],\ [1,2,5],\ [1,2,6],\ [1,2,7]$$

Theorem (EKR 1961, Frankl 1978, Wilson 1984)

Let $\mathcal{F}$ be an t-intersecting k-set system on an n-set. If $n > (t+1)(k-t+1)$, then

1. $|\mathcal{F}| \leq \binom{n-t}{k-t}$,
2. *and $\mathcal{F}$ meets this bound if and only if it is canonically t-intersecting.*

Weighted Adjacency matrices

Define a set of matrices A_i:

1. Vertices are k-subsets of $\{1, \ldots, n\}$.
2. Two vertices are adjacent if their intersection has size $k - i$.
3. The set of matrices $\{A_0, A_1, \ldots, A_k\}$ is an **association scheme**.
4. The A_i are simultaneously diagonalizable—they all have the same eigenspaces!

What is the largest coclique in graph X with

$$A(X) = \sum_{i=0}^{t-1} A_{k-i}?$$

The eigenvalues of $A(X)$ are a sum of the eigenvalues of A_{k-i}.

Wilson's proof of EKR theorem

1. The stabilizer of a canonical t-intersecting set is $\mathrm{Sym}(t) \times \mathrm{Sym}(n-t)$, and

$$\mathrm{ind}\left(1_{\mathrm{Sym}(t)\times\mathrm{Sym}(n-t)}\right)^{\mathrm{Sym}(n)} = [n] + [n-1,1] + [n-2,2] + \cdots + [n-t,t]$$

2. Find a **weighted adjacency matrix**, so a weighting a_i so that

$$A = \sum_{i=0}^{t-1} a_i A_{k-i}.$$

so the eigenvalues for the modules

$$[n-1,1], [n-2,2], [n-3,3], \ldots, [n-t,t]$$

are equal to -1 (this is the least eigenvalue),
and the row sum (eigenvalue for $\mathbf{1}$, so for the module $[n]$) is maximized.

This is what Wilson did in his 1984 paper.

Near-the-End-of-Talk Recap

What is the size of the largest set of “intersecting” objects?

Approach:

1. Model the problem with a graph.
2. Find the automorphism group G of the graph, each eigenspace of the graph is a G-module.
3. Find the stabilizer of the conjectured maximum cocliques H.
4. Find a weighting so the representations in $\mathrm{ind}\,(\mathbf{1}_H)^G$ (except trivial) are -1 and maximize the eigenvalue for the trivial representation (this up as a linear program).
5. Need the -1 to be the least eigenvalue and the ratio bound to be equal!!
6. The balanced characteristic vector of any maximum coclique is a eigenvector for the least eigenvalue

Places where this works

Example

1. k-subsets of $\{1, \ldots, n\}$, with n big
2. Two subsets are adjacent if they contain t common elements.
3. Automorphism group of the k-sets is $\mathrm{Sym}(n)$
4. Stabilizer group for a canonical coclique is $\mathrm{Sym}(n-t) \times \mathrm{Sym}(t)$

Example

1. k-subsets in a block design
2. Two blocks are adjacent if they contain a common element.
3. The intersection graph is a strongly regular graph.

More places where this works

Example

1. Perfect matching of $\{1, \ldots, 2k\}$
2. Two perfect matchings are adjacent if they contain a common edge.
3. Automorphism group of the perfect matchings is $\mathrm{Sym}(2k)$
4. Stabilizer group for a canonical coclique is $\mathrm{Sym}(2k-2) \times \mathrm{Sym}(2)$

Example

1. Permutations in $\mathrm{Sym}(n)$
2. Two σ and π are adjacent if there is an i with $i^\sigma = i^\pi$
3. Automorphism group contains $\mathrm{Sym}(n)$
4. Stabilizer group for a canonical coclique is s $\mathrm{Sym}(n-1) \times \mathrm{Sym}(1)$